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Research Article

$\mathbb{Z}_k$-zonal labeling of plane graphs

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Abstract: Zonal labeling in plane graphs plays a key role in the structural analysis of graph faces and is closely related to the Four Color Theorem. In this paper, we extend the classical zonal labeling framework from the modular group $\mathbb{Z}_3$ to $\mathbb{Z}_k$ for all $k \geq 2$. We show that this extension preserves the essential properties of zonality and provides increased flexibility for studying a broader class of plane graphs. We establish the necessary and sufficient conditions for $\mathbb{Z}_k$-zonal labeling and examine the corresponding inner zonal structures.We also characterize the implementation of inner zonal labeling within Halin graphs. This extended framework offers clear insights and practical methods for a deeper understanding of zonal characteristics in planar graph theory.

Keywords: $\mathbb{Z}_k$-zonal labeling, inner $\mathbb{Z}_k$-zonal, planar graph, Halin graph, wheel graph.

AMS Subject Classification: 05C10, 05C15, 05C78

1. Introduction

Zonal labeling is a modular facial labeling scheme for plane graphs in which each vertex is assigned a nonzero element of $\mathbb{Z}_k$, and the sum of the labels along every facial boundary must be zero modulo k. This requirement establishes a direct connection between local facial configurations and global modular constraints. The classical case $k = 3$, developed in earlier studies, showed how even a limited label set can reveal meaningful interactions between facial cycles, fixed planar embeddings, and the cycle space of a graph. These insights inspired broader investigation into how modular arithmetic and plane graph structure jointly influence facial zero-sum conditions. For

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previously obtained results on zonal labeling, the reader is referred to the references [1, 3–5]. The purpose of this paper is to extend the classical framework and formulate a comprehensive theory of $\mathbb{Z}_k$-zonal labeling for all integers $k \geq 2$. Increasing the modulus enlarges the set of feasible labels and allows richer structural patterns to emerge across different families of plane graphs. Prior work on modular facial constraints demonstrates structural properties that naturally extend to larger moduli and indicate the necessity of establishing a unified theoretical foundation capable of accommodating these broader modular behaviors.

The paper is organized as follows. Section 2 presents the definitions of $\mathbb{Z}_k$-zonal and inner zonal labeling and establishes key algebraic principles, including modulus expansion. It then provides complete zonality characterizations for several fundamental families of plane graphs, such as trees, cycles, wheels, complete bipartite plane graphs, unicyclic graphs, and plane graphs with small cycle rank. Section 3 investigates distinct-label $\mathbb{Z}_k$-zonal labelings and determines the precise modular conditions that allow or prohibit such assignments. Section 4 focuses on Halin graphs and develops an inductive method for constructing inner zonal labelings based on the structure of their characteristic trees. These results extend earlier developments in zonal labeling, bring together observations that previously appeared in separate contexts, and provide a coherent modular framework for understanding and analyzing facial zero-sum constraints in plane graphs.

2. $\mathbb{Z}_k$-zonal labeling

Definition 1. Let $G = (V, E, R)$ be a plane graph with the vertex set $V = V(G)$, the edge set $E = E(G)$, and the set of regions $R = R(G)$. A $\mathbb{Z}_k$-zonal labeling of G is a function $\ell : V(G) \to \{1, 2, \ldots, k-1\} \subset \mathbb{Z}_k$ such that for every region $R_i \in R(G)$, the sum of the labels on the boundary vertices satisfies $\sum_{v \in V(\partial R_i)} \ell(v) \equiv 0 \pmod{k}$, where ∂R_i denotes the boundary of R_i.

A plane graph admitting such a labeling is called $\mathbb{Z}_k$*-zonal*. For a plane graph G, a labeling $\ell(G)$ is called an *inner* $\mathbb{Z}_k$*-zonal labeling* if the value $\ell(R_i) = \sum_{v \in V(\partial R_i)} \ell(v)$ is equal to 0 in $\mathbb{Z}_k$ for all regions of G, with at most one exception. A plane graph G is said to be *inner* $\mathbb{Z}_k$*-zonal* if it admits such a labeling. Given any zonal labeling ℓ of G, the labeling $\bar{\ell}$ defined by $\bar{\ell}(v) = k - \ell(v)$ in $\mathbb{Z}_k$ for every $v \in V(G)$, is referred to as the *complementary labeling* of G.

When assigning nonzero labels from $\mathbb{Z}_k$ to the vertices of a graph such that the sum of the labels on the boundary of each region is congruent to zero modulo k, the graph cannot be trivial (i.e., $|V(G)| = 1$). Therefore, throughout this paper we restrict our attention to nontrivial plane graphs.

In the case $k = 2$, the only available vertex label is 1. Consequently, the $\mathbb{Z}_2$-zonal condition reduces to the requirement that the boundary of every face contains an even number of vertices. Hence, any $\mathbb{Z}_2$-zonal graph must consist exclusively of faces of even length. Although this condition is closely related to bipartiteness, the two notions are not equivalent in general. For instance, the complete bipartite graph $K_{1,2}$

does not admit a $\mathbb{Z}_2$-zonal labeling.

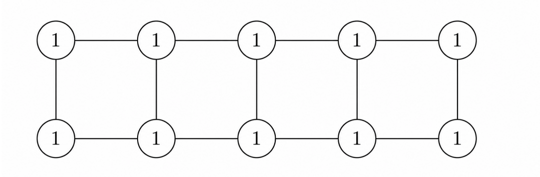

Figure 1. **The ladder graph** $L_5 = P_5 \,\square\, P_2$

Certain classes of plane graphs inherently satisfy the structural prerequisites for a $\mathbb{Z}_2$-zonal labeling due to their specific configurations. For example, in the grid graphs $L_{n,m} = P_n \square P_m$, all interior faces are 4-cycles. The outer face forms a cycle of length $2(m+n-2)$. Therefore, the combinatorial condition for a $\mathbb{Z}_2$-zonal labeling is automatically satisfied (see Figure 1). More generally, every 2-connected bipartite plane graph satisfies the conditions required for a $\mathbb{Z}_2$-zonal labeling, because each face boundary is a cycle and every cycle in a bipartite graph has even length.
Also, the complete bipartite graph $K_{1,n}$ is isomorphic to the star graph S_n on $n+1$ vertices. In this case, a $\mathbb{Z}_2$-zonal labeling exists if and only if n is odd. In the complete bipartite graph $K_{2,n}$ $(n \geq 2)$, every face is again bounded by a 4-cycle. It follows directly that a $\mathbb{Z}_2$-zonal labeling is guaranteed for all integers $n \geq 2$.

