

On graphs having proper $(1, k)$ -dominating sets

Mostafa Blidia^{1,*}, Ahmed Bouchou^{1,2}

¹LAMDA-RO Laboratory, Department of Mathematics, University of Blida,
B.P. 270, Blida, Algeria
*m_blidia@yahoo.fr

²University of Médéa, Algeria
ahmedbouchou14@gmail.com

Received: 21 November 2025; Accepted: 1 July 2026

Published Online: 3 July 2026

Dedicated to Odile Favaron

Abstract: For an integer $k \geq 1$, a $(1, k)$ -dominating set (or $(1, k)$ -dset for short) of a graph $G = (V, E)$ is a set S of vertices having the property that for every vertex $v \in V - S$, there is at least one vertex in S within distance 1 from v and a second vertex in S within distance at most k from v . For $k \geq 2$, a $(1, k)$ -dset D of a graph G is a proper $(1, k)$ -dset if it is not a $(1, k - 1)$ -dset. More precisely, D is a proper $(1, k)$ -dset (or $(1, \bar{k})$ -dset for short) if it is $(1, k)$ -dset and there is at least a vertex $u \in V - D$ for which there are two distinct vertices t and z in D such that t is adjacent to u, z and u are at distance exactly k apart from each other and no vertex in $D - \{t\}$ is at distance less than k from u . The $(1, k)$ -domination number $\gamma_{1,k}(G)$ (resp. $(1, \bar{k})$ -domination number $\gamma_{1,\bar{k}}(G)$) of G is the minimum cardinality among all $(1, k)$ -dsets (resp. $(1, \bar{k})$ -dsets) of G . In this paper, we are interested in the study of $(1, k)$ -dsets as well as their existence. We start by giving a necessary and sufficient condition for graphs having $(1, \bar{k})$ -dsets for $k \in \{3, 4\}$. Then we consider triangle-free graphs G with equal $\gamma_{1,k}(G)$ and $\gamma_{1,\bar{k}}(G)$ where characterizations are given when $k \in \{3, 4\}$. Finally, we show that the decision problems associated with $(1, 3)$ -domination, $(1, \bar{3})$ -domination and $(1, \bar{4})$ -domination are \mathcal{NP} -complete for bipartite graphs.

Keywords: $(1, k)$ -domination, $(1, \bar{k})$ -domination.

AMS Subject Classification: 05C69

1. Introduction

Let G be a simple graph with vertex set $V(G) = V$ and edge set $E(G) = E$. The *open neighborhood* of a vertex $v \in V$ is the set $N_G(v) = \{u \in V : uv \in E\}$ and the *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v of G is

* Corresponding Author

$d_G(v) = |N_G(v)|$. An *isolated vertex* is a vertex with degree zero. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A *universal vertex* in G is a vertex that is adjacent to all the other vertices in G . The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest (u, v) -path in G while the *diameter*, $\text{diam}(G)$, of G is the maximum distance among all pairs of vertices in G . The set of vertices that are within distance k from a vertex $v \in V$ is denoted by $N_G^k[v]$, that is $N_G^k[v] = \{x \in V : d_G(v, x) \leq k\}$. For any integer $i = 0, 1, \dots, k$, we define $A_G^i(v)$ as the set of all vertices that are at distance exactly i from v , that is $A_G^i(v) = \{x \in V : d_G(v, x) = i\}$. Hence $N_G^k[v] = \bigcup_{i=0}^k A_G^i(v)$, where $A_G^0(v) = \{v\}$. For a set $X \subseteq V(G)$, we denote the subgraph induced by X by $G[X]$. As usual, the *path* (*cycle*, *complete graph*, *star*, respectively) of order n is denoted by P_n (C_n , K_n , $K_{1, n-1}$, respectively). Let H be a graph. A graph G is said to be H -free if it has no induced subgraph isomorphic to H . In particular, a K_3 -free graph will be simply called a *triangle-free* graph. A *tree* is a connected graph with no cycle. A tree T is a *double star*, if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support is denoted $S_{p, q}$. Also a graph is *bipartite* if its vertex set can be partitioned into two independent sets.

A subset D of V is a *dominating set* of G if every vertex is either in D or has a neighbor in D . The cardinality of a minimum dominating set of G is the *domination number* of G , denoted by $\gamma(G)$.

