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Research Article

Graphs with equal domination, connected vertex cover, and watching numbers

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Abstract: This paper investigates how the parameters $\gamma(G)$, $\beta_c(G)$, and $\omega(G)$ influence and characterize the structure of a graph. We prove that if G is of the form $H \odot K_1$, then $\gamma(G) = \beta_c(G) = \omega(G)$. Furthermore, under the assumption that $\gamma(G) = \beta_c(G) = \omega(G)$ and by leveraging properties of connected vertex covers and dominating sets, we show that G belongs to a specific class of graphs.

Keywords: watching system, dominating set, connected vertex cover.

AMS Subject Classification: 05C69, 05C75

1. Introduction

In this paper, all graphs are assumed to be finite, simple, and undirected. We will often use the notation $G = (V, E)$ to denote the graph with non-empty vertex set $V = V(G)$ and edge set $E = E(G)$. The degree of a vertex v is denoted $deg(v)$. An edge of G with end vertices u and v is denoted by $u \sim v$. For every vertex $x \in V(G)$, the *open neighborhood* of vertex x is denoted by $N_G(x)$ and defined as $N_G(x) = \{y \in V(G) \ : \ x \sim y\}$, and the *closed neighborhood* of vertex $x \in V(G)$,

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$N_G[x]$, is $N_G[x] = N_G(x) \cup \{x\}$. For a set $T \subseteq V(G)$, the *open neighborhood* of T is $N_G(T) = \cup_{x \in T} N_G(x)$ and the closed neighborhood of T is $N_G[T] = N_G(T) \cup T$. A graph G is said to be a subgraph of a graph H, denoted by $G \leq H$, if $V(G) \subseteq V(H)$ and $E(G) \subseteq E(H)$. For a set $H \subseteq V(G)$, the *subgraph induced* by H is denoted by $G[H]$ while the graph $G \setminus H$ is the graph obtained from G by deleting the vertices in H and all edges incident with H. A subset C of $V(G)$ is a dominating set of G if every vertex in $V(G) \setminus C$ is adjacent to at least one vertex in C. The domination number of a graph G, denoted by $\gamma(G)$, is the minimum size of a dominating set of G. A set C of vertices G is an identifying code of G if for every two vertices x and y the sets $N_G[x] \cap C$ and $N_G[y] \cap C$ are non-empty and different. Given a graph G, the size of a smallest identifying code of G is called the identifying code number of G and is denoted by $\gamma^{ID}(G)$. If for two distinct vertices x and y, $N_G[x] = N_G[y]$, then they are called twins. Graph G is identifiable if and only if G is twin-free. The *watching system*, introduced in [2], is a generalization of identifying codes. A *watcher* ω of G is a pair $\omega = (v_i, Z_i)$ where v_i is a vertex and $Z_i \subseteq N_G[v_i]$. We will say that ω is located at v_i and that Z_i is its *watching area* or *watching zone*. A *watching system* in a graph G is a finite set $W = \{\omega_1, \omega_2, \ldots, \omega_k\}$ such that sets $L_W(v) = \{\omega_i \;:\; v \in Z_i\;,\; 1 \leq i \leq k\}$ are non-empty and distinct, for any $v \in V$. The *watching number* of G denoted by $\omega(G)$ is the minimum size of a watching system of G. Auger et al. [3], gave an upper bound of $\omega(G)$ for connected graphs of order n and characterized the trees attaining this bound. In 2014, Maimani et al. [12] studied the watching systems of triangular graphs. They proved that the watching number of triangular graph $T(n)$ is equal to $\lceil \frac{2n}{3} \rceil$. In 2017, Roozbayani et al. [11] studied identifying codes and watching systems of Keneser graphs.