Lemma 1. *Let G be a $\mathbb{Z}_k$-zonal graph and $k' = mk$ for some integer $m \geq 2$. Then G is $\mathbb{Z}_{k'}$-zonal.*

Proof. Following the idea as Observation 2.2 of [2], define the group homomorphism $\varphi : \mathbb{Z}_k \to \mathbb{Z}_{k'}$ by $\varphi(x) = mx$. Note that $\ker(\varphi) = \{0\}$, since $\varphi(x) = 0$ in $\mathbb{Z}_{k'}$ implies that $k' \mid mx$. Hence $k \mid x$ and therefore $x \equiv 0 \pmod{k}$. Let ℓ be a $\mathbb{Z}_k$-zonal labeling of G. Since $\ell(v) \notin \ker(\varphi)$ for every $v \in V(G)$, define $\ell' = \varphi \circ \ell : V(G) \to \mathbb{Z}_{k'} \setminus \{0\}$. For any face R of G, the zonal condition for ℓ gives $\sum_{v \in \partial R} \ell(v) \equiv 0 \pmod{k}$, so this sum equals $t_R k$ for some integer t_R. Then

$$\sum_{v \in V(\partial R)} \ell'(v) = \sum_{v \in V(\partial R)} \varphi(\ell(v)) = \varphi\left(\sum_{v \in V(\partial R)} \ell(v)\right) = \varphi(t_R k) = t_R \cdot mk = t_R k' \equiv 0 \pmod{k'}.$$

Hence ℓ' is a $\mathbb{Z}_{k'}$-zonal labeling of G. □

Remark 1. The converse of Lemma 1 does not necessarily hold, in general. For example, consider the windmill graph $Wd(k,n)$, defined as the graph obtained by joining n copies of the complete graph K_n at a single common vertex. In particular, the graph $Wd(3,3)$, shown in Figure 2, illustrates this construction.
We first construct a zonal labeling of the windmill graph $Wd(3,3)$ modulo 20, and then prove that no zonal labeling exists for this graph modulo 5. Define a labeling ℓ on $Wd(3,3)$

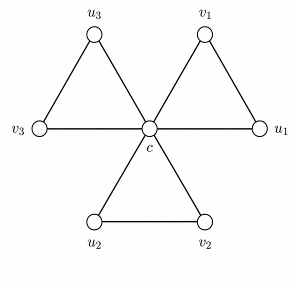

Figure 2. **The windmill graph Wd(3,3)**

by assigning the value 10 to the unique vertex of degree 6, and the value 5 to each vertex of degree 2. With respect to this labeling, the sum of the labels on the boundary of each of the three internal triangular faces, as well as on the boundary of the outer face, is congruent to zero modulo 20. Hence, $Wd(3,3)$ admits a $\mathbb{Z}_{20}$-zonal labeling.
Now we show that $Wd(3,3)$ admits no $\mathbb{Z}_5$-zonal labeling. Suppose, to the contrary, that such a labeling ℓ exists. For each of the three triangular faces, the $\mathbb{Z}_5$-zonal condition implies $\ell(v_1)+\ell(u_1)+\ell(c) \equiv \ell(v_2)+\ell(u_2)+\ell(c) \equiv \ell(v_3)+\ell(u_3)+\ell(c) \equiv 0 \pmod 5$. Consequently, $\ell(v_i)+\ell(u_i) \equiv \ell(v_j)+\ell(u_j) \pmod 5$ for all $1 \le i,j \le 3$. Applying the zonal condition to the outer face yields $\ell(v_1)+\ell(u_1)+\ell(v_2)+\ell(u_2)+\ell(v_3)+\ell(u_3)+\ell(c) \equiv 0 \pmod 5$. Using the relation $\ell(v_3)+\ell(u_3)+\ell(c) \equiv 0 \pmod 5$ together with the congruences $\ell(v_1)+\ell(u_1) \equiv \ell(v_2)+\ell(u_2) \pmod 5$, we obtain $2(\ell(v_1)+\ell(u_1)) \equiv 0 \pmod 5$, which implies $\ell(v_1)+\ell(u_1) \equiv 0 \pmod 5$. Substituting this back into the face condition for any triangular region yields $\ell(c) \equiv 0 \pmod 5$, contradicting the requirement that all vertex labels be nonzero in a zonal labeling. Therefore, $Wd(3,3)$ admits a $\mathbb{Z}_{20}$-zonal labeling but does not admit a $\mathbb{Z}_5$-zonal labeling.

Throughout this paper, we assume $k \ge 3$, unless explicitly noted otherwise for the case $k = 2$.

Remark 2. Let $n \ge 2$. Since $K_{2,n}$ is a 2-connected bipartite plane graph (see Figure 3), Proposition 2.7 in [2] implies that $K_{2,n}$ admits a $\mathbb{Z}_k$-zonal labeling for every $k \ge 3$.

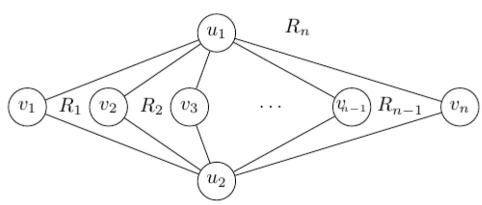

Figure 3. **A plannar embadding of $K_{2,n}$.**

For graphs whose internal faces are all triangles, the family of wheel graphs provides a notable example. The wheel graph W_n is obtained by joining each vertex of the cycle C_n to a single central vertex, forming a plane graph whose interior faces are all triangles except for the outer cycle (see Figure 4). The following theorem determines the necessary and sufficient conditions under which such graphs admit a $\mathbb{Z}_k$-zonal labeling.

Theorem 1. *Let n be an integer with $n \geq 3$, and $W_n = C_n \vee K_1$ be the wheel graph. Then W_n is $\mathbb{Z}_k$-zonal if and only if either $\gcd(n,k) = d \geq 3$ or $d = 2$ and $4 \mid n$.*

Proof. Let v_0 denote the central vertex of W_n and let $v_1, \ldots, v_n$ be the vertices of the cycle C_n in cyclic order.

