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Research Article

The strong total Roman domination in fuzzy graphs

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Dedicated to Odile Favaron

Abstract: In recent years, domination theory and its variants, including Roman domination, have been widely studied in fuzzy graphs due to their ability to model uncertainty in complex networks such as social, transportation, and biological systems. A strong total Roman dominating function (STRDF) on a fuzzy graph $G = (V, \sigma, \mu)$ is a mapping $f : V \to \{0, 1, 2\}$ such that every vertex u with $f(u) = 0$ has a strong neighbor labeled 2, and every vertex labeled 1 or 2 has at least one strong neighbor with a non-zero label. The strong total Roman domination number, $\gamma^t_{sR}(G)$, is defined as the minimum weight $\sum_{u\in V} f(u)\mu_s(u)$ among all STRDFs f, where $\mu_s(u)$ denotes the minimum membership value of the strong edges incident to u. In this paper, we introduce and study the strong total Roman domination number for fuzzy graphs. We establish several bounds, investigate its realizability, determine exact values for several standard families of fuzzy graphs, and present some applications.

Keywords: Domination, total Roman domination, fuzzy graphs.

AMS Subject Classification: 05C12, 05C76

1. Introduction

Graph theory is a very useful tool for addressing problems in scientific and technological fields, due to the ease with which complex networks and their behavior can be described using graphs and their variants. This modeling strategy allows us to analyze how they function, evaluate their efficiency, and develop models that facilitate their

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control and optimization. Typically, the behavioral parameters of a network are identified with invariants studied in deterministic graphs, without weights on edges and vertices. However, for a more suitable application to real-world models, it is equally necessary to broaden the scope of research to fuzzy graphs, especially in situations where there are elements of uncertainty in both the nodes and their connections.

Fuzzy graph theory allows us to model each element of a network (nodes and connections) with a degree of imprecision, using a set structure (vertices) and applications on the vertices and edges. This contrasts with a crisp graph, where vertices are fixed, and edges represent the result of a binary relationship between vertices (1 if the vertices are adjacent, and 0 if they are not adjacent). However, most real-world objects do not fit such a neat pattern. In social, logistical, chemical, or even biological networks, it is quite common for relationships to be only partial. Fuzzy graphs are best suited for modeling these networks because they allow connections to vary in intensity, rather than categorizing them as strictly binary.

In fuzzy set theory, introduced by L. Zadeh [35], membership degrees are weighted with real values between 0 and 1. By analogy, in fuzzy graph theory, in a network modeled by a graph, each vertex has a coefficient that measures its degree of relative importance in the network, and each edge likewise receives a weight value between 0 and 1, which quantifies the strength of the relationships between the nodes or their importance. Therefore, a fuzzy graph is a triple $G = (V, \sigma, \mu)$, where σ is a fuzzy subset of the non-empty set of vertices V, representing the degree of membership of each vertex in the graph, and μ is a fuzzy relation representing the degree of connection between pairs of vertices. Both σ and μ are represented by mappings over the real interval $[0, 1]$ that weight their hierarchy in the network.

Domination in fuzzy graph models has attracted considerable interest during the last decade, especially in generalized settings capable of modeling different types of uncertainty. Among these models, vague graphs have been one of the most studied extensions of classical fuzzy graphs leading to many domination parameters inspired by their crisp counterparts.

One of the first systematic studies of domination in fuzzy graphs was given in [5] by Borzooei and Rashmanlou. They introduced the concepts of cardinality, dominating set, total dominating set, independent dominating set and irredundance number in vague graphs, established their basic properties and illustrated the applicability of these concepts by an example. This work laid the groundwork for further study of domination in vague graph structures.

This line of research was continued by Borzooei and Rashmanlou [6] who proposed the concept of semi-global domination in vague graphs. In addition to defining the corresponding domination parameter, they presented some basic properties and relationships to other domination concepts and an illustrative application.

Later, Sahoo et al. [30] extended the theory to intuitionistic fuzzy graphs by introducing the notions of covering domination and paired domination. They studied the fundamental properties of these parameters and examined their relationships with the existing domination concepts in intuitionistic fuzzy graph models.

Talebi and Rashmanlou [33] also enriched the domination theory in vague graphs by introducing several new domination parameters motivated by optimization and facility location problems. They provided a theoretical basis for these parameters, derived the relations between them, and showed their applicability through representative examples.

Very recently, Kosari et al. [17] studied other domination variants in vague graphs motivated by medical applications. They presented new domination concepts and proved some theoretical properties. They demonstrated that these models are useful for supporting decision-making processes under uncertainty.

Beyond the domination-based contributions, some works have also extended the underlying models in fuzzy graph theory. Products of interval-valued fuzzy graphs have been investigated by Rashmanlou et al. [28]. Khan et al. [16] introduced interval-valued picture fuzzy hypergraphs for decision-making applications. Although these studies do not directly address domination, they provide more refined models of uncertainty for future studies of domination-related parameters.

Rao et al. [27] studied forcing parameters in fully connected cubic networks and obtained exact values and theoretical results for several forcing-related invariants. Although their contribution is not about domination, it shows the continuing interest in graph parameters related to monitoring, propagation and control processes in complex networks.

Fuzzy graphs have numerous applications due to their ability to manage uncertainty effectively. In social network analysis, fuzzy graphs are used to model relationships between people (friendship, trust, or influence), generating more realistic analyses of community structures and information diffusion [11, 13]. In transportation and logistics systems, traffic and weather, for example, can generate uncertainty in travel times and costs. These uncertain values can be modeled using a fuzzy graph, on which optimization and route scheduling strategies can be applied [26, 32]. Similarly, in biological systems such as protein interaction networks, the complex and uncertain interactions can be represented by fuzzy graphs in order to improve the diagnosis [22, 36]. In decision-making and expert systems, many processes rely on subjective judgments and incomplete information. Fuzzy graphs can model this information and facilitate multi-criteria decision-making and risk assessment [25, 37]. Finally, in image segmentation and pattern recognition, boundaries and features are often ambiguous. Weighting these patterns using a fuzzy graph improves the accuracy of the algorithms [7, 18].

The concept of graph domination has been extensively studied due to its suitability for analyzing the operational functioning and efficient control of networks. Selecting a set of nodes that can be used to control an entire network through its connections allows for its monitoring.

Roman domination in graphs is a variant of the domination concept, which not only identifies the nodes in a network that can be monitored but also selects a subset of those nodes that can provide coverage in the event of failures or disruptions. In both classical and Roman domination, the selection of these sets is done with an optimization goal in mind. The Roman domination concept was defined by Cockayne et al. [9], inspired by the problem of finding an optimal defense strategy for the Roman

Empire, in which legions could support one another when needed. Specifically, the vertices of the graph are assigned labels 0, 1, or 2 based on their capacity, such that a vertex with a value of 0 is considered incapable of self-protection, a vertex with a value of 1 can protect itself but would not have the capacity to provide defensive cover to others, and a vertex with a label of 2 would have the dual capacity of self-defense and of providing cover to others that require it.

The applications of Roman domination are not limited to the defense strategy that inspired the definition of the concept. There is a wide range of results and applications to all disciplines whose research starts from or is modeled using a network. The results of research on Roman domination have been useful for addressing problems of detecting nodes with high potential capacity in telecommunications networks, and for strengthening security and surveillance systems in a network that needs to be equipped with a strategic distribution of cameras or sensors. It is also widely applied to the design of logistics networks, where it is necessary to plan the location of distribution centers with different levels of capacity to provide an optimal solution in response times at a minimum cost. One widely studied variant of Roman domination in graphs is total Roman domination. Conceptually, this requires finding a Roman dominating function that classifies vertices according to the classic concept of Cockayne et al. [9], with the additional requirement that no control node (labeled 1 or 2) is isolated from the other control nodes. In [1, 29] we can find general bounds of the total Roman domination number of a graph and the exact value for some families of graphs. Other works have analyzed the resilience of the total Roman domination number of a graph under edge subdivision [2] and have investigated some variants of this parameter [3, 4, 10, 14, 34].

Combining the potential for control and monitoring of a network that exhibits graph domination with the suitability of fuzzy graphs for networks in which nodes and their interactions have distinct membership degrees and are subject to uncertain values, the concepts of domination and total domination in fuzzy graphs were introduced by Somasundaram et al. [31]. In a fuzzy graph, one vertex dominates itself, and two vertices dominate each other if there is a strong edge connecting them. In [31], the weight of a dominating set in a fuzzy graph is measured in the same way as in crisp graphs; that is, the domination number is the cardinality of a minimum dominating set. This definition overlooks the fact that the degree of membership in a fuzzy network is crucial for optimizing costs and enhancing the accuracy of the results obtained in this research. This fact led Manjusha and Sunitha [19] to introduce the strong domination [19] and strong total domination numbers [20] of a fuzzy graph. These are defined as the minimum cardinalities of the respective dominating and total dominating sets, weighting the unit value of each vertex by the minimum weight of the edges incident to it. In recent years, several variations of domination parameters in fuzzy graphs have been widely investigated due to their applicability in modeling real-world problems. For instance, Mojahedfar et al. [21] worked on the novel concept of Roman domination in fuzzy graphs, opening a new avenue for analyzing strategic protection and resource allocation within fuzzy network structures.