In [7], Hedetniemi et al. generalized the concept of domination to $(1, k)$ -domination, by defining a subset $D \subseteq V$ in a graph $G = (V, E)$ to be a $(1, k)$ -*dominating set* (abbreviated, $(1, k)$ -dset) if for every vertex $v \in V - D$ there are two distinct vertices $u, w \in D$ such that u is adjacent to v , and w is within distance k from v , that is $d_G(v, w) \leq k$. In this case the vertex v is said to be $(1, k)$ -dominated by D . The $(1, k)$ -*domination number* $\gamma_{1, k}(G)$ of G is the minimum cardinality of a $(1, k)$ -dominating set in G . A $(1, k)$ -dset of cardinality $\gamma_{1, k}(G)$ is called $\gamma_{1, k}(G)$ -set. It is worth noting that for $k = 1$, $\gamma_{1, k}(G)$ is the 2-domination number $\gamma_2(G)$ introduced by Fink and Jacobson [6].

It is essential to note that the study of $(1, k)$ -dssets in their general way remains open and only particular cases have been considered. Indeed, for $k = 2$, $(1, k)$ -dssets were studied by Hedetniemi et al. in [7], and also in [10, 12, 13]. Moreover, the independent version of $(1, k)$ -dssets have been considered in [7] for $k = 2$.

Hedetniemi et al. [7] presented the following general result about arbitrary dominating sets in connected graphs.

Theorem 1 ([7]). *Every dominating set in a connected graph G with $\gamma(G) \geq 2$ is a $(1, 4)$ -dset.*

The following observation is obvious.

Observation 2. Let G be a connected graph with $\gamma(G) \geq 2$, and let k be an integer with $1 \leq k \leq 4$. Then every $(1, j)$ -dominating set is a $(1, k)$ -dominating set for $j \leq k$, and thus

$$\gamma(G) = \gamma_{1,4}(G) \leq \gamma_{1,3}(G) \leq \gamma_{1,2}(G) \leq \gamma_{1,1}(G) = \gamma_2(G) \quad (1.1)$$

Later, in [9], Michalski introduced the concept of proper $(1, 2)$ -dominating sets. A $(1, 2)$ -dominating set D is a *proper* $(1, 2)$ -dominating set (or simply $(1, \bar{2})$ -dset) of a graph G if D is a $(1, 2)$ -dset but it is not a $(1, 1)$ -dset. In other words, D is a proper $(1, 2)$ -dominating set if it is $(1, 2)$ -dominating and there exists a vertex $x \in V - D$ such that x has exactly one adjacent vertex in D . The cardinality of a minimum $(1, \bar{2})$ -dset in a graph G is the *proper* $(1, 2)$ -domination number $\gamma_{1, \bar{2}}(G)$ of G .

In [11], Michalski et al. studied some properties of $(1, \bar{2})$ -dsets. In particular a necessary and sufficient condition for the existence of such sets in a connected graph was given. In [8], Kosiorowska et al. studied intersections of the $(1, 1)$ -sets and $(1, \bar{2})$ -dsets, where they introduced the $(1, \bar{2})$ -intersection index defined as the minimum possible cardinality of such intersections. The authors [8] determined the exact value of $(1, \bar{2})$ -intersection index for some classes of graphs.

We generalize the concept of proper $(1, 2)$ -dominating sets to proper $(1, k)$ -dominating sets. For $k \geq 2$, a $(1, k)$ -dset D of a graph G is a *proper* $(1, k)$ -dset if it is not a $(1, k - 1)$ -dset. In other words, D is a proper $(1, k)$ -dset, abbreviated $(1, \bar{k})$ -dset, if it is $(1, k)$ -dset and there is at least a vertex $u \in V - D$ for which there are two distinct vertices t and z in D such that t is adjacent to u , $d_G(u, z) = k$ and no vertex in $D - \{t\}$ is at distance less than k from u . In this case the vertex u is said to be $(1, \bar{k})$ -dominated by D . The cardinality of a minimum $(1, \bar{k})$ -dset in a graph G is the *proper* $(1, k)$ -domination number $\gamma_{1, \bar{k}}(G)$ of G . A $(1, \bar{k})$ -dset of cardinality $\gamma_{1, \bar{k}}(G)$ is called $\gamma_{1, \bar{k}}(G)$ -set. Since every $(1, \bar{k})$ -dset of G is a $(1, k)$ -dset of G , we have $\gamma_{1, k}(G) \leq \gamma_{1, \bar{k}}(G)$. Moreover, by Theorem 1, the following result is provided.