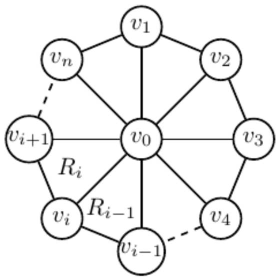

Figure 4. **The wheel graph W_n**

First suppose $d = \gcd(n,k) \geq 3$. Define a labeling ℓ on $V(W_n)$ by

$$\ell(v_0) = \frac{(d-2)k}{d}, \qquad \ell(v_i) = \frac{k}{d} \quad (1 \leq i \leq n).$$

For any triangular face $v_0 v_i v_{i+1}$, we have $\ell(v_0) + \ell(v_i) + \ell(v_{i+1}) = \frac{(d-2)k}{d} + \frac{k}{d} + \frac{k}{d} = k \equiv 0 \pmod{k}$. Moreover, the sum of the labels on the outer face C_n is $\sum_{i=1}^{n} \ell(v_i) = n \cdot \frac{k}{d} = k \cdot \frac{n}{d} \equiv 0 \pmod{k}$, since $d \mid n$. Hence, ℓ is a $\mathbb{Z}_k$-zonal labeling of W_n.
Next assume that $d = 2$ and $4 \mid n$. Assign the labels 1 and $\frac{k}{2} - 1$ alternately to the vertices of the cycle and set $\ell(v_0) = \frac{k}{2}$. Then, for each triangular face $v_0 v_i v_{i+1}$, the sum is $1 + \left(\frac{k}{2} - 1\right) + \frac{k}{2} = k \equiv 0 \pmod{k}$. Furthermore, the outer face sum equals $\frac{n}{2}\left(1 + \frac{k}{2} - 1\right) = \frac{n}{2} \cdot \frac{k}{2} = \frac{nk}{4} \equiv 0 \pmod{k}$ because $4 \mid n$. Therefore, W_n is also $\mathbb{Z}_k$-zonal in this case.
Conversely, suppose that either $d = 1$, or $d = 2$ and $n \equiv 2 \pmod{4}$. Assume, for contradiction, that W_n admits a $\mathbb{Z}_k$-zonal labeling ℓ. Let $\ell(v_0) = c$. Considering the triangular faces of W_n, the labels on the vertices of the cycle must either all be equal or alternate between two distinct values a and b.

Case 1. $\ell(v_i) = a$ for all $1 \leq i \leq n$. The outer face condition yields $na \equiv 0 \pmod{k}$. If $d = 1$, then $k \mid a$. Consequently, $a \equiv 0 \pmod{k}$, a contradiction. Now suppose $d = 2$. Since $1 \leq a \leq k-1$ and $na \equiv 0 \pmod{k}$, it follows that $a = \frac{k}{2}$. Applying the triangular face condition gives $\frac{k}{2} + \frac{k}{2} + c \equiv 0 \pmod{k}$, which implies $c \equiv 0$, again a contradiction.

Case 2. The labels on C_n alternate between two distinct values a and b. Necessarily, n is even. The outer face condition becomes $\frac{n}{2}(a+b) \equiv 0 \pmod{k}$. Under the present assumptions, we have $\gcd(\frac{n}{2}, k) = 1$. Therefore, $a + b \equiv 0 \pmod{k}$. Substituting this relation into the triangular face condition yields $c \equiv 0 \pmod{k}$, a contradiction. Since both cases lead to a contradiction, no $\mathbb{Z}_k$-zonal labeling exists under the stated conditions. Consequently, W_n admits a $\mathbb{Z}_k$-zonal labeling if and only if either $\gcd(n, k) \geq 3$, or $\gcd(n, k) = 2$ and $4 \mid n$. □

We note that analogous results for abelian groups were established in Propositions 2.5 and 2.6 of [2], where the existence of a Γ-zonal labeling of W_n is characterized in terms of the existence of an element of appropriate order in Γ. The formulation presented here specializes those results to the cyclic group $\mathbb{Z}_k$ and provides a self-contained elementary proof.

Theorem 2. *Let T be a tree. Then T admits a $\mathbb{Z}_k$-zonal labeling.*

Proof. Since trees are planar graphs with no cycles and exactly one unbounded exterior region, a zonal labeling for a tree needs only to satisfy the zonal condition on this outer face. We construct an explicit labeling for every integer $k \geq 3$, and for every tree T with vertex set $V(T)$. Let $\ell : V(T) \to \mathbb{Z}_k \setminus \{0\} = \{1, \ldots, k-1\}$ denote the labeling to be defined. We consider separately the cases where the number of vertices of T is even or odd.
Assume first that the tree T has an even number of vertices, say $|V(T)| = 2m$. Partition the vertex set into two equal parts: $V(T) = V_1 \cup V_2$, $|V_1| = |V_2| = m$. Choose an arbitrary nonzero label $a \in \{1, \ldots, k-1\}$, and assign it to every vertex of V_1. For the second part V_2, assign the additive inverse of a in $\mathbb{Z}_k$, namely $\bar{a} = k - a$. Thus the labeling is defined by

$$\ell(v) = \begin{cases} a & v \in V_1 \\ \bar{a} & v \in V_2 \end{cases}.$$

The sum of all vertex labels is then $\sum_{v \in V(T)} \ell(v) = m \cdot a + m \cdot (k - a) = m \cdot k \equiv 0 \pmod{k}$. So the $\mathbb{Z}_k$-zonal condition for the exterior face is satisfied. Now assume that $|V(T)| = 2m + 1$. To achieve this, we create a partition with slightly different sizes. Split the vertex set as $V(T) = V_1 \cup V_2 \cup \{u\}$, where $|V_1| = m + 1, |V_2| = m - 1$. We assign the following labels: every vertex of V_1 receives the label 1, every vertex of V_2 receives its additive inverse in $\mathbb{Z}_k$ i.e. $k - 1$, and the remaining vertex u receives the label $k - 2$. Formally,

$$\ell(v) = \begin{cases} 1 & v \in V_1 \\ k-1 & v \in V_2 \\ k-2 & v = u \end{cases}.$$

We now compute the sum of all vertex labels: $(m+1)\cdot 1+(m-1)\cdot(k-1)+1\cdot(k-2) = mk$. Reducing modulo k, the full expression simplifies to $\sum_{v\in V(T)} \ell(v) = mk \equiv 0 \pmod{k}$. Thus the zonal condition holds in this case as well (see Figure 5). □

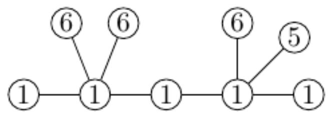

Figure 5. **A tree of order 9 with a $\mathbb{Z}_7$-zonal labeling.**

Theorem 3. *For every integer $n \geq 3$, the cycle graph C_n is $\mathbb{Z}_k$-zonal.*

Proof. Since a cycle has two faces whose boundaries consist of the same vertex set, a labeling analogous to the construction in Theorem 2 may be applied. Specifically, one assigns labels to the vertices of the cycle so that the sum of the labels on each face is congruent to zero modulo k. This yields the required labeling. An analogous result for arbitrary abelian groups is given in Proposition 2.4 of [2]. □

In 2022, Bowling and Zhang [4] investigated zonal labelings on graphs of small cycle rank. Building upon their results and proof techniques, we extend their approach to the class of unicyclic and bicyclic graphs.