Let $G = (V, E)$ be a simple connected graph. Three classical graph invariants play essential roles in modeling monitoring and control processes over network structures: the *domination number* $\gamma(G)$, the *connected vertex cover number* $\beta_c(G)$, and the *watching number* $\omega(G)$. Each parameter captures a distinct but complementary aspect of structure coverage, connectivity, and distinguishability. Understanding the relationships among them provides insight into the balance between redundancy and efficiency in network supervision. Efficient monitoring and control of complex systems such as sensor grids, communication frameworks, and surveillance infrastructures can be modeled using these parameters. The domination number $\gamma(G)$ measures the minimal set of monitors ensuring global observability. The connected vertex cover number $\beta_c(G)$ quantifies the smallest connected subset securing all edges. The watching number $\omega(G)$ identifies the minimal family of watchers uniquely distinguishing every vertex. Graphs achieving equality among these parameters,

$$\gamma(G) = \beta_c(G) = \omega(G),$$

represent optimal configurations balancing minimal coverage, connectivity, and identification capacity. A corona graph is a graph construction used to model systems with a central structure connected to many local or peripheral structures, it has useful applications in network design, communication systems, biology, and distributed computing [6, 8, 13]. In this study, we consider a combination involving corona graphs to investigate the relationship and equality among the domination number, connected vertex cover number, and watching numbers. The domination number and connected domination number of a graph have many practical applications in networks such as communication systems, sensor placement, transportation networks, and social networks [9]. In particular, a minimum dominating set can help reduce energy consumption, minimize control overhead, improve routing efficiency, decrease communication redundancy, and enable faster communication [1, 15]. These concepts are widely applied in Wi-Fi mesh systems, military communication systems, and drone networks [4, 15]. In a watching system, a watcher monitors its own vertex, may also monitor nearby vertices, and helps identify where an event or fault occurs. In wireless sensor networks, watching systems help identify the exact affected region, reduce redundant sensors, and improve tracking accuracy.

2. Main results

In this Section, we have defined a family of graphs, as outlined in Definition 1, and demonstrated the veracity of Theorem 4. The following results are quoted from [2], [7] and [10].

Theorem 1. *[2] Let G be a graph of order n. Then*

(i) If G is twin free graph, then $\gamma(G) \leq \omega(G) \leq \gamma^{ID}(G)$,

(ii) $\lceil log_2(n+1) \rceil \leq \omega(G) \leq \gamma(G)\lceil log_2(\Delta(G)+2) \rceil$.

The *corona product* of G and H is the graph $G \odot H$ obtained by taking one copy of G, called the center graph, $|V(G)|$ copies of H, and making the ith vertex of G adjacent to every vertex of the ith copy of H, where $1 \leq i \leq |V(G)|$. Frucht and Harary [8] introduced this graph product in 1970.

Theorem 2. *[10] Let G be a graph with no isolated vertex. Then $\gamma(G) \leq \frac{n}{2}$.*

Theorem 3. *[7] Let G be a graph without isolated vertices on n vertices such that n is even. Then, $\gamma(G) = \frac{n}{2}$ if and only if the components of G are C_4 or $H \odot K_1$, where H is a connected graph.*

A vertex cover of G is a set $L \subseteq V$ such that for each edge $u \sim v \in E$ at least one of u and v is in L. The vertex cover number of G, is denoted by $\beta(G)$ and is defined as

$$\beta(G) = \min\{|L| \ : \ L \text{ is a vertex cover of } G\}.$$

A vertex cover T of G is a connected vertex cover if the induced subgraph on T is a connected graph. The connected vertex cover number of G, is denoted by $\beta_c(G)$ and is defined as

$$\beta_c(G) = \min\{|T| \ : \ T \textit{ is a connected vertex cover of } G\}.$$

Remark 1. It is easy to see that every vertex cover set of G forms a dominating set of G. Moreover, if $C \subseteq V(G)$ is a connected vertex cover of G, then the induced subgraph $V(G) \setminus C$ contains no edges; that is, it is an empty graph. Consider a vertex $v \in V(G)$ with degree one, connected only to a vertex u. The vertex v cannot belong to any minimum connected vertex cover of G, whereas u always does. If v is removed, the remaining vertices still form a connected vertex cover. Therefore, a vertex of degree one does not belong to any minimum connected vertex cover of G, while its neighbor always does.