Corollary 1. *If G is a connected graph having a $(1, \bar{k})$ -dset, then $k \leq 4$.*

From this Corollary, we deduce that a proper $(1, k)$ -dominating set does not exist, for $k \geq 5$.

It is worth mentioning that an inequality chain similar to (1.1) cannot apply to $\gamma_{1, \bar{k}}(G)$, as can be seen by considering the tree $S_{p, q}^*$ obtained from a double star $S_{p, q}$, with $p \leq q$, by subdividing once the edge joining the support vertices of $S_{p, q}$. One can see that $\gamma_{1, \bar{4}}(S_{p, q}^*) = p + 1$, $\gamma_{1, \bar{3}}(S_{p, q}^*) = 2$ and $\gamma_{1, \bar{2}}(S_{p, q}^*) = 3$.

In this paper, we are interested in the study and the existence of $(1, \bar{k})$ -dsets. We start by giving a characterization of connected graphs having such sets for $k \in \{3, 4\}$. Then we study connected triangle-free graphs G with $\gamma_{1, k}(G) = \gamma_{1, \bar{k}}(G)$ for $k \in \{3, 4\}$. Finally, we show that decision problem corresponding to the problem of computing $\gamma_{1, k}(G)$ and $\gamma_{1, \bar{k}}(G)$ are \mathcal{NP} -complete even when restricted to bipartite graphs.

2. Graphs having $(1, \bar{k})$ -dsets

First, let us recall a result of Michalski et al. [11] providing a characterization of connected graphs having a $(1, \bar{2})$ -dset.

Theorem 3 ([11]). *A connected graph G has a $(1, \bar{2})$ -dset if and only if G is not a complete graph.*

In what follows we investigate the cases $k = 3$ and $k = 4$ for connected graphs.

Theorem 4. *Let G be a connected graph. Then G has a $(1, \bar{3})$ -dset if and only if $\text{diam}(G) \geq 3$ and there exist two adjacent vertices t and u such that $N_G(u) \subset N_G[t]$ and for all $y \in A_G^2(u)$ we have $N_G(y) \cap A_G^3(u) = \emptyset$, $y \in N_G(t)$.*

Proof. Assume that G has a $(1, \bar{3})$ -dset D . By the definition of a $(1, \bar{3})$ -dset, D is a $(1, 3)$ -set of G for which there exist a vertex $u \in V - D$ and two vertices t and z in D such that $d_G(u, t) = 1$, $d_G(u, z) = 3$ and no vertex $y \in D - \{t\}$ satisfies $d_G(u, y) \leq 2$. Clearly $\text{diam}(G) \geq 3$ and the vertices u and t are adjacent. Moreover, since D is a dominating set, every neighbor y of u must be adjacent to t , for otherwise some neighbor of y in D will be at distance 2 from u , contradicting the definition of $(1, 3)$ -sets. Hence $N_G(u) \subset N_G[t]$. Now let $y \in A_G^2(u)$ and $N_G(y) \cap A_G^3(u) = \emptyset$. Since D is a dominating set of G and there is no vertex $y \in D - \{t\}$ such that $d_G(u, y) = 2$, $y \in N_G(t)$.

Conversely, assume that $\text{diam}(G) \geq 3$, and there exist two adjacent vertices t and u such that $N_G(u) \subset N_G[t]$ where for every $y \in A_G^2(u)$ such that $N_G(y) \cap A_G^3(u) = \emptyset$, $y \in N_G(t)$. Then $A_G^3(u) \neq \emptyset$, otherwise t is a universal vertex; meaning that $\text{diam}(G) \leq 2$ or G is not connected, a contradiction. Now let $u-y_1-y_2-z$ be a shortest path between u and z such that $y_1 \in A_G^1(u)$, $y_2 \in A_G^2(u)$ and $z \in A_G^3(u)$. We shall show that either $D = (V - N_G^2[u]) \cup \{t\}$ or $D' = D \cup \{y_2\}$, is a $(1, \bar{3})$ -dset of G . Note that $A_G^3(u) \subset D$. We distinguish between three cases.