A unicyclic graph $G \neq C_n$ can be expressed as the union of a cycle and a forest, that is, $G = C \cup F$, where each tree of the forest is attached to a vertex of the unique cycle C.
Every planar embedding of a unicyclic graph has exactly two faces. If $|V(F)| = 1$, then one face is bounded by the vertices of C, while the other face contains all vertices of G. Consequently, the two facial boundaries contain $|V(C)|$ and $|V(C)|+1$ vertices, respectively. It follows that no labeling can simultaneously make the sums of the labels on both faces congruent to zero modulo k. Hence, G does not admit a $\mathbb{Z}_k$-zonal labeling. Now assume that $|V(F)| \geq 2$. Then there exists a planar embedding of G in which the bounded face has boundary precisely C, while the unbounded face contains all vertices of the graph. Since there are at least two vertices outside the cycle, Theorems 2 and 3 imply that one may assign labels to the vertices of C together with the vertices outside the cycle so that the facial sums on both the inner and outer faces are congruent to zero modulo k. Therefore, G admits a $\mathbb{Z}_k$-zonal labeling in this case.

In the following section, we investigate the existence of $\mathbb{Z}_k$-zonal labelings in graphs containing two cycles with the minimum degree $\delta(G) \geq 2$. For this class of graphs, if at least one of the cycles is a triangle, then the number of vertices common to the two cycles is at most three. It can be easily verified that, if the number of common vertices is between 1 and 3, then no $\mathbb{Z}_k$-zonal labeling exists. Therefore, in the next theorem, we restrict our attention to bicyclic graphs that contain no triangle.

Theorem 4. *Let G be a triangle-free bicyclic graph with minimum degree $\delta(G) \geq 2$, containing two cycles C and C' that share exactly p common vertices, where $p \geq 1$. Then G admits a $\mathbb{Z}_k$-zonal labeling if and only if $p \neq 1$.*

Proof. Assume first that $p = 1$. Then the cycles C and C' have exactly one common vertex v. In a planar embedding of G, the two cycles determine two distinct bounded regions, and the boundary of the third face is obtained by traversing the vertices of both cycles while counting the shared vertex v only once. Applying the zonal condition to this face yields

$$\sum_{u \in V(C)} \ell(u) + \sum_{u \in V(C')} \ell(u) - \ell(v) \equiv -\ell(v) \not\equiv 0 \pmod{k},$$

since the zonal condition on the two inner faces implies $\sum_{u \in V(C)} \ell(u) = \sum_{u \in V(C')} \ell(u) \equiv 0 \pmod{k}$. This contradicts the zonal requirement for the third face. Hence, G does not admit a $\mathbb{Z}_k$-zonal labeling when $p = 1$.
Conversely, suppose $p = 2$. Then the cycles C and C' share exactly two vertices, denoted by x and y. The graph $G \setminus \{x, y\}$ consists of two path components, say $G \setminus \{x, y\} = P_r \cup P_s$, for suitable integers $r, s > 1$ with $r + s = n - 2$. Assign the labels 1 and $k - 1$ to x and y, respectively. Label the vertices of each path component according to Theorem 2. Then the sum of the labels on both paths and on the vertices x and y is zero modulo k. It follows that the zonal condition holds on every face, and hence G admits a $\mathbb{Z}_k$-zonal labeling.
Assume now that $p \geq 3$. Let x and y denote the first and last common vertices of C and C', respectively. In this case, $G \setminus \{x, y\} = P_r \cup P_s \cup P_{p-2}$, where $r, s \geq 0$ and $r + s + p = n$. Denote these paths by $P_r : u_1 u_2 \dots u_r$, $P_s : v_1 v_2 \dots v_s$ and $P_{p-2} : w_1 w_2 \dots w_{p-2}$, as shown in Figure 6.

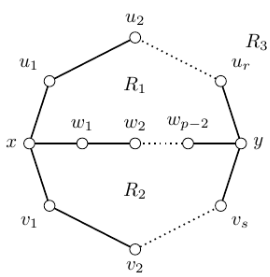

Figure 6. **A bicyclic graph G with $p \geq 3$ and no bridges.**

Suppose first that $r = 0$. If $s = 0$, then the graph G would contain a multiple edge between x and y, which is a contradiction. If $s = 1$, then G contains the triangle

$\triangle xyv_1$, again a contradiction. Hence, $s \geq 2$. In addition, if $p = 3$, then the path xw_1y together with the edge xy forms a triangle, contradicting the assumptions. Therefore, it follows that $p \geq 4$. We now assign labels to the path P_{p-2} according to Theorem 2, and proceed analogously for the path P_s. It remains to assign label 1 to x and label $k-1$ to y, respectively. This implies that the total sum of labels in each of the three regions R_1, R_2 and R_3 of G is congruent to zero modulo k. If $s = 0$, the argument reduces to the earlier case $r = 0$. Now assume that $r, s \geq 1$. There exist positive integers a, b, c satisfying

$$r \leq ak-1 \leq r(k-1), \qquad s \leq bk-1 \leq s(k-1), \qquad p-2 \leq ck-1 \leq (p-2)(k-1).$$

Indeed, if $r = 1$, we set $a = 1$. If $2 \leq r \leq k-1$, then $r > \frac{k}{k-1}$ and hence $r(k-1) \geq k$. Therefore, the interval $[r, r(k-1)]$ contains $k-1$, and we may take $a = 1$. Now suppose that $r \geq k$. Choose $a = \lceil \dfrac{r+1}{k} \rceil$. Then $a \geq \dfrac{r+1}{k}$, and hence $ak-1 \geq r$. Moreover, since $a \leq \dfrac{r+1}{k} + 1$, we have $ak-1 \leq r+k$. On the other hand, $r \geq k \geq 3$ implies that $r(k-2) \geq k$, and therefore $r+k \leq r(k-1)$. Consequently, $ak-1 \leq r(k-1)$, and the desired inequalities hold.
The same argument applies to s and $p-2$. Then the following Diophantine equations admit integer solutions:

$$z_1 + \cdots + z_r = ak-1, \qquad z_1 + \cdots + z_s = bk-1, \qquad z_1 + \cdots + z_{p-2} = ck-1,$$

where $1 \leq z_i \leq k-1$. Let $(a_1, \ldots, a_r)$, $(b_1, \ldots, b_s)$, $(c_1, \ldots, c_{p-2})$ be corresponding solutions. Define a labeling $\ell : V(H) \longrightarrow \mathbb{Z}_k \setminus \{0\}$ by:

$$\ell(g) = \begin{cases} 1, & \text{if } g = x \text{ or } g = y, \\ a_i, & \text{if } g = u_i, \\ b_j, & \text{if } g = v_j, \\ c_l, & \text{if } g = w_l. \end{cases}$$

We verify that the sum of labels along the boundary of the interior regions R_1, R_2 and of the exterior region R_3 is congruent to zero modulo k, Indeed

$$\sum_{g \in V(\partial R_1)} \ell(g) = 1 + (ak-1) + 1 + (ck-1) \equiv 0 \pmod{k},$$

$$\sum_{g \in V(\partial R_2)} \ell(g) = 1 + (ck-1) + 1 + (bk-1) \equiv 0 \pmod{k},$$

$$\sum_{g \in V(\partial R_3)} \ell(g) = 1 + (ak-1) + 1 + (bk-1) \equiv 0 \pmod{k}.$$

This completes the proof for the case $r, s \geq 1$. Therefore, the above assignment is a $\mathbb{Z}_k$-zonal labeling of G. $\square$

Theorem 5. *Let G be a graph obtained from two disjoint cycles C and C' by adding a path of length $r \geq 1$ between a vertex of C to a vertex of C'. Then G admits a $\mathbb{Z}_k$-zonal labeling if and only if $r \neq 2$.*

Proof. Assume first that $r = 1$ (see Figure 7). In this case, the two cycles C and C' are joined by a single edge not belonging to either cycle. By Theorem 3, the cycles C and C' can be labeled so that their boundary sums are congruent to zero modulo k. Hence G is $\mathbb{Z}_k$-zonal.