Definition 1. Let G be a graph and set

$$\mathcal{B} = \Big\{G : G \leq H \odot K_1 \text{ for some graph } H \text{ with } V(H) \subseteq V(G)\Big\}.$$

We define the following set

$$\mathcal{A} = \Big\{G \in \mathcal{B} : |V(G)| \geq |V(H)| + 3\Big\}.$$

Lemma 1. *Let G be a connected graph. Then the following conditions hold.*

(i) *If $\gamma(G) = \omega(G)$, then every $x \in V(G)$ has at most one pendant vertex as a neighbor.*

(ii) *If $\gamma(G) = \beta_c(G)$ and $G_1 = [G \setminus y] \in \mathcal{A}$, then y is adjacent to at most one copy of K_1 in G_1.*

(iii) *Let $\gamma(G) = \beta_c(G)$ and $G_1 = [G \setminus y] \in \mathcal{A}$. If $y \notin C \cup D$, where D is a dominating set of G and C is a connected vertex cover of G, such that $|D| = \gamma(G) = \beta_c(G) = |C|$, then either $deg_G(y) = 1$ or y is not adjacent to any copies of K_1 in G_1.*

Proof. (i) Assume that there exists a vertex $x \in V(G)$ having at least two pendant neighbors. Let u and v be two such neighbors of x. Then $u, v \in N_G(x)$ and $\deg_G(u) = \deg_G(v) = 1$. Let $W = \{\omega_i = (x_i, Z_i) \mid x_i \in V(G), Z_i \subseteq N_G(x_i)\}$ be a minimum cardinality watching system for G, and define $D = \{x_i \mid \omega_i \in W\}$, i.e., D is the set of all watcher locations in G. We have $|W| = \omega(G)$ and $|D| \leq |W|$. Since D is a dominating set of G, $\gamma(G) \leq |D|$. Hence $|W| = \omega(G) = \gamma(G) \leq |D| \leq |W|$. Thus $\gamma(G) = |D|$. We claim that $|\{v, u, x\} \cap D| \geq 2$. Assume, to the contrary, that $|\{v, u, x\} \cap D| = 1$. Since v and u are both degree-1 vertices whose only neighbor is x, they can be dominated only by x. Therefore, under the assumption that $|\{v, u, x\} \cap D| = 1$, the only possible watcher among $\{v, u, x\}$ is x; that is, $D \cap \{v, u, x\} = \{x\}$. If $x \in D$, then x is the x_i for some ω_i. Denote this watcher by $\omega_x = (x, Z_x)$. This implies that $L_W(u) = L_W(v) = \{\omega_x\}$, which contradicts the defining property of a watching system that requires each vertex to have a unique label. Hence, we arrive at a contradiction. Therefore, $|\{v,u,x\} \cap D| \geq 2$. Now it is possible that $x \notin D$. So define

$$D_1 = (D \setminus \{v,u,x\}) \cup \{x\}.$$

Since x dominates u and v, and every vertex in $V(G) \setminus \{u,v\}$ remains dominated by D_1, it follows that D_1 is a dominating set of G. Hence,

$$|D| = \gamma(G) \leq |D_1| \leq |D| - 1,$$

which is a contradiction.

(ii) Let C be a connected vertex cover of G such that $\gamma(G) = \beta_c(G) = |C|$. Let $G_1 = [G \setminus y] \in \mathcal{A}$. Since $G_1 \in \mathcal{A} \subseteq \mathcal{B}$, by Definition 1, we have $G_1 \leq H \odot K_1$ for some graph H. Hence,

$$V(G_1) = V(H) \cup \{u_1, u_2, \ldots, u_\ell\} \quad (3 \leq \ell \leq p),$$

where $V(H) = \{v_1, \ldots, v_p\}$ and each u_i is a pendant vertex adjacent to exactly one vertex $v_i \in V(H)$; that is, $N_{G_1}(u_i) = \{v_i\}$, $1 \leq i \leq \ell$. We assume the opposite of what we want to prove: that y is adjacent to at least two pendant vertices in G. Without loss of generality, assume that $N_G(y) \cap \{u_1, u_2, \cdots, u_\ell\} = \{u_1, u_2, \cdots, u_r\}$ where $r \geq 2$. We consider the following cases:

A) Let $y \in C$. Since u_i and v_i are connected by an edge in G_1, at least one of them (or both, for $1 \leq i \leq r$) must be in C. This follows directly from the definition of a vertex cover. By Remark 1, for every $r+1 \leq i \leq \ell$, we have $u_i \notin C$.