Total domination, in general, and in fuzzy graphs in particular, introduces a total

control indicator for network monitoring. This is because it requires not only that the set be dominating—and thus monitor the network—but also that the control nodes be connected (at least in pairs). This ensures that monitoring can also occur among the control nodes themselves in the event of potential failures or disruptions.
In this article, we define and study a parameter that extends total domination to the Roman version within the context of fuzzy graphs. We introduce the concept of strong total Roman domination in fuzzy graphs as a robustness parameter for networks that require both monitoring and resilience. Therefore, following the logic of Roman domination in crisp graphs, we aim to select nodes within the fuzzy network that not only have the capacity to control network operations but can also provide coverage to nodes with lower capacity or those established under a cost-optimization strategy.
To this end, in a fuzzy graph G, we consider a function $f : V(G) \to \{0, 1, 2\}$ that assigns values, $f(v)$, to the vertices according to the following criteria:

- $f(v) = 2$ if the vertex has the dual capacity to monitor and provide service coverage to other nodes in the network;
- $f(v) = 1$ if the vertex has the capacity to monitor other nodes in the network;
- $f(v) = 0$ if the vertex has a lower monitoring capacity that requires support.

The classification of vertices through the mapping f is carried out to optimize cost while ensuring that the dominating vertices (those with labels 1 or 2) are connected, at least in pairs, via a strong edge. In the following section, we formally define the concept of strong total Roman domination in fuzzy graphs.
In this work, we provide several general bounds for the strong total Roman domination number in fuzzy graphs and determine the exact values for specific families of such graphs.

2. Notations and basic concepts

Throughout this article, only undirected simple graphs and fuzzy graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the terminology and definitions provided in [12, 15, 22].
In graph theory, a graph G is defined as a pair (V, E), where V is a set of vertices and E is a set of unordered pairs of vertices, known as edges. Extending this concept, a fuzzy graph $G = (V, \sigma, \mu)$ is defined as a triple consisting of a non-empty set V and two membership functions, $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$. In this framework, the membership value of an edge, $\mu(u, v)$, is bounded by the minimum membership value of its incident vertices, ensuring that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$ for all $u, v \in V$, in which case μ is termed a fuzzy relation on σ.
Unless otherwise stated, V is assumed to be finite. Here, σ is referred to as the fuzzy vertex set and μ as the fuzzy edge set, where μ serves as a fuzzy relation on σ. For simplicity, the fuzzy graph $G = (V, \sigma, \mu)$ is often denoted by G or (σ, μ). Additionally,

the underlying graph is denoted by $G^* = (\sigma^*, \mu^*)$, where $\sigma^* = \{u \in V : \sigma(u) > 0\}$ and $\mu^* = \{(u,v) \in V \times V : \mu(u,v) > 0\}$.

Two vertices u and v in a fuzzy graph $G = (V, \sigma, \mu)$ are called adjacent or neighbors if $\mu(u,v) > 0$. A $u - v$ path P in G is a sequence of vertices $u = u_0, u_1, \ldots, u_n = v$ such that $\mu(u_{i-1}, u_i) > 0$, for $i = 1, \ldots, n$; the value n is the length of the path. The strength of P, denoted by $s(P)$, is defined as the membership value of its weakest edge by $s(P) = \wedge\{\mu(u_{i-1}, u_i) : i = 1, \ldots, n\}$.

The strength of connectedness between u and v in G, denoted by $CONN_G(u,v)$, is the maximum strength among all $u - v$ paths. A path P is a strongest $u - v$ path if $s(P) = CONN_G(u,v)$. A fuzzy graph G is connected if $CONN_G(u,v) > 0$ for all $u, v \in \sigma^*$.

An edge (u,v) is called a strong edge if its membership value equals the strength of connectedness between its endpoints, i.e., $\mu(u,v) = CONN_G(u,v)$. If (u,v) is strong, then u and v are strong neighbors. An edge that is not a strong edge is known as a δ-edge. Let $N(u)$ denote the set of all neighbors of u, and $N_s(u)$ denote the strong neighborhood of u. The closed strong neighborhood is $N_s[u] = N_s(u) \cup \{u\}$.

The degree $d(u)$ and strong degree $d_s(u)$ of a vertex u are defined as

$$d(u) = \sum_{v \in N(u)} \mu(u,v) \quad \text{and} \quad d_s(u) = \sum_{v \in N_s(u)} \mu(u,v).$$

That is, $d(u)$ is the sum of membership values of all incident edges, while $d_s(u)$ considers only strong incident edges. The minimum and maximum strong degrees of G are $\delta_s(G) = \wedge\{d_s(u) : u \in V\}$ and $\Delta_s(G) = \vee\{d_s(u) : u \in V\}$, respectively.

Furthermore, the strong neighborhood degree of u is $d_{SN}(u) = \sum_{v \in N_s(u)} \sigma(v)$. Its global minimum and maximum are denoted by $\delta_{SN}(G)$ and $\Delta_{SN}(G)$, respectively. We also define the minimum strong incident strength of a vertex u as $\mu_s(u) = \wedge\{\mu(u,v) : v \in N_s(u)\}$. By convention, we set $\mu_s(u) = 0$ if $N_s(u) = \emptyset$.

An edge (u,v) is effective (or M-strong) if $\mu(u,v) = \sigma(u) \wedge \sigma(v)$, in which case u and v are effective neighbors. A fuzzy graph with $|\sigma^*| = n$ is called a complete fuzzy graph, K_n, if every pair of vertices consists of effective neighbors. G is bipartite if V can be partitioned into two non-empty sets X and Y such that $\mu(u,v) = 0$ if $u, v \in X$ or $u, v \in Y$. If, in addition, $\mu(u,v) = \sigma(u) \wedge \sigma(v)$ for all $u \in X$ and $v \in Y$, then G is a complete bipartite fuzzy graph, denoted by K_{σ_1, σ_2}, where $|X| = \sigma_1$ and $|Y| = \sigma_2$.

A vertex u is isolated if $\mu(u,v) = 0$ for all $v \in V \setminus \{u\}$.

The order p and size q of a fuzzy graph $G = (V, \sigma, \mu)$ are defined as $p = \sum_{u \in V} \sigma(u)$ and $q = \sum_{(u,v) \in V \times V} \mu(u,v)$, respectively. For any subset $S \subseteq V$, $|S|$ denotes the cardinality of S, while its scalar cardinality is given by $|S|_s = \sum_{u \in S} \sigma(u)$.

The complement of G is defined as $\overline{G} = (V, \sigma, \overline{\mu})$, where the membership function of the edges is given by

$$\overline{\mu}(u,v) = \sigma(u) \wedge \sigma(v) - \mu(u,v) \quad \text{for all } u, v \in V.$$

Notably, the complement of a complete fuzzy graph is a fuzzy graph in which every vertex is isolated (i.e., $\overline{\mu}(u,v) = 0$ for all u, v).

We denote the minimum and maximum weights of a strong edge in G as $\mu_{\min}(G)$ and $\mu_{\max}(G)$, respectively, defined as

$$\mu_{\min} = \wedge\{\mu(u,v) : v \in N_s(u), u \in V\} \quad \text{and} \quad \mu_{\max} = \vee\{\mu(u,v) : v \in N_s(u), u \in V\}.$$

Correspondingly, let $\overline{\mu}_{\min} = \mu_{\min}(\overline{G})$ and $\overline{\mu}_{\max} = \mu_{\max}(\overline{G})$ represent these values for the complement graph.

In graph theory, given a graph $G = (V, E)$, a vertex $v \in V$ is said to dominate itself and all its neighbors. A subset $S \subseteq V$ is a dominating set of G if every vertex in $V \setminus S$ is adjacent to at least one vertex in S. The minimum cardinality of a dominating set is known as *the domination number* of G, denoted by $\gamma(G)$. A more restrictive version of domination requires every vertex to be dominated by another vertex, excluding itself. Formally, given a graph $G = (V, E)$ without isolated vertices, a subset $S \subseteq V$ is a *total dominating set* if every vertex $v \in V$ is adjacent to at least one vertex in S. In other words, the open neighborhood of S must contain all vertices of G, meaning that $N(S) = V$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. The concept of total domination was formally introduced and studied by Cockayne et al. in 1980 (see [8]).