Now let $r = 2$. Then G contains exactly one vertex outside the union of the two cycles. The zonal conditions on the faces bounded by C and C' force both boundary sums to vanish modulo k, leaving the single outside vertex with label 0, which is not allowed. Therefore, no labeling can simultaneously satisfy the $\mathbb{Z}_k$-zonal conditions for G, C, and C'.

Next, assume that $r \geq 3$. Label the vertices of each cycle independently according to Theorem 3 so that their boundary sums vanish modulo k. The internal vertices of the path joining C and C' are then labeled using Theorem 2 so that the sum of their labels is congruent to 0 modulo k. It follows that every bounded face, as well as the outer face, satisfies the zonal condition. Hence, the resulting labeling is a $\mathbb{Z}_k$-zonal labeling of G. Combining all cases, G admits a $\mathbb{Z}_k$-zonal labeling if and only if $r \neq 2$. □

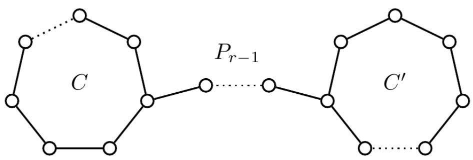

Figure 7. **A bicyclic graph G with no vertices of degree one**

Question 1. What are the necessary and sufficient conditions for a bicyclic graph to admit a $\mathbb{Z}_k$-zonal labeling?

3. $\mathbb{Z}_k$-zonal labeling with distinct labels

In graph labeling, using distinct labels plays an important role because it ensures that each graph has its own identifiable numerical pattern. If labels are allowed to repeat, then many graphs can easily satisfy the labeling requirements, and the problem loses much of its mathematical value and interest. As noted by Gallian [6], the use of unique labels is a key principle in distinguishing, comparing, and enumerating different types of labeling, allowing researchers to better understand the underlying structure of graphs.

From an applied perspective, in areas such as network design or coding systems, distinct labels are also essential. Kovář [7] observes that when two nodes or edges in

a network share the same label, it becomes difficult to identify routes or connections reliably. Likewise, Xin et al. [8] show that when all labels are different, the structure of the problem becomes richer and more meaningful, providing stronger mathematical insight.
We now investigate whether it is possible to assign distinct labels to all vertices of a graph, or to all vertices around each face, while still satisfying the $\mathbb{Z}_k$-zonal condition. Here, "distinct" means that no two vertices receive the same label. A basic necessary condition is that the graph has at most $k-1$ vertices, since the available labels lie in the set $\{1, 2, \ldots, k-1\}$. However, this condition alone is not sufficient: the requirements imposed by the regions may prevent the existence of a zonal labeling with non-repeating labels.

Theorem 6. *Let G be a graph of order n such that G is either a tree or a cycle, and suppose that $2 \leq n \leq k-1$. Then G admits a $\mathbb{Z}_k$-zonal labeling with distinct vertex labels if and only if $n \neq 2\left\lfloor \frac{k}{2} \right\rfloor - 1$.*

Proof. Let G be a tree or a cycle on n vertices with $2 \leq n \leq k-1$. Since a tree has one face and a cycle has two faces sharing the same boundary, the $\mathbb{Z}_k$-zonal condition simply requires that the sum of all vertex labels be congruent to zero modulo k. Given that the labeling must use distinct nonzero elements of $\mathbb{Z}_k$, the proof reduces to determining when one can select n distinct labels whose total sum is divisible by k. Let $n = 2\left\lfloor \frac{k}{2} \right\rfloor - 1$. If $k = 2m$, then $n = 2m - 1 = k - 1$, which forces the labeling to use all elements of $\{1, 2, \ldots, 2m-1\}$. The total sum of these labels is $\frac{(2m-1)(2m)}{2} = m(2m-1) \equiv m \pmod{k}$, which is nonzero. Therefore, no distinct $\mathbb{Z}_{2m}$-zonal labeling exists when $n = k-1$. Next, consider the case $k = 2m+1$. Here $n = 2m-1 = k-2$, and the set $\{1, \ldots, 2m\}$ has total sum $\frac{2m(2m+1)}{2} = m(2m+1) \equiv 0 \pmod{k}$. Any selection of $n = 2m-1$ distinct labels must omit exactly one value r, producing a sum $0 - r \equiv -r \not\equiv 0 \pmod{k}$, so again the zonal condition cannot be satisfied.
Conversely, let $n \neq 2\left\lfloor \frac{k}{2} \right\rfloor - 1$. We show that one can select n distinct labels whose total sum is congruent to 0 modulo k. Consider first $k = 2m$. The set $\{1, \ldots, 2m-1\}$ contains $m-1$ complementary pairs $(1, 2m-1), (2, 2m-2), \ldots, (m-1, m+1)$. If n is even with $2 \leq n \leq 2m-2$, selecting $\frac{n}{2}$ such pairs immediately provides a valid zero-sum assignment. If n is odd with $3 \leq n \leq 2m-3$, begin with the triple $\{1,\ m,\ m-1\}$, whose sum is zero modulo k. The remaining $n-3$ labels are taken from complementary pairs. For the largest odd value $n = 2m-3$, at most $\frac{n-3}{2} = m-3$ pairs are needed. Also, using labels $1, m$, and $m-1$, exactly $m-3$ of the mentioned pairs have not yet been used. Thus the construction is always attainable. Now consider $k = 2m+1$. The set $\{1, \ldots, 2m\}$ contains m complementary pairs. If n is even $(2 \leq n \leq 2m)$, choosing $\frac{n}{2}$ pairs yields an immediate zero-sum labeling. If n is odd $(3 \leq n \leq 2m-3)$, start with the triple $\{1,\ 2,\ 2m-2\}$, whose sum is congruent to zero modulo k. The remaining labels are again supplied by complementary pairs. Since at most $\frac{n-3}{2} = m-3$ pairs are required, and by omitting $(1, 2m), (2, 2m-1)$ and

$(3, 2m-2)$; $m-3$ such pairs are available, thus a zero-sum selection can always be completed. □

Remark 3. According to Theorem 1, wheel graphs never admit a distinct labeling. Applying Theorem 6 to the cycle C_3 implies that it admits a distinct labeling if and only if $k \geq 6$.