Case 1. Every vertex in $A_G^1(u) - \{t\}$ has a neighbor in $A_G^2(u)$ and every vertex in $A_G^2(u)$ has a neighbor in $A_G^3(u)$. Then u is adjacent to t and since $z \in A_G^3(u)$, u is $(1, \bar{3})$ -dominated by D . Moreover, since $N_G(u) \subset N_G[t]$, every vertex y in $A_G^1(u) - \{t\}$ is adjacent to t and there exists $z' \in A_G^3(u)$ such that $d_G(y, z') \leq 2$. So, y is $(1, 3)$ -dominated by D . Also, since $N_G^2[u] - \{t\} \subset V - D$ and $A_G^3(u) \subset D$, every vertex y' in $A_G^2(u) - \{t\}$ has a neighbor in $A_G^3(u)$ and since $d_G(y', t) \leq 2$, y' is $(1, 3)$ -dominated by D . Hence D is a $(1, \bar{3})$ -dset of G .

Case 2. There exists a vertex $y \in A_G^2(u)$ such that $N_G(y) \cap A_G^3(u) = \emptyset$, $y \in N_G(t)$ and y is not $(1, 3)$ -dominated by D . Then $yy_1, yy_2, y_2t \notin E$. Moreover $y'y_2 \notin E$ for every vertex $y' \in A_G^1(u) \cap N_G(y)$. We will show that $D' = D \cup \{y_2\}$ is a $(1, \bar{3})$ -dset of G . Clearly $y \in N_G(t)$ and the path $y-t-y_1-y_2$ is a shortest path between y and y_2 . Hence y is $(1, \bar{3})$ -dominated by D' . Thus, it remains to show that every vertex x of $(V - D') - \{u\}$ is $(1, 3)$ -dominated by D' . Note that $N_G[y] \subset N_G^2(u)$. If

$x \in A_G^1(u) - \{t\}$, then x is adjacent to t , because $N_G(u) \subset N_G[t]$, and it is easy to see that $d_G(x, y_2) \leq 3$. Now assume that $x \in A_G^2(u)$. Then either x has no neighbor in $A_G^3(u)$, and thus $x \in N_G(t)$ and $d_G(x, y_2) \leq 3$, or x has a neighbor in $A_G^3(u)$, and thus x has a neighbor in $A_G^3(u)$ and $d_G(x, t) \leq 2$. In both cases, x is $(1, 3)$ -dominated by D' .

Case 3. There exists a vertex $y \in A_G^1(u) - \{t\}$ such that y has no neighbor in $A_G^2(u)$ and y is not $(1, 3)$ -dominated by D . Then $yy_1, yy_2, y_2t \notin E$. In this case, the same argument applied in Case 2 shows that $D' = D \cup \{y_2\}$ is a $(1, \bar{3})$ -dset of G , and this completes the proof. \square

Theorem 5. *Let G be a connected graph. Then G has a $(1, \bar{4})$ -dset if and only if $\text{diam}(G) \geq 4$, there exist two adjacent vertices t and u such that $N_G^2(u) \subset N_G[t]$ and for all $y \in A_G^3(u)$, $N_G(y) \cap A_G^4(u) \neq \emptyset$.*

Proof. Assume that G contains a $(1, \bar{4})$ -dset, say D . By the definition of a $(1, \bar{4})$ -dset, D is a $(1, 4)$ -dset of G , there exists a vertex $u \in V - D$ and there exist two distinct vertices t and $z \in D$ such that $N_G(u) \cap D = \{t\}$, $d_G(u, z) = 4$ and there is no vertex $z' \in D - \{t\}$ such that $d_G(u, z') \leq 3$. Therefore, $\text{diam}(G) \geq 4$, u and t are adjacent and $N_G[y] \cap (D - \{t\}) = \emptyset$, for all $y \in N_G^2[u] - \{t\}$. Moreover, since D is a dominating set of G , $N_G[y] \cap D = \{t\}$, for all $y \in N_G^2[u] - \{t\}$. Hence $N_G^2(u) \subset N_G[t]$. Now let $y \in A_G^3(u)$. Clearly $y \in V - D$, and y is not adjacent to t , for otherwise $y \in A_G^2(u)$. Since D is a dominating set of G , $N_G(y) \cap D \neq \emptyset$, and thus we have $N_G(y) \cap A_G^4(u) \neq \emptyset$ for all $y \in A_G^3(u)$.