One of the most prominent variants of this concept is Roman domination. A *Roman dominating function* (RDF) on a graph G is a mapping $f : V \to \{0, 1, 2\}$ such that every vertex u with $f(u) = 0$ is adjacent to at least one vertex v with $f(v) = 2$. The weight of an RDF is defined as $w(f) = \sum_{u \in V} f(u)$. The *Roman domination number* of G, denoted by $\gamma_R(G)$, is the minimum weight among all Roman dominating functions on G. The *total Roman domination number of a graph* $G = (V, E)$, denoted by $\gamma_{tR}(G)$, is the minimum weight $w(f) = \sum_{v \in V} f(v)$ among all total Roman dominating functions. A Roman dominating function $f : V \to \{0, 1, 2\}$ is called a *total Roman dominating function* if the subgraph induced by the set of vertices $\{v \in V \mid f(v) > 0\}$ has no isolated vertices. The concepts of Roman and total Roman domination were formally introduced by Cockayne et al. in [9].

In fuzzy graph theory, the concept of domination was first extended to fuzzy graphs by Nagoor Gani and Chandrasekaran [23] using the notion of strong edges. Given a fuzzy graph $G = (V, \sigma, \mu)$, every vertex $v \in V$ strongly dominates itself and each of its strong neighbors. Accordingly, a subset $D \subseteq V$ is a strong dominating set of G if every vertex in $V \setminus D$ is a strong neighbor of at least one vertex in D.

While Nagoor Gani and Vijayalakshmi [24] initially defined the strong domination number based on the scalar cardinality of the set, Manjusha and Sunitha [19] noted that in fuzzy location problems, the membership values of strong edges often yield more optimal results than vertex membership values. Consequently, the weight of a strong dominating set D is defined as $W(D) = \sum_{u \in D} \mu_s(u)$, where $\mu_s(u)$ represents the minimum membership value among the strong edges incident to u.

The *strong domination number of a fuzzy graph* G, denoted by $\gamma_s(G)$, is defined as the minimum weight among all strong dominating sets of G. A strong dominating set that achieves this minimum weight is referred to as a minimum strong dominating set, and it is called a $\gamma_s(G)$-set.

Following the criteria established by Manjusha and Sunitha [19], *a strong Roman dominating function (SRDF) on a fuzzy graph* $G = (V, \sigma, \mu)$ is defined as a function $f : V \to \{0, 1, 2\}$ such that every vertex $u \in V$ with $f(u) = 0$ has at least one strong neighbor $v \in N_s(u)$ with $f(v) = 2$. The *weight of a SRDF* f is given by $w(f) = \sum_{u \in V} f(u)\mu_s(u)$. The *strong Roman domination number* of G, denoted by $\gamma_{sR}(G)$, is the minimum weight among all possible SRDFs on G.
Any SRDF f on a fuzzy graph G induces a partition of V into the sets (V_0, V_1, V_2), where $V_i = \{u \in V : f(u) = i\}$ for $i \in \{0, 1, 2\}$. Since there is a bijective correspondence between a function f and its associated partition, we will hereafter represent a SRDF by its partition (V_0, V_1, V_2). A SRDF on G with weight $\gamma_{sR}(G)$ is referred to as a γ_{sR}-function of G or a $\gamma_{sR}(G)$-function.
The concept of the *strong total domination number* in fuzzy graphs was introduced by Manjusha and Sunitha in [20].

Definition 1. [20] Let $G = (V, \sigma, \mu)$ be a fuzzy graph without isolated vertices. A strong dominating set $D \subseteq V$ is called a *strong total dominating set* if every vertex of V has at least one strong neighbor in D. The minimum weight of a strong total dominating set is the *strong total domination number* of G, denoted by $\gamma_s^t(G)$.

Example 1. Figure 1 below shows the fuzzy cycle graph $C_4 = (V, \sigma, \mu)$ together with the corresponding $\sigma(v)$ values (in blue) and $\mu(u, v)$ values (in red).

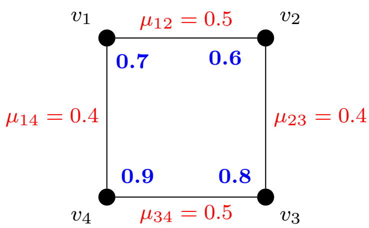

Figure 1. **Strong RDF in a fuzzy cycle.**

Clearly, all edges in the fuzzy graph are strong, which implies that the strong neighborhood of each vertex matches its open neighborhood in the cycle: $N_s(v) = N(v)$. We define the function f by assigning the value 2 to two adjacent vertices, v_1 and v_2, and 0 to the remaining two, v_3 and v_4:

$$f(v_1) = 2, \quad f(v_2) = 2, \quad f(v_3) = 0, \quad f(v_4) = 0$$

Since all edges are strong, f qualifies as a SRDF.

The *strong total Roman domination number* in fuzzy graphs is introduced and studied herein.

Definition 2. Let $G = (V, \sigma, \mu)$ be a fuzzy graph without isolated vertices. A strong Roman dominating function (SRDF) f on G is called a *strong total Roman dominating function* (STRDF) if every vertex u with $f(u) > 0$ has at least one strong neighbor v such that $f(v) > 0$. The weight of a STRDF is defined as $w(f) = \sum_{u \in V} f(u)\mu_s(u)$. The *strong total Roman domination number* of G, denoted by $\gamma_{sR}^t(G)$, is the minimum weight among all possible STRDFs on G.

Note that every strong total dominating set is, by definition, a strong dominating set of G. Similarly, every strong total Roman dominating function is also a strong Roman dominating function. Consequently, the following inequalities hold:

Remark 1. For any fuzzy graph $G = (V, \sigma, \mu)$, the following relationships are satisfied:

(i) $\gamma_s(G) \leq \gamma_s^t(G)$.

(ii) $\gamma_{sR}(G) \leq \gamma_{sR}^t(G)$.

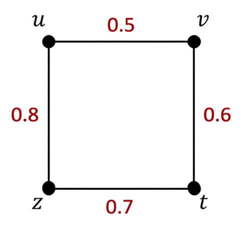

Figure 2. **Fuzzy graph.**

Example 2. The fuzzy graph illustrated in Figure 2 contains three strong edges: (u, z), (z, t), and (v, t). The strong dominating sets of G are $\{u, v\}$, $\{u, t\}$, $\{z, v\}$, $\{z, t\}$, $\{u, v, z\}$, $\{u, v, t\}$, $\{v, z, t\}$, $\{u, z, t\}$, and $\{u, v, z, t\}$. Among these, the only strong total dominating sets are $D_1 = \{z, t\}$, $D_2 = \{u, z, t\}$, $D_3 = \{v, z, t\}$ and $D_4 = \{u, v, z, t\}$, which implies that $\gamma_s^t(G) = W(D_1) = 0.7 + 0.6 = 1.3$, because $D_1 \subset D_i$ for $i = 2, 3, 4$.
Since (u, v) is a δ-edge, we have $\mu_s(u) = 0.8 > 0.7 = \mu_s(z)$ and $\mu_s(v) = 0.6 = \mu_s(t)$. Given that every STRDF f on G must satisfy $f(z) > 0$ and $f(t) > 0$, the unique function f attaining $\gamma_{sR}^t(G)$ is defined by $f(z) = f(t) = 2$ and $f(u) = f(v) = 0$. Its weight is $\gamma_{sR}^t(G) = w(f) = 2\mu_s(z) + 2\mu_s(t) = 2.6$.

3. Results on strong total Roman domination in fuzzy graphs

This section establishes general lower and upper bounds for the strong total Roman domination number of a fuzzy graph. We begin by providing fundamental bounds that relate this parameter to the strong total domination number. Let us start by giving some motivation for the derived bounds.

Example 3. Let $G = (V, \sigma, \mu)$ be a fuzzy path P_5 with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$, and edge memberships $\mu(v_1v_2) = 0.7$, $\mu(v_2v_3) = 0.5$, $\mu(v_3v_4) = 0.6$, $\mu(v_4v_5) = 0.4$. We may readily check that every edge is a strong edge because there are no cycles in the graph. So the strong neighbourhood of every internal vertex consists of both its path-neighbours. The strong minimum edge weight at each vertex is $\mu_s(v_1) = 0.7$, $\mu_s(v_2) = 0.5$, $\mu_s(v_3) =$

0.5, $\mu_s(v_4) = 0.4$, $\mu_s(v_5) = 0.4$. Define $f\colon V \to \{0,1,2\}$ by $f(v_1) = 0$, $f(v_2) = 2$, $f(v_3) = 0$, $f(v_4) = 2$, $f(v_5) = 0$. Then $\{v_1, v_3\}$ have v_2 as a strong neighbour, with $f(v_2) = 2$; v_5 has v_4 as a strong neighbour, with $f(v_4) = 2$; and therefore, f is a SRDF of G which lead us to $\gamma_{sR}(G) \leq 1.8$.
Since $f(v_2) > 0$ but all strong neighbours of v_2 carry label 0, the function above is not a valid STRDF. The same happens with v_4. We may easily check that the function $g(v) = 1$, for all $v \in V$ is a STRDF with minimum possible weight $w(g) = 2.3$.
This example illustrates how the total condition in the definition may strictly increase the domination number compared with the non-total variant, and motivates the general bound $\gamma^t_{sR}(G) \geq \gamma'_s(G)$ proved in the following theorem.