4. $\mathbb{Z}_k$ zonality of Halin Graphs

In this section, we investigate $\mathbb{Z}_k$-zonal labelings on an important family of planar graphs known as Halin graphs. In our earlier work [5], we established that every Halin graph admits an inner $\mathbb{Z}_3$-zonal labeling, and moreover, that this labeling is unique up to taking complements. Motivated by that result, we now extend the framework by allowing an arbitrary modulus $k \geq 3$. Our aim is to characterize when a Halin graph supports an inner $\mathbb{Z}_k$-zonal labeling and to identify the structural features that determine the existence of such labelings. This generalization not only includes the case $k = 3$ but also illustrates how the interaction between the underlying tree of a Halin graph and the cycle formed by its leaves affects the possibility of admitting an inner $\mathbb{Z}_k$-zonal labeling.

Definition 2. A Halin graph is a planar graph obtained from a tree T with no vertices of degree two and with at least four vertices, by joining all leaves of T into a cycle C according to their circular order in a planar embedding of T, so that $H = T \cup C$. Halin graphs are planar, 3-connected, Hamiltonian (indeed, 1-Hamiltonian), Hamiltonian-connected, and (almost) pancyclic.

Remark 4. No Halin graph admits an inner $\mathbb{Z}_2$-zonal labeling.

Indeed, by the structural properties of Halin graphs, every Halin graph contains at least two triangular faces, and the zonal condition modulo 2 cannot be satisfied on such faces. Hence, an inner $\mathbb{Z}_2$-zonal labeling does not exist.

Theorem 7. *Let k be an integer with $k \geq 3$, and let H be a Halin graph. Then:*

(i) If k is odd, then H admits at least $k-1$ distinct inner $\mathbb{Z}_k$-zonal labelings.

(ii) If k is even, then H admits at least $k-2$ distinct inner $\mathbb{Z}_k$-zonal labelings.

Proof. Let $a \in \mathbb{Z}_k \setminus \{0\}$ with $2a \not\equiv 0 \pmod{k}$ when k is even (if k is odd, then every nonzero a satisfies this condition) and set $b = \overline{2a} \in \mathbb{Z}_k$. Then $b \neq 0 \pmod{k}$, and both a and b are admissible nonzero labels. Let $H = T \cup C$ be a Halin graph, where T is its characteristic tree and C the outer cycle formed by the leaves of T. By the definition of Halin graphs, H contains at least one triangle. Let x, y, z be the vertices of such a triangle, where x and y are leaves of T (so $d_T(x) = d_T(y) = 1$) and z is their common support vertex in T. Since T is a tree, it admits a proper 2-coloring. Fix such a coloring, and let B and W denote the corresponding color classes, with $z \in B$. We now define a labeling $\ell_{H,a} : V(H) \longrightarrow \mathbb{Z}_k \setminus \{0\}$ as follows.

- For each internal vertex $v \in V(T) \setminus V(C)$, set

$$\ell_{H,a}(v) = \begin{cases} b & \text{if } v \in B, \\ \bar{b} = 2a & \text{if } v \in W. \end{cases}$$

- For each leaf $u \in V(C)$, let

$$\ell_{H,a}(u) = \begin{cases} a & \text{if } u \in W, \\ \bar{a} & \text{if } u \in B. \end{cases}$$

Although choosing a different reference triangle $\triangle xyz$ may alter the assignment, the label of the support vertex z is always either b or $\bar{b}$ under the labeling rule. Consequently, both the leaves x and y receive the same label a or $\bar{a}$, depending on the triangle $\triangle xyz$. Thus, these labelings are precisely $\ell_{H,a}$ and its complement $\ell_{H,\bar{a}}$, which account for the claimed $k-1$ (resp. $k-2$) distinct inner $\mathbb{Z}_k$-zonal labelings. We show that $\ell_{H,a}$ is an inner $\mathbb{Z}_k$-zonal labeling by induction on $n = |V(H)|$. When $n = 4$, the characteristic tree is the star $T = S_4 = K_{1,3}$, and $H = W_3 = C_3 \cup S_4$ is the wheel on four vertices. The labeling $\ell_{H,a}$ assigns the label b to the central vertex and the label a to each of the remaining three vertices. One can easily check that $\ell_{H,a}$ is an inner $\mathbb{Z}_k$-zonal labeling in this case.
Next, assume that for every Halin graph $H' = T' \cup C'$ with fewer than n vertices, the labeling $\ell_{H',a}$ is an inner $\mathbb{Z}_k$-zonal labeling. Let $H = T \cup C$ be a Halin graph with $|V(T)| = n$. We consider two structural cases based on the diameter of T (see Figure 8).

(a) Halin graph H (b) Halin graph H'

Figure 8. **Two Halin graphs H and H'**

Case 1. $\mathrm{diam}(T) = 2$. Here T is the star S_n, and thus $H = W_{n-1}$, the wheel on n vertices. Exactly as in the base case, the labeling $\ell_{H,a}$ satisfies the zonal condition on each triangular face. Hence $\ell_{H,a}$ is inner $\mathbb{Z}_k$-zonal.

Case 2. $\mathrm{diam}(T) \geq 3$. Let $P = (x_1, x_2, \ldots, x_{d+1})$ be a diameter path of T. Then x_1 is a leaf with support vertex x_2. Since T is the characteristic tree of H, the vertex x_2 has at least one additional leaf neighbor besides x_1. Let $L(x_2) = \{x_1 = y_1, y_2, \ldots, y_s\}$ denote the set of leaves adjacent to the support vertex x_2, where $s = d_T(x_2) - 1 \geq 2$. Remove all leaves in $L(x_2)$ to obtain a smaller characteristic tree $T' = T - L(x_2)$ of order $n - s$, and let $H' = T' \cup C'$ be the corresponding smaller Halin graph. Assign to each vertex of $T' \setminus \{x_2\}$ the label it receives under $\ell_{H,a}$. Also, if $\ell_{H,a}(x_3) = b$, assign the label a to x_2; otherwise if $\ell_{H,a}(x_3) = \bar{b}$, assign the label $\bar{a}$ to x_2. The restricted labeling on H' is then either $\ell_{H',a}$ or $\ell_{H',\bar{a}}$. Without loss of generality we assume that $\ell_{H',a}(x_2) = a$. The graph H has exactly $s-1$ more triangular faces than H'. After disregarding these $s - 1$ triangles, the remaining faces of H coincide with those of H' except for the adjacent regions R_1 and R_2 in H that are incident with x_2, which correspond to regions R'_1 and R'_2 in H'. Since all regions of H' satisfy the zonal condition by the induction hypothesis, it remains only to verify that R_1 and R_2 also satisfy it in H. If $\ell_{H',a}(x_3) = b$, then

$$\begin{aligned}
\sum_{u \in V(\partial R_1)} \ell_{H,a}(u) &= \ell_{H,a}(x_1) + \ell_{H,a}(x_2) + \sum_{u \in V(\partial R'_1) \setminus \{x_2\}} \ell_{H',a}(u) \\
&= \bar{a} + \bar{b} + \sum_{u \in V(\partial R'_1) \setminus \{x_2\}} \ell_{H',a}(u) \\
&= a + \sum_{u \in V(\partial R'_1) \setminus \{x_2\}} \ell_{H',a}(u) \\
&= \sum_{u \in V(\partial R'_1)} \ell_{H',a}(u) \equiv 0 \pmod{k}.
\end{aligned}$$