First, suppose $N_G(y) \cap C \cap V(H) = \emptyset$. In other words, all of the neighbors of y that are also in C are pendant vertices. If no u_i is in C for any $1 \leq i \leq r$, then y would be an isolated vertex in the induced subgraph $G[C]$, contradicting the connectivity of C. Similarly, for each i $(1 \leq i \leq r)$, there must exist some $v_j \in C$, because otherwise C would be disconnected. However, this is impossible since the induced subgraph on C is connected and $y \in C$, implying there exists a path between some u_i and v_j. Since $N_G(u_i) = \{y, v_i\}$ for all $1 \leq i \leq r$, there exists at least one i such that $\{u_i, v_i\} \subseteq C$. Since u_i only covers the edges u_iv_i, u_iy, and $\{v_i, y\}$ are already in C, removing u_i does not leave any uncovered edges. Thus, $C \setminus \{u_i\}$ is a vertex cover set of G, contradicting the minimality of C given $\gamma(G) = \beta_c(G)$.

Now if $N_G(y) \cap C \cap V(H) \neq \emptyset$, then $C \setminus \{y\}$ is a dominating set of G, because for every $x \in N_G(y) \cap V(H)$, we have $deg_{G_1}(x) \geq 2$. However, this is a contradiction.

B) Let $y \notin C$. By the definition of a vertex cover, we have $\{u_1, u_2, \cdots, u_r\} \subseteq C$. The definition of a connected vertex cover C implies that $N_{G_1}(\{u_1, \cdots, u_r\}) \subseteq C$. Obviously, $(C \cup \{y\}) \setminus \{u_1, u_2, \cdots, u_r\}$ is a vertex cover of G where $r \geq 2$. In both Cases A and B, our assumption that y is adjacent to at least two pendant vertices (u_i) leads to a contradiction. Therefore, our assumption must be false. It must be the case that y is adjacent to at most one copy of K_1 (pendant vertex u_i) in G_1.

(iii) Assume that $N_G(u) = \{x, y\}$ where $deg_{G_1}(u) = 1$. If $deg_G(y) = 1$, then there is nothing to prove. So, let v be an element of the neighborhood of y in G. By part (ii), $deg_{G_1}(v) \geq 2$. Since $G[C]$ is connected and $y \notin C$, it follows that $\{x, u, v\} \subseteq C$. Now consider $C \setminus \{u\}$. It is clear that $x \in N_{G_1}[u] \cap C$ and $v \in N_G[y] \cap C$. Thus, $C \setminus \{u\}$ is a dominating set of G. This is impossible. □

Corollary 1. *Suppose that G is a connected graph such that $\gamma(G) = \beta_c(G) = \omega(G)$ and x is a vertex of it such that $x \notin D \cup C$ where D and C are a dominating set and a connected vertex cover set of G, respectively, such that $|D| = \gamma(G) = \beta_c(G) = |C|$. If $G_1 = [G \setminus x] \in \mathcal{A}$, then $G \in \mathcal{A}$.*

Proof. This is a straightforward consequence of Lemma 1. □

Example 1. In this example, we consider the graph G in Figure 1. It is clear that $G \notin \mathcal{A}$ because, $|V(G)| < |H|+3$ where $H = [G \setminus u_2]$. Since $\Delta(G) = 5$ and $|V(G)| = 8$, we obtain the lower bound $\gamma(G) \geq 2$. The set $\{v_1, v_4\}$ dominates G, hence $\gamma(G) = 2$. Next, any connected vertex cover of G must contain at least four vertices. For example, $C = \{v_1, v_2, v_3, v_5\}$ is a connected vertex cover of minimum size, implying $\beta_c(G) = 4$. By Theorem 1(ii), since $|V(G)| = 8$, we have $\omega(G) \geq 4$. Consider the set of watchers

$$W_G = \{\omega_1 = (v_1, N_G(v_1)), \omega_2 = (v_2, N_G(v_2)), \omega_3 = (v_3, N_G(v_3)), \omega_4 = (v_4, N_G(v_4))\}.$$