Theorem 1. *For any fuzzy graph $G = (V, \sigma, \mu)$, the following inequalities hold:*

$$\gamma^t_s(G) \leq \gamma^t_{sR}(G) \leq 2\gamma^t_s(G).$$

Proof. First, we show that $\gamma^t_s(G) \leq \gamma^t_{sR}(G)$. Let $f = (V_0, V_1, V_2)$ be a γ^t_{sR}-function on G. By definition, every vertex in V_0 has at least one strong neighbor in V_2, which implies that the set $D = V_1 \cup V_2$ is a strong dominating set of G. Furthermore, since f is a STRDF, every vertex in $V_1 \cup V_2$ must have at least one strong neighbor in $V_1 \cup V_2$. Consequently, D is a strong total dominating set, and the following holds:

$$\begin{aligned}
\gamma^t_s(G) \leq W(D) &= \sum_{u \in V_1} \mu_s(u) + \sum_{u \in V_2} \mu_s(u) \\
&\leq \sum_{u \in V_1} \mu_s(u) + 2\sum_{u \in V_2} \mu_s(u) \\
&= \sum_{i=1}^{2} \sum_{u \in V_i} f(u)\mu_s(u) = w(f) = \gamma^t_{sR}(G).
\end{aligned} \tag{3.1}$$

Next, we prove that $\gamma^t_{sR}(G) \leq 2\gamma^t_s(G)$. Let D be a minimum strong total dominating set of G (a $\gamma^t_s(G)$-set). We define a function $f : V \to \{0, 1, 2\}$ as $f(u) = 2$ if $u \in D$, and $f(u) = 0$ otherwise. Since D is a strong dominating set, f is a strong Roman dominating function (SRDF). Moreover, because D is a strong total dominating set, every vertex in D has a strong neighbor in D. Given that $V_2 = D$ and $V_1 = \emptyset$, the condition for f to be a STRDF is satisfied. Therefore,

$$\gamma^t_{sR}(G) \leq w(f) = \sum_{u \in D} 2\mu_s(u) = 2W(D) = 2\gamma^t_s(G).$$

This completes the proof. □

Example 4. Let $G = (V, \sigma, \mu)$ be a fuzzy graph on $V = \{v_1, v_2, v_3, v_4\}$ with $\sigma(v_1) = 0.8$, $\sigma(v_2) = 0.7$, $\sigma(v_3) = 0.9$, $\sigma(v_4) = 0.6$, and edge weights $\mu(v_1, v_2) = 0.6$, $\mu(v_2, v_3) = 0.6$, $\mu(v_3, v_4) = 0.6$, $\mu(v_4, v_1) = 0.5$ (see Figure 3). The strongest path from v_4 to v_1 not using the direct edge has strength $\min\{0.6, 0.6, 0.6\} = 0.6 > 0.5 = \mu(v_4, v_1)$, so (v_4, v_1) is not strong. Consequently, $\mu_s(v_i) = 0.6$ for all i.

The set $D = \{v_2, v_3\}$ is a strong total dominating set: v_1 and v_4 are strongly dominated, and v_2, v_3 are mutually strong neighbors. Its weight is $\gamma_s^t(G) \leq 0.6 + 0.6 = 1.2$; since no single vertex can form a valid total dominating set, $\gamma_s^t(G) = 1.2$.

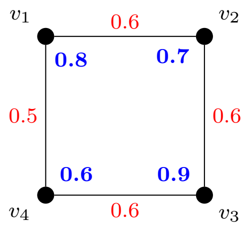

Figure 3. Fuzzy graph of Example 4. Edge (v_1, v_4) is non-strong.

The function $f(v_2) = f(v_3) = 2$, $f(v_1) = f(v_4) = 0$ is a STRDF: v_1 and v_4 are strongly dominated by v_2 and v_3 respectively, and the support $\{v_2, v_3\}$ is connected by a strong edge. Its weight is $w(f) = 2{\cdot}0.6 + 2{\cdot}0.6 = 2.4$. No STRDF of weight less than 2.4 exists, so $\gamma_{sR}^t(G) = 2.4$. The bounds of Theorem 1 are satisfied, with the upper bound achieved:

$$1.2 \;=\; \gamma_s^t(G) \;\leq\; \gamma_{sR}^t(G) \;=\; 2.4 \;=\; 2\,\gamma_s^t(G).$$

In the following theorem, we examine the families of fuzzy graphs for which the parameters γ_s^t and γ_{sR}^t coincide. Specifically, we characterize the fuzzy graphs that attain the lower bound for γ_{sR}^t established in Theorem 1.

Theorem 2. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph of order $n = |\sigma^*|$ with no isolated vertices. The following statements are equivalent:*

1. $\gamma_s^t(G) = \gamma_{sR}^t(G)$.
2. *n is an even integer and G is formed by $n/2$ disjoint strong edges.*

Proof. Suppose first that G is formed by the disjoint union of $n/2$ strong edges. In this case, the only strong total dominating set of G is $D = V$, and the unique $\gamma_{sR}^t(G)$-function is $f : V \to \{0, 1, 2\}$ defined by $f(u) = 1$ for all $u \in V$. Thus, we obtain:

$$\gamma_s^t(G) = W(V) = \sum_{u \in V} \mu_s(u) = w(f) = \gamma_{sR}^t(G).$$

Now, assume that $\gamma_s^t(G) = \gamma_{sR}^t(G)$. In this scenario, the inequality in (3.1) must become an equality, which implies $V_2 = \emptyset$ and $V_0 = \emptyset$. This leads to $V_1 = V$. To complete the proof, it suffices to show that every vertex in V has exactly one strong neighbor.

Suppose, to the contrary, that there exists at least one vertex with more than one strong neighbor. Let $\mathcal{B} = \{u \in V : |N_s(u)| \geq 2\}$. We distinguish two cases:

Case 1. Suppose that for every vertex $u \in \mathcal{B}$, all its strong neighbors $v \in N_s(u)$ also satisfy $|N_s(v)| \geq 2$. Let $u \in \mathcal{B}$ and $v \in N_s(u)$ such that, without loss of generality, $\mu_s(u) \leq \mu_s(v)$. Consider the function $g : V \to \{0, 1, 2\}$ defined by $g(u) = 2$, $g(v) = 0$, and $g(w) = 1$ for all $w \in V \setminus \{u, v\}$. Clearly, g is a SRDF since v is strongly dominated by u. In addition, g is a STRDF because every vertex in $V \setminus \{v\}$ has at least one strong neighbor other than v, ensuring that $G[V \setminus \{v\}]$ has no isolated vertices. However, since $|V_2| = 1$, g cannot be a γ^t_{sR}-function (as we established that any γ^t_{sR}-function must have $V_2 = \emptyset$). This leads to

$$\begin{aligned}\gamma^t_{sR}(G) < w(g) &= 2\mu_s(u) + \sum_{w \in V \setminus u,v} \mu_s(w) \\ &\leq \mu_s(u) + \mu_s(v) + \sum_{w \in V \setminus u,v} \mu_s(w) \\ &= w(f) = \gamma^t_{sR}(G),\end{aligned}$$

which is a contradiction.

Case 2. Suppose there exists $u \in \mathcal{B}$ and $v \in N_s(u)$ such that $N_s(v) = \{u\}$. We define g as in Case 1. Since $u \in \mathcal{B}$, u has at least one strong neighbor other than v; thus $G[V \setminus \{v\}]$ has no isolated vertices, and g is a STRDF. Moreover, the definition of a fuzzy graph implies $\mu_s(u) \leq \mu(u, v) = \mu_s(v)$. Following the same derivation as in Case 1, we obtain $\gamma^t_{sR}(G) < w(g) \leq \gamma^t_{sR}(G)$, another contradiction.

Consequently, every vertex in V has exactly one strong neighbor, meaning that G is a disjoint union of strong edges. Since G has no isolated vertices, it must consist of $n/2$ such edges, implying that n is even. □

Additionally, we provide both upper and lower bounds for the strong total Roman domination number in a fuzzy graph, this time in terms of the strong domination number.