If $\ell_{H',a}(x_3) = \bar{b}$, then

$$\begin{aligned}
\sum_{u \in V(\partial R_1)} \ell_{H,a}(u) &= a + b + \sum_{u \in V(\partial R'_1) \setminus \{x_2\}} \ell_{H',a}(u) \\
&= \bar{a} + \sum_{u \in V(\partial R'_1) \setminus \{x_2\}} \ell_{H',a}(u) \\
&= \sum_{u \in V(\partial R'_1)} \ell_{H',a}(u) \\
&\equiv 0 \pmod{k}.
\end{aligned}$$

An entirely analogous calculation shows that $\sum_{u \in \partial R_2} \ell_{H,a}(u) \equiv 0 \pmod{k}$. Finally, by the labeling rule, each of the $s - 1$ triangular regions removed when forming H' also has label-sum congruent to zero modulo k. Therefore every internal region of H satisfies the zonal condition, and $\ell_{H,a}$ is an inner $\mathbb{Z}_k$-zonal labeling of H. □

Corollary 1. *If every internal region of the Halin graph H has odd length, then under the labeling $\ell_{H,a}$ each vertex on the outer face of H receives the label a. If $k \mid a \cdot l(T)$, then $\ell_{H,a}$ is a zonal labeling, where $l(T)$ denotes the number of leaves of its characteristic tree T.*

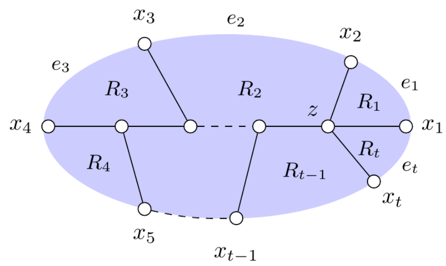

Figure 9. **An illustration of the construction of $\ell_{H,a,b}$.**

For a Halin graph $H = T \cup C$, let the vertices of the outer cycle C be labeled cyclically as $x_1, x_2, \ldots, x_t$, and let the edges of C be $e_i = x_i x_{i+1} \quad (1 \le i \le t-1)$, and $e_t = x_t x_1$. For each i, let R_i denote the inner face whose boundary contains the edge e_i. (see Figure 9). Choose arbitrary nonzero labels $a, b \in \mathbb{Z}_k \setminus \{0\}$ such that $a \neq \bar{b}$, equivalently, $a + b \neq 0 \pmod{k}$. Set $c = \overline{a+b}$, which is also nonzero. Without loss of generality, assume that x_1, x_2 and z form a triangle. Let B and W denote the two color classes of a proper 2-coloring of T, chosen so that $z \in B$. We now define a labeling $\ell_{H,a,b} : V(H) \to \mathbb{Z}_k \setminus \{0\}$ as follows:

- Set $\ell_{H,a,b}(x_1) = a, \quad \ell_{H,a,b}(x_2) = b,$
- For each internal vertex $v \in V(T) \setminus V(C)$, define

$$\ell_{H,a,b}(v) = \begin{cases} c & \text{if } v \in B, \\ \bar{c} & \text{if } v \in W. \end{cases}$$

Next, we recursively assign labels to the vertices $x_3, \ldots, x_t$. For each $i \ge 3$, we distinguish cases according to whether x_i and x_{i-1} belong to the same color class in the fixed proper 2-coloring of T.

- If x_i and x_{i-1} belong to different color classes, then set $\ell(x_i) = \overline{\ell(x_{i-1})}$.
- If x_i and x_{i-1} belong to the same color class, then define

$$\ell_{H,a,b}(x_i) = \begin{cases} b & \text{if } \ell_{H,a,b}(x_{i-1}) = a, \\ a & \text{if } \ell_{H,a,b}(x_{i-1}) = b, \\ \bar{b} & \text{if } \ell_{H,a,b}(x_{i-1}) = \bar{a}, \\ \bar{a} & \text{if } \ell_{H,a,b}(x_{i-1}) = \bar{b}. \end{cases}$$

By construction, the $\mathbb{Z}_k$-zonal condition is automatically satisfied for the faces $R_1, R_2, \ldots, R_{t-1}$. It therefore remains only to verify the condition for the final inner face R_t. If the sum of the labels on the boundary of R_t is congruent to zero modulo k, then $\ell_{H,a,b}$ is a valid inner $\mathbb{Z}_k$-zonal labeling of H. Consequently, H is inner $\mathbb{Z}_k$-zonal. In this case, the above construction yields an inner $\mathbb{Z}_k$-zonal labeling extending the alternating $c, \bar{c}$ pattern from the internal vertices to the characteristic tree T.

Corollary 2. *(i) If every region of the Halin graph H has odd boundary length, then under the labeling $\ell_{H,a,b}$, all vertices on the outer cycle of H receive only the labels a and b. (ii) If H admits an inner $\mathbb{Z}_k$-zonal labeling of the form $\ell_{H,a,b}$ with n vertices, then H possesses at least $(k-1)(k-2)$ distinct inner $\mathbb{Z}_k$-zonal labelings.*

Remark 5. If $a = b$, then clearly $\ell_{H,a,b} = \ell_{H,a}$. Suppose that $a \neq b$. Then the labeling $\ell_{H,a,b}$ assigns at least three distinct labels to the vertices whenever H is not a wheel. Indeed, as shown in Figure 10, the labels a, b, and $\bar{a}$ are pairwise distinct.

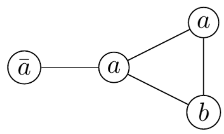

Figure 10. **If $c = a$ and $a \neq b$.**

In the following, we determine the number of distinct labels that appear in the labeling of the Halin graph H under $\ell_{H,a}$.

Theorem 8. *A Halin graph H admits an inner $\mathbb{Z}_k$-zonal labeling with a single label if and only if $H = W_n$ and $3 \mid k$.*

Proof. First assume that $H = W_n$. In a wheel graph, every inner face is a triangle. If all vertices receive the same label $a \in \mathbb{Z}_k \setminus \{0\}$, then the boundary sum of each inner face equals $3a$. Hence, the inner zonal condition is satisfied if and only if $3a \equiv 0 \pmod{k}$, which holds for a nonzero label a precisely when $3 \mid k$. Thus, in this case an inner $\mathbb{Z}_k$-zonal labeling with a single label exists.