A direct verification shows that the labels $L_W(x)$ are non-empty pairwise distinct for all vertices $x \in V(G)$. For example, $L_W(u_1) = \{\omega_1, \omega_2\}$, $L_W(u_2) = \{\omega_3\}$, while

$$L_W(v_1) = \{\omega_2, \omega_3, \omega_4\} \ \ \textit{and} \ \ L_W(v_5) = \{\omega_1, \omega_2, \omega_3, \omega_4\}.$$

Thus W_G is a watching system of size 4, and therefore $\omega(G) = 4$. Consequently,

$$\gamma(G) + 2 = \beta_c(G) = \omega(G) = 4.$$

Define the graph $G_1 = [G \setminus v_1]$ obtained by removing the vertex v_1. Note that $G_1 \in \mathcal{A}$ because, $|V(G_1)| = |H_1| + 3$ where $H_1 = K_4$. Let $S = \{v_2, v_3, v_5\}$. One can verify that S simultaneously satisfies the following properties:

1. S is a dominating set of G_1.
2. S is a connected vertex cover of G_1.
3. S can be extended to a watching system W_{G_1} of size 3 by placing watchers at the vertices of S with their open neighborhoods as watching areas.

Each of these sets is minimum for its respective parameter. Hence,

$$\gamma(G_1) = \beta_c(G_1) = \omega(G_1) = 3.$$

This example is included to illustrate the role of the condition $\gamma(G) = \beta_c(G) = \omega(G)$ in Corollary 1. Since this condition is not satisfied by G, Corollary 1 is not applicable to G.

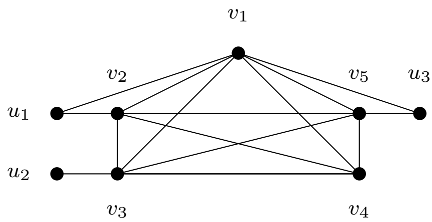

Figure 1. **Graph of G**

Remark 2. If a vertex is of degree one and is included in a minimum watching system of the graph, we can move that watcher to its neighbor to obtain a new minimum watching system. Also, if G is not an empty graph and $\gamma(G) = \omega(G)$, then by Theorems 1, part (ii), and 2, we have $|V(G)| \geq 6$. For $K_1 \odot K_1$ and $K_2 \odot K_1$ we have $\gamma = \beta_c = \omega - 1$.

Theorem 4. *Let G be a connected graph of order $n \geq 6$. Then the following conditions hold.*

(i) If G is $H \odot K_1$, then $\gamma(G) = \beta_c(G) = \omega(G)$,

(ii) If $\gamma(G) = \beta_c(G) = \omega(G)$, then $G \in \mathcal{A}$.

Proof. (i) Assume that $V(H) = \{v_1, \cdots, v_s\}$ and $V(G) = V(H) \cup \{v_{s+1}, \cdots, v_{2s}\}$. It is clear that H is a connected vertex cover of G. So $\beta_c(G) \leq |H| = s$. On the other hand, we have $\gamma(G) \leq \beta_c(G)$. By Theorem 3,

$$s = \gamma(G) \leq \beta_c(G) \leq s.$$

To complete the proof, we must prove $\gamma(G) = \omega(G)$. First, suppose H is twin-free, and define $W = \{\omega_1, \cdots, \omega_s\}$ where $\omega_i = (v_i, Z_i)$ and $Z_i = N_G[v_i]$ for $1 \leq i \leq s$. We claim that W is a watching system of G, meaning for any two vertices $x, y \in V(G)$, $L_W(x) \neq L_W(y)$. If x, y are from the pendant vertex set $\{v_{s+1}, \ldots, v_{2s}\}$, then $x = v_{i+s}$ and $y = v_{j+s}$ are adjacent only to v_i and v_j, respectively. Thus, $L_W(v_{i+s}) = \{\omega_i\}$, which are distinct for different i. Since H is connected, for $x = v_i$ and $y = v_{j+s}$,

$$2 \leq |L_W(v_i)| \neq |L_W(v_{j+s})| = 1.$$

Because H is twin-free, the sets $N_G[v_i]$ are all distinct. This shows that if H is twin-free, then $\omega(G) \leq s$.
Now suppose H is not identifiable (i.e., H has twins). Define $W' = \{\omega'_1, \cdots, \omega'_s\}$

$$where \quad \omega'_i = (v_i, Z_i),\ Z_i = \begin{cases} N_G[v_i] & \text{if } N_H[v_i] \neq N_H[v_j]\ 1 \leq i \neq j \leq s, \\ N_G(v_i) & \text{if } N_H[v_i] = N_H[v_j] \text{ for some } j \neq i. \end{cases}$$