Theorem 3. *For any fuzzy graph $G = (V, \sigma, \mu)$, the following relationship holds:*

$$2\gamma_s(G) \leq \gamma^t_{sR}(G) \leq 3\gamma_s(G).$$

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma^t_{sR}(G)$-function that minimizes the number of vertices assigned a label 1. By definition, the weight of f is given by

$$w(f) = \sum_{u \in V_2} 2\mu_s(u) + \sum_{u \in V_1} \mu_s(u) = \gamma^t_{sR}(G).$$

• Lower Bound: $2\gamma_s(G) \leq \gamma^t_{sR}(G)$.

If V_2 is a strong dominating set, then

$$\gamma_s(G) \leq \sum_{u \in V_2} \mu_s(u) \leq \frac{1}{2} w(f).$$

and hence $2\gamma_s(G) \leq \gamma_{sR}^t(G)$. Otherwise, let A be the set of vertices not dominated by V_2. Since f is a STRDF, it follows that $A \subseteq V_1$.
We claim that the induced subgraph $G[A]$ consists of a disjoint union of strong edges. First, $G[A]$ contains no isolated vertices, as f is a total function and no vertex in A is dominated by V_2.
Suppose that $|N_s(u) \cap A| \geq 2$ for every $u \in A$. Let $u \in A$ and $v \in N_s(u) \cap A$ be arbitrary vertices. By the assumption, it also holds that $|N_s(v) \cap A| \geq 2$. Without loss of generality, assume that $\mu_s(u) \leq \mu_s(v)$. Now, consider the function $g : V \to \{0, 1, 2\}$ defined by $g(u) = 2$, $g(v) = 0$, and $g(z) = f(z)$ for all $z \in V \setminus \{u, v\}$. It is clear that g is a SRDF. Furthermore, g is a STRDF because every strong neighbor of v contains at least one strong neighbor other than v; thus, the induced subgraph $G[V \setminus \{v\}]$ contains no isolated vertices.
The weight of g satisfies

$$w(g) = w(f) - \mu_s(v) + \mu_s(u) \leq w(f).$$

This implies that g is a $\gamma_{sR}^t(G)$-function. However, g has fewer vertices with label 1 than f (since $v \in A \subseteq V_1$ was changed to $g(v) = 0$), which contradicts the choice of f.

Suppose now that there exist two strong neighbors $u, v \in A$ such that $|N_s(u) \cap A| \geq 2$ and $|N_s(v) \cap A| = 1$. Following the same argument as above, we define the function $g : V \to \{0, 1, 2\}$ as $g(u) = 2$, $g(v) = 0$, and $g(z) = f(z)$ for all $z \in V \setminus \{u, v\}$. In this case, g remains a STRDF on G because v has no strong neighbors other than u, and the condition $|(N_s(u) \setminus \{v\}) \cap A| \geq 1$ ensures that u still has at least one strong neighbor in the support of g. Moreover, since $\mu_s(u) \leq \mu(u, v) = \mu_s(v)$, the weight of g satisfies

$$w(g) = w(f) - \mu_s(v) + \mu_s(u) \leq w(f).$$

This leads to the conclusion that g is a $\gamma_{sR}^t(G)$-function with fewer labels equal to 1 than f, which again contradicts the minimality of f.
Consequently, $G[A]$ consists of disjoint strong edges. Let D be a set comprising V_2 and exactly one vertex from each edge in $G[A]$. Then D is a strong dominating set and

$$\begin{aligned}
\gamma_s(G) \leq \sum_{u \in D} \mu_s(u) &= \sum_{u \in V_2} \mu_s(u) + \frac{1}{2} \sum_{u \in A} \mu_s(u) \\
&\leq \sum_{u \in V_2} \mu_s(u) + \frac{1}{2} \sum_{u \in V_1} \mu_s(u) \\
&\leq \frac{1}{2} \left(\sum_{u \in V_2} 2\mu_s(u) + \sum_{u \in V_1} \mu_s(u) \right) = \frac{1}{2} \gamma_{sR}^t(G).
\end{aligned}$$

• Upper Bound: $\gamma_{sR}^t(G) \leq 3\gamma_s(G)$.
Let D be a γ_s-set of G. Define $\mathcal{A} = \{u \in D : N_s(u) \cap D = \emptyset\}$. For each $u \in \mathcal{A}$, choose a neighbor $u^* \in N_s(u)$ such that $\mu_s(u) = \mu(u, u^*)$. Since $\mu_s(u^*) \leq \mu(u, u^*) = \mu_s(u)$,

we define $f : V \to \{0, 1, 2\}$ as $f(u) = 2$ for all $u \in D$, $f(u^*) = 1$ for all $u \in \mathcal{A}$, and $f(z) = 0$ otherwise. This f is a STRDF and its weight is

$$\gamma_{sR}^t(G) \leq w(f) = \sum_{u \in D} 2\mu_s(u) + \sum_{u \in \mathcal{A}} \mu_s(u^*) \leq \sum_{u \in D} 3\mu_s(u) = 3\gamma_s(G).$$

The proof is now complete. □

The following two results establish upper bounds for the strong total Roman domination parameter in fuzzy graphs in terms of the strong Roman domination number.

Theorem 4. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with no isolated vertex. Then*

$$\gamma_{sR}^t(G) < 2\gamma_{sR}(G).$$

Proof. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{sR}(G)$-function that minimizes the number of vertices assigned a label 1. By definition, the weight of f is given by

$$w(f) = \sum_{u \in V_2} 2\mu_s(u) + \sum_{u \in V_1} \mu_s(u) = \gamma_{sR}(G).$$

Clearly, no vertex $u \in V_1$ can be a strong neighbor of a vertex $v \in V_2$, otherwise the function $g = (V_0 \cup \{u\}, V_1 \setminus \{u\}, V_2)$ would be a SRD-function with weight $w(g) = w(f) - \mu_s(u) < w(f)$, which is a contradiction. In fact, no two vertices in V_1 can be strong neighbors. Indeed, if $u, v \in V_1$ were strong neighbors, assuming without loss of generality that $\mu_s(u) \leq \mu_s(v)$, we could define the SRD-function $g = (V_0 \cup \{v\}, V_1 \setminus \{u, v\}, V_2 \cup \{u\})$ with weight $w(g) = w(f) + \mu_s(u) - \mu_s(v) \leq w(f)$. This contradicts the fact that f is a $\gamma_{sR}(G)$-function that minimizes the number of vertices labeled 1. Therefore, if $V_1 \neq \emptyset$, then V_1 is an independent set and there are no strong edges between the vertices of V_1 and the vertices of V_2, which implies that $V_2 \neq \emptyset$. Let $Z \subseteq V_2$ be the set of vertices in V_2 that have no strong neighbors in the subgraph $G[V_2]$. We define the sets $W^* = \{u^* \in N_s(u) : u \in V_1 \text{ and } \mu_s(u) = \mu(u, u^*)\}$ and $Z^* = \{u^* \in N_s(u) : u \in Z \text{ and } \mu_s(u) = \mu(u, u^*)\}$. Clearly, the following holds:

$$\mu_s(u^*) \leq \mu_s(u), \text{ for every } u \in V_1 \cup Z, \tag{3.2}$$

since $\mu_s(u^*) \leq \mu(u^*, u) = \mu_s(u)$. Furthermore, we note that $W^* \cup Z^* \subseteq V_0$, since there are no strong edges between the vertices of V_1 and $V_1 \cup V_2$, and the vertices of Z do not have strong neighbors in V_2. Consequently, the function $g = (V_0 \setminus (W^* \cup Z^*), V_1 \cup W^* \cup Z^*, V_2)$ is a STRD-function, because every vertex in V_1 has at least one strong neighbor in W^*, and every vertex in V_2 has at least one strong neighbor in $V_2 \cup Z^*$. From this fact, along with (3.2) and $V_2 \neq \emptyset$, it follows that:

$$\begin{aligned}
\gamma^t_{sR}(G) \leq w(g) &= w(f) + \sum_{u^* \in W^*} \mu_s(u^*) + \sum_{u^* \in Z^*} \mu_s(u^*) \leq \gamma_{sR}(G) + \sum_{u \in V_1} \mu_s(u) \\
&+ \textstyle\sum_{u \in Z} \mu_s(u) \\
&< \gamma_{sR}(G) + \sum_{u \in V_1} \mu_s(u) + \sum_{u \in Z} 2\mu_s(u) \leq \gamma_{sR}(G) + \sum_{u \in V_1} \mu_s(u) \\
&+ \textstyle\sum_{u \in V_2} 2\mu_s(u) \\
&= 2\gamma_{sR}(G),
\end{aligned}$$

which completes the proof. □

Corollary 1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph such that $\gamma_{sR}(G) = 2\gamma_s(G)$. Then:*