Conversely, suppose that H is a Halin graph that is not a wheel. Let T be its characteristic tree. Then $\operatorname{diam}(T) \geq 3$. Assume, for contradiction, that all vertices of H receive the same label a. Let $P = (u_0, u_1, \ldots, u_{d+1})$ be a diametral path in T, where $d = \operatorname{diam}(T)$. The vertex u_1 is incident with $t = d_T(u_1) - 1$ triangular faces, denoted by $R_1, \ldots, R_t$ (see Figure 11). For each such face, $\sum_{v \in R_i} \ell(v) = 3a \equiv 0 \pmod{k}$. Since H is not a wheel and every vertex has degree at least three, the planar embedding contains at least one additional inner face R whose boundary length is either 4 or 5 (see Figure 12). For this face we obtain $\sum_{v \in R} \ell(v) = 4a \equiv 0 \pmod{k}$ or $5a \equiv 0 \pmod{k}$. Together with $3a \equiv 0 \pmod{k}$, this implies $a \equiv 0 \pmod{k}$, contradicting the assumption that the labeling uses a nonzero label. Therefore, no such labeling exists. $\square$

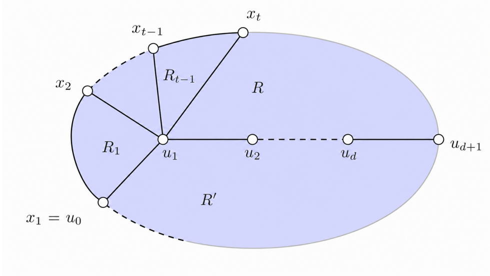

Figure 11. **An illustration of Theorem 8.**

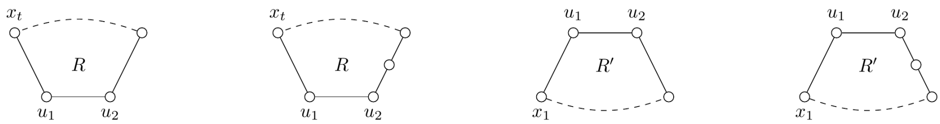

Figure 12. **Inner faces of lengths 4 and 5 in non-wheel Halin graphs**

Remark 6. Let $H = W_n$. According to Theorem 2.3 in [5], there exist exactly two inner $\mathbb{Z}_3$-zonal labelings for H, denoted by $\ell_{H,1}$ and $\ell_{H,2}$. In the former, all vertices receive the label 1, whereas in the latter, all vertices receive the label 2. Now suppose that $k \geq 4$. In this case, $\ell_{H,1}$ is an inner $\mathbb{Z}_k$-zonal labeling of H in which the leaves of the characteristic tree T receive the label 1 and the central vertex receives the label $k-2$.

Theorem 9. *Let $H \neq W_n$ be a Halin graph. Under an inner $\mathbb{Z}_k$-zonal labeling $\ell_{H,a}$, exactly two distinct vertex labels are assigned to the vertices of H if and only if either $3 \mid k$, or $4 \mid k$ and the boundary of each interior region has odd length.*

Proof. If $3 \mid k$, then the inner $\mathbb{Z}_k$-zonal labeling $\ell_{H,\frac{k}{3}}$ exists and assigns to the vertices of the Halin graph H exactly the two labels $\frac{k}{3}$ and $\frac{2k}{3}$. If $4 \mid k$ and the boundary of each interior region of H has odd length, then under the corresponding inner $\mathbb{Z}_k$-zonal labeling $\ell_{H,\frac{k}{4}}$, the vertices of H receive exactly the two labels $\frac{k}{4}$ and $\frac{k}{2}$. Conversely, suppose that H admits an inner $\mathbb{Z}_k$-zonal labeling of the form $\ell_{H,a}$ in which exactly two distinct labels a and c appear, and assume that $3 \nmid k$. Consider a triangular face of H whose two leaf vertices are labeled a. Clearly, the support vertex cannot be labeled a, because this would imply $3 \mid k$, a contradiction. Hence, the support vertex must receive the label c, since $H \neq W_n$. The support vertex must then be adjacent to a vertex labeled $\bar{c}$, as $\bar{c} \neq a$. It follows that $\bar{c} = c$, and consequently $c = \frac{k}{2}$. The sum of the labels of the triangle satisfies $2a + c \equiv 0 \pmod{k}$, which gives $2a + \frac{k}{2} = tk$ for some integer t, hence $4a = (2t-1)k$. This implies $4 \mid k$. It also follows that all interior vertices of the characteristic tree T are labeled c. Moreover, if the boundary of any interior region of H has even length, the two leaves of that region would receive the labels a and $\bar{a}$. Since $a, c \neq \bar{a}$, at least three distinct labels $a, \bar{a}$, and c would appear in the labeling of H, contradicting the assumption of two labels.

Therefore, the boundary of every interior region of H must have odd length. $\square$

In view of the preceding results, we are motivated to consider the following problem.

Question 2. Let k be an integer such that $3 \nmid k$ and $4 \nmid k$. Does there exist an algorithm that produces an inner $\mathbb{Z}_K$-zonal labeling of a Halin graph H using exactly two distinct labels?

Remark 7. Let k be an integer such that $3 \nmid k$ and $4 \nmid k$, and let $a \in \mathbb{Z}_k \setminus \{0\}$. When k is even, assume further that $a \neq \frac{k}{2}$. If H contains at least one face of even length, then the labeling $\ell_{H,a}$ assigns exactly four distinct labels, namely a, $\bar{a}$, $c = \overline{2a}$, and $\bar{c} = 2a$. On the other hand, if every face of H has odd length, then only three distinct labels are used. In this case, all leaves of the characteristic tree T receive the label a. Consequently, $\ell_{H,a}$ is a $\mathbb{Z}_k$-zonal labeling if and only if $k \mid l(T)\, a$.

Remark 8. The internal vertices of H are not, in general, required to receive only the labels c or $\bar{c}$ (for example, see Figure 13).

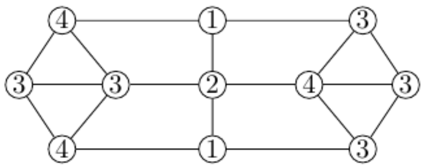

Figure 13. **inner $\mathbb{Z}_5$-zonal labeling**

We conclude this paper with the following two open questions.

Question 3. What are the necessary and sufficient conditions for a Halin graph to admit a $\mathbb{Z}_k$-zonal labeling?

Question 4. If a graph G admits a $\mathbb{Z}_k$-zonal labeling, does there exist a divisor $p \mid k$ such that G is also $\mathbb{Z}_p$-zonal?

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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