We claim that W' is a watching system for G. Let v_i and v_j be two vertices of G. If $s+1 \leq i \neq j \leq 2s$, then it is clear that

$$\omega_i' = L_{W'}(v_i) \neq L_{W'}(v_j) = \omega_j'.$$

Let $1 \leq i \leq s$ and $s+1 \leq j \leq 2s$ such that $L_W(v_i) = L_W(v_{j+s})$. Then

$$|L_W(v_i)| = |L_W(v_{j+s})| = 1.$$

Since $deg_G(v_i) \geq 2$, so $i \neq j$ and $L_W(v_i) = L_W(v_{j+s}) = \omega(j)$. Thus $N_H[v_i] = N_H[v_j]$ and $deg_H(v_j) = deg_H(v_i) = 1$,. contradicting the connectivity of H with $|V(G)| \geq 6$. For $1 \leq i \neq j \leq s$, if there does not exist ℓ such that $N_H[v_i] = N_H[v_\ell]$ or $N_H[v_j] = N_H[v_\ell]$, then by definition W',

$$L_{W'}(v_i) \neq L_{W'}(v_j).$$

Without loss of generality, assume $N_H[v_i] = N_H[v_1]$. Consider two cases:
a) If $v_i \not\sim v_j$, then $\omega_j \in L_{W'}(v_j)$ but $\omega_j \notin L_{W'}(v_i)$, so

$$L_{W'}(v_i) \neq L_{W'}(v_j).$$

b) If $v_i \sim v_j$, then $\omega_i \in L_{W'}(v_j)$ and $\omega_i \notin L_{W'}(v_i)$, so again

$$L_{W'}(v_i) \neq L_{W'}(v_j).$$

Hence, W' is a watching system for G. Therefore, $\omega(G) \leq s$. By Theorems 1 part(i) and 3, the proof is complete.

(ii) Let $\gamma(G) = \beta_c(G) = \omega(G)$. If $\gamma(G) = \frac{n}{2}$, then by Theorem 3, there is nothing to prove. We prove the result by induction on n. First, let $n = 6$. Then by Theorem 1, part (ii), we have $\gamma(G) = \beta_c(G) = \omega(G) \geq \lceil log_2(6+1) \rceil = 3$. By Theorem 2, $\gamma(G) = \frac{n}{2}$, hence by Theorem 3, we have $G \in \mathcal{A}$. So we suppose that $n > 6$ and that the result has been proved for smaller values of n. Let C be a connected vertex cover such that $\beta_c(G) = |C|$ and $D = \{x_i \ ; \ \omega_i \in W\}$ where $W = \{\omega_1, \omega_2, \cdots, \omega_\ell\}$ is a minimum watching system of G. By Remark 2, we can assume that $deg_G(x_i) \geq 2$ for all $x_i \in D$. It is clear that D is a dominating set of G. Since $\beta_c(G) = \gamma(G) < \frac{n}{2}$, there exist $x \in V(G)$ such that $x \notin D \cup C$. Now let G_1 be the induced subgraph $G \setminus x$. In this case, it is easy to see that D and C are dominating and the connected vertex cover sets of G_1, respectively. Therefore,

$$\gamma(G_1) \leq \beta_c(G_1) \leq \gamma(G) = \beta_c(G) = \omega(G).$$

Let $N_G(x) = \{x_1, \cdots, x_r\}$. Since C is a vertex cover, so $N_G(x) \subseteq C$. Now suppose D_1 is the minimum dominating set of G_1 with $|D_1| < |D|$ under the contrary assumption. It is clear that $D_1 \cup \{x\}$ is a dominating set for G. So

$$\omega(G) - 1 = \beta_c(G) - 1 = \gamma(G) - 1 \leq \gamma(G_1) \leq \beta_c(G_1) \leq \gamma(G) = \beta_c(G) = \omega(G).$$