$$\gamma^t_{sR}(G) \leq \frac{3}{2}\gamma_{sR}(G) = 3\gamma_s(G).$$

Proof. Let $D \subseteq V(G)$ be a $\gamma_s(G)$-set. Define the function $f = (V_0, V_1, V_2)$ on G as $f(u) = 2$ if $u \in D$, and $f(u) = 0$ otherwise. Clearly, f is a SRD-function such that:

$$w(f) = \sum_{u \in D} 2\mu_s(u) = 2\gamma_s(G) = \gamma_{sR}(G).$$

Thus, f is a γ_{sR}-function on G with $V_1 = \emptyset$.
Following the reasoning in the proof of Theorem 4, let Z be the subset of V_2 consisting of vertices that do not have a strong neighbor in V_2. Define $Z^* = \{u^* \in N_s(u) : u \in Z \text{ and } \mu_s(u) = \mu(u, u^*)\}$. By definition, $\mu_s(u^*) \leq \mu_s(u)$ for all $u \in Z$. Since $V_1 = \emptyset$, it is evident that $Z^* \subseteq V_0$. Therefore, the function $g = (V_0 \setminus Z^*, V_1 \cup Z^*, V_2)$ is a TSRD-function, because every vertex in $V_2 \setminus Z$ has a strong neighbor in V_2, and every vertex in Z has a strong neighbor in Z^*. This implies:

$$\begin{aligned}
\gamma^t_{sR}(G) \leq w(g) &= w(f) + \sum_{u^* \in Z^*} \mu_s(u^*) \leq \gamma_{sR}(G) + \sum_{u \in Z} \mu_s(u) \\
&\leq \gamma_{sR}(G) + \sum_{u \in V_2} \mu_s(u) = \gamma_{sR}(G) + \frac{1}{2}\gamma_{sR}(G) = \frac{3}{2}\gamma_{sR}(G),
\end{aligned}$$

which completes the proof. □

4. The strong total Roman domination number of some families of graphs

In this section, we determine the exact value of the strong total Roman domination number for several families of fuzzy graphs, including complete fuzzy graphs, complete bipartite fuzzy graphs, fuzzy cycles, and fuzzy graphs containing at least one universal vertex.
Given a fuzzy graph $G = (V, \sigma, \mu)$, let $\mathcal{U}_s(G) = \{v \in V : N_s[v] = V\}$ denote the set of universal vertices in G.

Proposition 1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with $|\sigma^*| \geq 3$. If $\mathcal{U}_s(G) \neq \emptyset$, then*

$$\gamma^t_{sR}(G) = 3 \min\{\mu_s(v) : v \in \mathcal{U}_s(G)\}.$$

Proof. Let $u \in \mathcal{U}_s(G)$ be a vertex such that $\mu_s(u) = \min\{\mu_s(v) : v \in \mathcal{U}_s(G)\}$. We first show that $\mu_s(u) \leq \mu_s(v)$ for all $v \in V$. Suppose, for the sake of contradiction, that there exists a vertex $w \in V$ such that $\mu_s(w) < \mu_s(u)$. By definition, there exists a vertex $z \in N_s(w) \setminus \{u\}$ such that $\mu(w, z) = \mu_s(w)$. Since $u \in \mathcal{U}_s(G)$, it follows that $N_s[u] = V$, which implies $z \in N_s(u)$. In this case, we can define a $w - z$ path $P : w, u, z$ with strength $s(P) = \min\{\mu(w, u), \mu(u, z)\}$. Given that both (w, u) and (u, z) are strong edges, the weights of both edges are at least $\mu_s(u)$. Thus, $s(P) \geq \mu_s(u) > \mu_s(w)$, which leads to

$$\mu(w, z) = \mu_s(w) < s(P) \leq \mathrm{CONN}_G(w, z).$$

This implies that (w, z) is not a strong edge, which leads to a contradiction. Therefore, $\mu_s(u) \leq \mu_s(v)$ for all $v \in V$. Now, let $u^* \in N_s(u)$ be such that $\mu_s(u) = \mu(u, u^*)$. It follows that $\mu_s(u^*) \leq \mu(u, u^*) = \mu_s(u)$ and, since we previously established that $\mu_s(u) \leq \mu_s(u^*)$, we have $\mu_s(u^*) = \mu_s(u) \leq \mu_s(v)$ for all $v \in V$. Consequently, the function $f : V \to \{0, 1, 2\}$ defined by $f(u) = 2$, $f(u^*) = 1$, and $f(v) = 0$ for every $v \in V \setminus \{u, u^*\}$ is a γ^t_{sR}-function on G with weight $\gamma^t_{sR}(G) = \omega(f) = 3\mu_s(u) = 3 \min\{\mu_s(v) : v \in \mathcal{U}_s(G)\}$. □

As a direct consequence of Proposition 1, we determine the exact value of the strong total Roman domination number for any complete fuzzy graph.

Corollary 2. *For a complete fuzzy graph $K_n = (V, \sigma, \mu)$ with $|\sigma^*| = n \geq 3$,*

$$\gamma^t_{sR}(K_n) = 3 \min\{\mu_s(v) : v \in V\}.$$

Proof. The result follows directly from Proposition 1, noting that in a complete fuzzy graph, every vertex is a universal vertex since all edges are strong. □

The following theorem establishes the strong total Roman domination number of a complete bipartite fuzzy graph based on the weight of its weakest strong edge.

Theorem 5. *Let $K_{n_1,n_2} = (X \cup Y, \sigma, \mu)$ be a complete bipartite fuzzy graph with $|X| = n_1$ and $|Y| = n_2$. Assume, without loss of generality, that $n_1 \leq n_2$. Then:*

$$\gamma_{sR}^t(K_{n_1,n_2}) = \begin{cases} \min\{3, 1+n_2\} \cdot \mu_{\min}(K_{n_1,n_2}) & \text{if } n_1 = 1, \\ 4 \cdot \mu_{\min}(K_{n_1,n_2}) & \text{if } n_1 \geq 2. \end{cases}$$

Proof. In a complete bipartite fuzzy graph K_{n_1,n_2}, all edges are strong. Let $X = \{x_1, \dots, x_{n_1}\}$ and $Y = \{y_1, \dots, y_{n_2}\}$ be the partite sets of V, ordered such that $\mu_s(x_i) \leq \mu_s(x_{i+1})$ and $\mu_s(y_j) \leq \mu_s(y_{j+1})$. It follows that $\mu_{\min}(K_{n_1,n_2}) = \min\{\mu(x_i, y_j)\} = \mu(x_1, y_1) = \mu_s(x_1) = \mu_s(y_1)$.
If $n_1 = n_2 = 1$, then K_{n_1,n_2} consists of a single edge, and clearly $\gamma_{sR}^t(K_{n_1,n_2}) = 2\mu_s(x_1) = 2\mu_{\min}(K_{n_1,n_2})$. Now, assume $n_1 = 1$ and $n_2 \geq 2$. In this case, x_1 is a universal vertex (i.e., $x_1 \in \mathcal{U}_s(K_{1,n_2})$). By Proposition 1, the result follows.
Next, consider the case where $n_1 \geq 2$. Define a function $f : V \to \{0, 1, 2\}$ such that $f(x_1) = 2$, $f(y_1) = 2$, and $f(v) = 0$ for all other vertices. This function is a strong total Roman dominating function (STRDF) on K_{n_1,n_2}, which implies $\gamma_{sR}^t(K_{n_1,n_2}) \leq \omega(f) = 4\mu_s(x_1) = 4\mu_{\min}(K_{n_1,n_2})$.
To prove the reverse inequality, let f be a γ_{sR}^t-function. Since $\mu_s(x_1) = \min\{\mu_s(u) : u \in X \cup Y\}$, we analyze the size of $V_2 = \{u \in V : f(u) = 2\}$. If $|V_2| \neq 1$, then either $V_2 = \emptyset$ (implying $V_1 = X \cup Y$) or $|V_2| \geq 2$. In either case, the weight satisfies $\omega(f) \geq \min\{n_1 + n_2, 2|V_2|\}\mu_s(x_1) \geq 4\mu_s(x_1)$, given that $n_1, n_2 \geq 2$.
If $|V_2| = 1$, the vertex in V_2 must have at least one neighbor in V_1 to satisfy the total domination property. Furthermore, since every vertex in the opposing partite set must be dominated and $n_1, n_2 \geq 2$, it follows that $V_1 \cap X \neq \emptyset$ and $V_1 \cap Y \neq \emptyset$. Thus, $|V_1| \geq 2$, leading to

$$\gamma_{sR}^t(K_{n_1,n_2}) = \omega(f) = \sum_{u \in V} f(u)\mu_s(u) \geq (|V_1| + 2|V_2|)\mu_s(x_1) \geq (2+2)\mu_s(x_1) = 4\mu_s(x_1).$$

This completes the proof. □

The exact value of the total Roman domination number for a crisp cycle with n vertices is known to be n (see [1]). However, for fuzzy cycles, the proof is considerably more complex, as the distribution of weights among the fuzzy edges is a determining factor in selecting an optimal γ_{sR}^t-function. In the following theorem, we establish an upper bound for $\gamma_{sR}^t(C_n)$ in terms of the size of a fuzzy cycle and the maximum and minimum weights of its strong edges.

Theorem 6. *Let $n \geq 3$ and $k = \lfloor n/3 \rfloor$ be integers. Let $C_n = (V, \sigma, \mu)$ be a fuzzy cycle with $|\sigma| = n$, represented by the sequence $u_1, u_2, \dots, u_n, u_1$. If C_n contains at least two edges with minimum weight, then:*

$$\gamma_{sR}^t(C_n) \leq q - (\mu_{\max} - \mu_{\min}).$$

Proof. Since C_n has at least two edges with minimum weight, it follows that all its edges are strong. For the sake of convenience, all indices are taken modulo n, so that $u_{n+1} = u_1$. We analyze three distinct cases.

Case 1. $n = 3k$. For each $m \in \{1, \dots, n\}$, let $f_m : V \to \{0, 1, 2\}$ be a function defined by $f_m(u_{m+3i}) = 2$, $f_m(u_{m+3i+1}) = 1$, and $f_m(u_{m+3i+2}) = 0$ for $i = 0, \dots, k-1$. These functions $\{f_m\}$ are STRDFs on C_n and satisfy $\gamma^t_{sR}(C_n) \leq \omega(f_m)$ for all m.
Consider an arbitrary vertex u_j. We claim that $\sum_{m=1}^{n} f_m(u_j) = n$. To prove this, observe that the functions f_m systematically generate all cyclic shifts of the base pattern $(2, 1, 0, 2, 1, 0, \dots, 2, 1, 0)$. For each fixed m, exactly one of the following holds for the position of u_j:

- $j \equiv m + 3i \pmod{n}$ for some $i \in \{0, \dots, k-1\} \implies f_m(u_j) = 2$,
- $j \equiv m + 3i + 1 \pmod{n}$ for some $i \in \{0, \dots, k-1\} \implies f_m(u_j) = 1$,
- $j \equiv m + 3i + 2 \pmod{n}$ for some $i \in \{0, \dots, k-1\} \implies f_m(u_j) = 0$.

Since the sets $\{u_{m+3i}\}_{i=0}^{k-1}$, $\{u_{m+3i+1}\}_{i=0}^{k-1}$, and $\{u_{m+3i+2}\}_{i=0}^{k-1}$ partition V into three equal subsets of size $k = n/3$, exactly one i satisfies each congruence for every m. Consequently, each vertex is assigned each of the labels 0, 1, and 2 exactly $n/3$ times. Therefore, $\sum_{m=1}^{n} f_m(u_j) = (0 + 1 + 2) \cdot n/3 = n$.
Then, we have that

$$\begin{aligned} n \cdot \gamma^t_{sR}(C_n) &\leq \sum_{m=1}^{n} \omega(f_m) = \sum_{m=1}^{n} \sum_{j=1}^{n} f_m(u_j)\mu_s(u_j) \\ &= \sum_{j=1}^{n} \mu_s(u_j) \sum_{m=1}^{n} f_m(u_j) \leq n \sum_{j=1}^{n} \mu_s(u_j) \\ &\leq n\left(q - (\mu_{\max} - \mu_{\min})\right). \end{aligned} \tag{4.1}$$

Consequently, $\gamma^t_{sR}(C_n) \leq q - (\mu_{\max} - \mu_{\min})$.

Case 2. $n = 3k + 1$. We proceed analogously. For each $m \in \{1, \dots, n\}$, we define f_m as follows $f_m(u_m) = 1$, $f_m(u_{m+1+3i}) = 1$, $f_m(u_{m+2+3i}) = 2$ and $f_m(u_{m+3+3i}) = 0$ for $i = 0, \dots, k-1$. Since each f_m is a STRDF on C_n, we conclude that

$$n \cdot \gamma^t_{sR}(C_n) \leq \sum_{m=1}^{n} \omega(f_m) = (3k+1) \sum_{j=1}^{n} \mu_s(u_j) \leq (3k+1)\left(q - (\mu_{\max} - \mu_{\min})\right). \tag{4.2}$$

Thus, we obtain: $\gamma^t_{sR}(C_n) \leq \frac{3k+1}{n}\left(q - (\mu_{\max} - \mu_{\min})\right) = q - (\mu_{\max} - \mu_{\min})$.

Case 3. $n = 3k + 2$. Similarly, for each $m \in \{1, \dots, n\}$, let f_m be defined by: $f_m(u_m) = f_m(u_{m-1}) = 1$, $f_m(u_{m+1+3i}) = 2$, $f_m(u_{m+2+3i}) = 0$, and $f_m(u_{m+3+3i}) = 1$ for $i = 0, \dots, k-1$. The set $\{f_m\}$ consists of STRDFs on C_n and

$$n \cdot \gamma^t_{sR}(C_n) \leq \sum_{m=1}^{n} \omega(f_m) = (3k+2) \sum_{j=1}^{n} \mu_s(u_j) \leq (3k+2)\left(q - (\mu_{\max} - \mu_{\min})\right). \tag{4.3}$$

This implies that $\gamma^t_{sR}(C_n) \leq \frac{3k+2}{n}\left(q - (\mu_{\max} - \mu_{\min})\right) = q - (\mu_{\max} - \mu_{\min})$. $\square$

Theorem 7. *Let $n, k \geq 1$ be integers such that $n = 3k+\ell$ for $\ell \in \{1, 2\}$. Let $C_n = (V, \sigma, \mu)$ be a fuzzy cycle of size q with n vertices. If C_n contains at least two edges with minimum weight $\mu_{\min}$, then the following assertions are equivalent:*

1. *$\gamma^t_{sR}(C_n) = q - (\mu_{\max} - \mu_{\min})$.*
2. *At least $n-1$ edges have weight equal to $\mu_{\min}$ and at most one edge has weight equal to $\mu_{\max}$.*

Proof. Let the fuzzy cycle be denoted by $C_n : u_1, u_2, \ldots, u_n, u_1$. Since C_n contains at least two edges with minimum weight, it follows that all its edges are strong. If all edge weights are equal to $\mu_{\min}$, except for at most one edge with weight $\mu_{\max}$, then $\mu_s(u_i) = \mu_{\min}$ for all $i = 1, \ldots, n$. In this case, the underlying graph cycle of C_n can be identified as an unweighted cycle C'_n scaled by $\mu_{\min}$. It is known that the total Roman domination number of an unweighted cycle of order n is equal to n. Thus, we have

$$\gamma^t_{sR}(C_n) = \gamma_{tR}(C'_n)\mu_{\min} = n\mu_{\min} = q - (\mu_{\max} - \mu_{\min}).$$

To prove the converse, assume that $\gamma^t_{sR}(C_n) = q - (\mu_{\max} - \mu_{\min})$. Under this assumption, all inequalities in Theorem 6 must hold as equalities—specifically those in (4.2) if $n = 3k + 1$, and those in (4.3) if $n = 3k + 2$. Hence, every function in the set $\{f_m : m = 1, \ldots, n\}$ is a $\gamma^t_{sR}(C_n)$-function. The set of n identities $\{w(f_m) = \gamma^t_{sR}(C_n)\}_{m=1}^n$ can be expressed as a linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

$$\mathbf{x} = (\mu_s(u_1), \ldots, \mu_s(u_n))^T, \quad \mathbf{b} = (\gamma^t_{sR}(C_n), \ldots, \gamma^t_{sR}(C_n))^T,$$

and $\mathbf{A}$ is a circulant matrix whose rows correspond to the labels $\{0, 1, 2\}$ assigned by each f_m. According to the Rouché-Frobenius Theorem, this system has a unique solution provided that $\operatorname{rank}(\mathbf{A}) = n$ (i.e., $\mathbf{A}$ is non-singular), a condition that holds for all cases except $n = 3k + 2$ with k even, which we treat separately. Since the sum of the entries in each row of $\mathbf{A}$ is constant and equals n, and $\mathbf{b}$ is a constant vector, the unique solution is $\mu_s(u_i) = \frac{1}{n}\gamma^t_{sR}(C_n)$ for all $i = 1, \ldots, n$.
For the case $n = 3k + 2$ with k even, the rank of $\mathbf{A}$ is $3k + 1$, yielding infinitely many solutions of the form

$$\mathbf{x} = \left(\frac{2}{n}\gamma^t_{sR}(C_n) - \lambda, \lambda, \ldots, \frac{2}{n}\gamma^t_{sR}(C_n) - \lambda, \lambda\right)^T, \quad \lambda \in \mathbb{R}.$$

Since C_n is a fuzzy cycle, λ must satisfy $\lambda \in (0, \frac{2}{n}\gamma^t_{sR}(C_n))$. We now show that $\lambda = \frac{1}{n}\gamma^t_{sR}(C_n)$. If all the values $\mu(u_j, u_{j+1})$ are equal, then we are done. Otherwise, $\mathbf{x} = (\alpha, \beta, \ldots, \alpha, \beta)^T$ for distinct values α, β. Suppose without loss of generality that $\alpha < \beta$ and that $\mu(u_1, u_2) < \mu(u_2, u_3)$, which implies that $\mu_s(u_2) = \beta = \mu(u_1, u_2)$.