We show that we can assume that $r \geq 2$.
Let, for every $y \in V(G) \setminus C$, we have $|N_G(y) \cap C| = 1$. We define $f : V(G) \setminus C \to C$ by $f(v) = v'$, where $N_G(v) \cap C = \{v'\}$. If $y_1 \in V(G) \setminus C$ and $y_2 \in V(G) \setminus C$ such that $f(y_1) = f(y_2)$, then by Lemma 1, part (i), we have a contradiction. Hence, f is one-to-one and so $|V(G) \setminus C| \leq |C|$. Since $|V(G) \setminus C| + |C| = n$, it follows that $n \leq 2|C|$ and hence $|C| \geq \frac{n}{2}$.
On the other hand, by the assumption of the theorem $\beta_c(G) = \gamma(G)$. Since $|C| = \beta_c(G)$ and by Theorem 2, $\gamma(G) \leq \frac{n}{2}$, we obtain $|C| \leq \frac{n}{2}$. Therefore, $|C| = \frac{n}{2}$. By Theorem 3, $G = H \odot K_1$.
Let there be $y \in V(G) \setminus C$ such that $|N_G(y) \cap C| \geq 2$ and $y\prime, y\prime\prime \in N_G(y) \cap C$. We prove the following claims.

Claim 1. We claim that there exist $y_1\prime \in N_G(y\prime), y_1\prime\prime \in N_G(y\prime\prime)$ such that $deg_G(y_1\prime) = deg_G(y_1\prime\prime) = 1$. If every neighbor $y_1\prime \in N_G(y\prime)$ had $deg_G(y_1\prime) \geq 2$, then we show that $C \setminus y\prime$ is a dominating set of G, which leads to a contradiction. Since the induced subgraph on C is connected and $y\prime \in C$, there exists $u \in C$ such that $u \in N_G(y\prime)$, so $y\prime$ is dominated by u. On the other hand $deg_G(y_1\prime) \geq 2$, hence $y_1\prime \in C$ or there exists $y_2\prime \in C \cap N_G(y_1\prime)$. Hence, all neighbors of $y\prime$ are also dominated by $C \setminus y\prime$. Hence, the claim holds. ♦
Therefore, each open neighbor of y has a pendant vertex, so all neighbors of y belong to D, and thus $y \notin D$. Recall that $y \notin C$ and $deg_G(y) \geq 2$. Hence, we can assume $x = y$ and $r \geq 2$.
If $N_G(x) \subseteq D_1$, then it is clear that $|D_1| = |D|$. Which is a contradiction. Without loss of generality, assume $N_G(x) \nsubseteq D$, and $x_1 \notin D$.

Claim 2. We claim that for each vertex $w \in N_{G_1}(x_1)$ with $w \neq x$, we have $deg_{G_1}(w) \geq 2$. Indeed if $\deg_{G_1}(w) = 1$, then $w \in D$. Now, consider the set

$$D' = (D \setminus \{w\}) \cup \{x_1\}.$$

Since x_1 dominates w, the set D' is also a dominating set of G_1 and satisfies $|D'| = |D|$. Moreover, $N_G(x) \subseteq D'$, because x_1 has been added to the dominating set in place of w. Hence, we can easily see that D' is a minimum dominating set of G, so $\gamma(G) \leq |D'| = |D|$. Which is a contradiction. This establishes our claim. ♦
Now we show that $C \setminus \{x_1\}$ is a dominating set for G, which is a contradiction to the fact that $|C| = |D| = \gamma(G)$, and it proves that $|D_1| = |D|$. Since the neighbors of x_1 are not pendant vertices (By Claim 2) and C is a vertex cover set, every vertex belonging to the open neighborhood of x_1 is dominated by $C \setminus \{x_1\}$. Also, since the induced subgraph on C is connected and $x_1 \in C$, there exists $t \in C \cap N(x_1)$, hence

x_1 is dominated by t. Thus, $C \setminus \{x_1\}$ is a dominating set of G.
Summarizing, we have

$$\gamma(G_1) = \gamma(G) \leq \beta_c(G_1) \leq \gamma(G) = \beta_c(G) = \omega(G),$$

and

$$\gamma(G_1) = \gamma(G) \leq \omega(G_1) \leq \gamma(G) = \beta_c(G) = \omega(G).$$