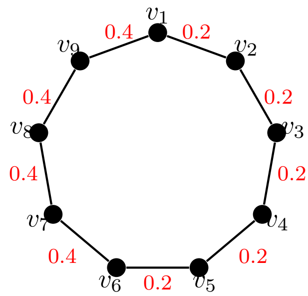

Figure 4. **A fuzzy cycle C_9 where the equality in Theorem 6 holds.**

Since $\mu_s(u_3) = \alpha = \min\{\mu(u_2, u_3), \mu(u_3, u_4)\} < \beta$, it must be $\alpha = \mu(u_3, u_4)$ which contradicts the fact that $\mu_s(u_4) = \beta$ because $\alpha < \beta$.
This confirms that even when $n = 3k + 2$ with k even, the only valid solution for the fuzzy cycle is $\mu_s(u_i) = \frac{1}{n}\gamma^t_{sR}(C_n)$ for all i.
Thus, we have proven that $\mu_s(u_i) = \frac{1}{n}\gamma^t_{sR}(C_n)$ for all i, which implies

$$\gamma^t_{sR}(C_n) = n\mu_{\min}. \tag{4.4}$$

By substituting the hypothesis $\gamma^t_{sR}(C_n) = q - \mu_{\max} + \mu_{\min}$ into (4.4), we obtain $q - \mu_{\max} + \mu_{\min} = n\mu_{\min}$, leading to $q = (n-1)\mu_{\min} + \mu_{\max}$. Given that C_n has n strong edges, this confirms that at least $n-1$ edges have weight $\mu_{\min} = \frac{1}{n}\gamma^t_{sR}(C_n)$ and at most one edge has weight $\mu_{\max} \geq \mu_{\min}$. This completes the proof. □

The reasoning employed in the proof of Theorem 7 can be extended to the case $n = 3k$ to show that $\gamma^t_{sR}(C_n) = q - (\mu_{\max} - \mu_{\min})$ does not uniquely imply that at least $n-1$ edges have a membership value of $\mu_{\min}$ and at most one edge has a value of $\mu_{\max}$. This is because the circulant matrix associated with the linear system $\{w(f_m) = \gamma^t_{sR}(C_n)\}_{m=1}^n$, for $n = 3k$ (with $n > 3$), has rank equal to 3. As a consequence, in the case $n = 3k$, there are infinitely many fuzzy cycles for which the bound in Theorem 6 is attained. Figure 4 illustrates this by showing a fuzzy cycle C_9 with size $q = 2.6$, $\mu_{\max} = 0.4$, and $\mu_{\min} = 0.2$. A $\gamma^t_{sR}(C_9)$-function $f : V \to \{0, 1, 2\}$ is defined by $f(v_1) = f(v_4) = f(v_7) = 0$, $f(v_2) = f(v_5) = f(v_8) = 2$, and $f(v_3) = f(v_6) = f(v_9) = 1$. The strong total Roman domination number of $C_9 = (V, \sigma, \mu)$ is $\gamma^t_{sR}(C_9) = 2.4 = q - \mu_{\max} + \mu_{\min}$. Notably, this fuzzy cycle contains more than one edge with a membership value equal to $\mu_{\max}$.

5. Concluding remarks.

Research on crisp graphs is extensive, but its application to real-world networks is often limited because points of interest in real-world structures usually have varying degrees of relevance, which are more accurately modeled by fuzzy graphs.

This study introduces the concept of strong Roman domination in fuzzy graphs. Specifically, we have established several bounds for this parameter in terms of other fuzzy graph invariants such as the strong domination and the strong total domination numbers.
Furthermore, we have determined the exact values of γ_{sR}^{t} for specific families of fuzzy graphs, including complete fuzzy graphs, complete bipartite fuzzy graphs, fuzzy cycles, and fuzzy graphs containing at least one universal vertex.
As an application of the proposed theory, we consider the problem of emergency response network planning under uncertainty. The strong total Roman domination model provides a natural optimization framework for determining the location and operational status of emergency facilities while accounting for uncertain connectivity and the need for mutual support among response centers.
Municipal emergency networks must guarantee rapid response coverage across all stations despite uncertain communication quality due to congestion, partial outages, or distance. We model the network as a fuzzy graph $G = (V, \sigma, \mu)$ where:

- each vertex $v \in V$ represents an emergency station (hospital, fire post, police unit, etc.) with $\sigma(v) \in [0, 1]$ representing its operational reliability;
- each edge (u, v) models a communication/response link with $\mu(u, v) \in [0, 1]$ indicating its reliability, which is bounded by $\min\{\sigma(u), \sigma(v)\}$;
- an edge is *strong* if it represents the most reliable connection between its endpoints, i.e., no indirect route achieves higher connectivity.

A STRDF $f : V \to \{0, 1, 2\}$ assigns roles as follows: $f(v) = 2$ to Full Response Unit (FRU); $f(v) = 1$ to Support Unit (SU); and $f(v) = 0$ to monitoring post. The STRDF conditions ensure that:

1. every monitoring post ($f = 0$) is reachable from a FRU via strong communication,
2. every active station ($f > 0$) has at least one other active neighbor, preventing isolated command posts.

The weight $w(f) = \sum_v f(v) \cdot \mu_s(v)$ minimizes the total *deployment cost*, where $\mu_s(v)$ reflects the weakest reliable link incident to v, that is, a proxy for communication load.
Consider a district with five emergency stations: Hospital (H), Fire Station (F), Police HQ (P), Ambulance Center (A), and Guard Post (G), with membership values:

$$\sigma(H) = 0.9,\ \sigma(F) = 0.8,\ \sigma(P) = 0.7,\ \sigma(A) = 0.8,\ \sigma(G) = 0.6,$$

and links forming a cycle H-F-P-G-A-H with weights $\mu(H, F) = 0.8$, $\mu(F, P) = 0.6$, $\mu(P, G) = 0.5$, $\mu(G, A) = 0.6$, $\mu(A, H) = 0.8$ (Figure 5). The strongest path from P to G not using the direct link has strength $\min\{0.6, 0.8, 0.8, 0.6\} = 0.6 > 0.5$, so (P, G)

is not strong. All other edges are strong. The strong graph is the path P-F-H-A-G with

$$\mu_s(H) = 0.8, \quad \mu_s(F) = \mu_s(A) = \mu_s(P) = \mu_s(G) = 0.6.$$

The function $f(F) = 2$, $f(A) = 2$, $f(H) = 1$, $f(P) = f(G) = 0$ is a STRDF of weight $w(f) = 2{\cdot}0.6 + 2{\cdot}0.6 + 1{\cdot}0.8 = 3.2$. One can readily verify that P is dominated by F (FRU); G is dominated by A (FRU); the support $\{F, H, A\}$ forms a connected path F-H-A in the strong graph, meaning that no isolated active station exists. This is the optimal STRDF, and hence $\gamma^t_{sR}(G) = 3.2$.
The optimal deployment assigns FRUs to the Fire Station and Ambulance Center, with the Hospital serving as a Support Unit bridging them. The Police HQ and Guard Post operate as monitoring posts, each reliably served by a nearby FRU. This solution minimizes the total operational cost while guaranteeing that every station is within one strong communication hop of full emergency support, and that no command post becomes an isolated link in the response chain. The fuzzy model naturally captures the inherent uncertainty in real communication reliability, providing robust guarantees even under degraded link conditions.

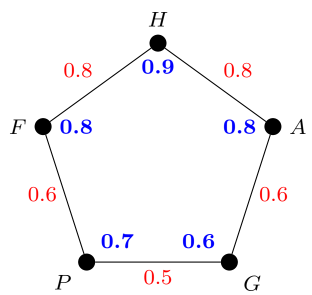

Figure 5. Emergency network of the application.

As a continuation of this research, we suggest the following lines of investigation:

- To establish new lower bounds for γ^t_{sR} in terms of the minimum strong degree.
- To explore further refinements of the general bounds presented in this study.
- To obtain exact values for other specific families of fuzzy graphs.
- To study the complexity of the associated decision problem.

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