Hence, $\gamma(G_1) = \omega(G_1) = \beta_c(G_1)$. Thus, G_1 in the condition induction applies and $G_1 \in \mathcal{A}$. Since $G_1 = G[G \backslash x]$ and $x \notin D \cup C$, where D and C are the dominating set and connected vertex cover set of G, respectively, such that $|D| = \gamma(G) = \beta_c(G) = |C|$. Therefore, by Corollary 1, $G \in \mathcal{A}$. $\square$

Example 2. In this example, we show that the converse of Theorem 4, part (ii) need not be true. Consider the graph G shown in Figure 2. It is clear that $\gamma(G) = 3$. By Theorem 1, part (ii), we have $\omega(G) \geq 4$. Suppose that $W = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, where

$$\omega_1 = (v_1, \{v_1, v_2, v_5\}), \quad \omega_2 = (v_2, N(v_2)), \quad \omega_3 = (v_3, N(v_3) \setminus \{v_5\}), \quad \omega_4 = (v_6, N(v_6)).$$

It is obvious that W is a watching system of G. Thus, $\omega(G) = 4$. By Remark 1, a vertex of degree one does not belong to any minimum connected vertex cover C, while its neighbor must belong to C. Hence, $\{v_2, v_3, v_6\} \subseteq C$ for every minimum connected vertex cover C of G. Moreover, at least two vertices from the set $\{v_1, v_4, v_5\}$ must belong to C; otherwise some edge remains uncovered. Therefore, $|C| = \beta_c(G) \geq 5$. Consequently, $\gamma(G) < \omega(G) < \beta_c(G)$, but $G \in \mathcal{A}$.

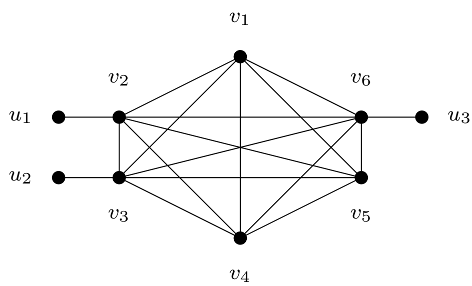

Figure 2. **Graph of G**

Conclusion

In this paper, we investigated the structural properties of graphs in which three fundamental graph parameters — the domination number $\gamma(G)$, the connected vertex cover number $\beta_c(G)$, and the watching number $\omega(G)$ — are equal. We focused on the class of graphs formed as the corona product $G = H \odot K_1$, and we proved that for such graphs, these three parameters coincide. This led to the definition of a new class

of graphs, denoted by $\mathcal{A}$, which is characterized by specific structural features based on vertex neighborhoods and stable parameter behavior under local transformations. Our analysis, supported by precise theorems and lemmas, such as upper and lower bounds for $\omega(G)$ [2], and restrictions on vertex neighborhoods, revealed that graphs in class $\mathcal{A}$ possess a unique structural coherence. These results are not only theoretically significant but also applicable in various domains.
In *wireless sensor networks*, the domination number plays a key role in identifying the minimum number of monitoring nodes [14]. The connected vertex cover ensures network connectivity while maintaining coverage [5]. The watching number, introduced as a generalization of identifying codes, offers a framework for targeted monitoring [2]. The equality of these parameters in a single graph structure enables the design of efficient, robust, and resource-optimal networks. The equality considered here refers only to the corresponding minimum cardinalities and does not imply the existence of a single vertex set that simultaneously constitutes a minimum dominating set, a minimum connected vertex cover, and a minimum watching system. These results have potential applications in network design, distributed systems, fault-tolerant computing, and algorithmic graph theory.
Overall, this work bridges structural graph theory with practical optimization problems, providing a foundation for further interdisciplinary research on graphs with unified parameter properties. Finally, we remark that Example 2, shows that the converse of Theorem 4(ii) does not hold in general. This suggests a natural direction for future research: to determine classes of graphs for which the converse of Theorem 4(ii) holds.

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