Conversely, assume that $\text{diam}(G) \geq 4$, and there exist two adjacent vertices t and u such that $N_G^2(u) \subset N_G[t]$, where for every $y \in A_G^3(u)$, $N_G(y) \cap A_G^4(u) \neq \emptyset$. Then $A_G^4(u) \neq \emptyset$, otherwise t is a universal vertex; meaning that $\text{diam}(G) \leq 2$ or G is not connected, a contradiction. We shall show that the set $D = (V - N_G^3[u]) \cup \{t\}$ is a $(1, \bar{4})$ -dset of G . First we prove that u is $(1, \bar{4})$ -dominated by D . Clearly u is adjacent to t , and since $A_G^4(u) \subset D$ and $N_G(y) \cap A_G^4(u) \neq \emptyset$ for every $y \in A_G^3(u)$, $N_G(y) \cap A_G^4(u) \subset D$. Hence, u is $(1, \bar{4})$ -dominated by D . Now, it remains to show that every vertex y of $N_G^3(u) - \{t\}$ is $(1, 4)$ -dominated by D . Observe that $N_G^3(u) - \{t\} = (N_G^2(u) - \{t\}) \cup A_G^3(u)$. Now, let $u-y_1-y_2-y_3-z$ be a shortest path such that $y_i \in A_G^i(u)$ for $i = 1, 2, 3$ and $z \in A_G^4(u)$. Since $N_G^2(u) \subset N_G[t]$, $y_i \in N_G(t)$ for $i = 1, 2$. Moreover $y_3 \notin N_G(t)$, for otherwise $y_3 \in A_G^2(u)$. Now, if $y \in N_G^2(u) - \{t\}$, then obviously $N_G(y) \cap D = \{t\}$. Moreover, we have the path $y-t-y_2-y_3-z$ with $z \in D$ and $yz \notin E$. Hence $2 \leq d_G(y, z) \leq 4$, which means that y is $(1, 4)$ -dominated by D . Finally, assume that $y \in A_G^3(u)$. Then $N_G(y) \cap A_G^4(u) \neq \emptyset$ and $d_G(y, t) = 2$. Therefore there exists a vertex $w \in A_G^4(u) \cap D$ such that $yw \in E$. Hence y is $(1, 4)$ -dominated by D , and this completes the proof. \square

Since a cycle C_n of order n has no vertices t and u satisfying conditions in Theorems 4 and 5, we deduce the following result.

Corollary 2. *Let C_n be a cycle of order n . Then C_n has neither a $(1, \bar{3})$ -dset nor a $(1, \bar{4})$ -dset.*

Restricted to the class of triangle-free graphs, we have the following necessary condition for the existence of $(1, \bar{k})$ -dsets for $k \in \{3, 4\}$.

Proposition 1. *Let $k \in \{3, 4\}$ and let G be a connected triangle-free graph with $\text{diam}(G) \geq k$. If G has a $(1, \bar{k})$ -dset D , then every vertex which is $(1, \bar{k})$ -dominated by D is a leaf.*

Proof. Assume that G has a $(1, \bar{k})$ -dset D , and let u be a vertex of $V - D$ which is $(1, \bar{k})$ -dominated by D . Then there exist two distinct vertices $t, z \in D$ such that $N_G(u) \cap D = \{t\}$, $d_G(u, z) = k$ and no vertex $y \in D - \{t\}$ is at distance at most $k - 1$ from u . Thus, if u is not a leaf, then u has another neighbor v which belongs to $V - D$. Since D is a dominating set and $d_G(u, y) \geq k$ for every vertex $y \in D - \{t\}$, v has to be adjacent to t . Hence $\{u, t, v\}$ induces a triangle, a contradiction. \square

The following corollaries are easily deduced from Proposition 1.

Corollary 3. *Let $k \in \{3, 4\}$ and let G be a connected bipartite graph with $\text{diam}(G) \geq k$. If G has a $(1, \bar{k})$ -dset D , then every vertex that is $(1, \bar{k})$ -dominated by D is a leaf.*

Corollary 4. *Let $k \in \{3, 4\}$ and let T be a tree with $\text{diam}(T) \geq k$. If T has a $(1, \bar{k})$ -dset D , then every vertex that is $(1, \bar{k})$ -dominated by D is a leaf.*

Corollary 5. *If G is a connected triangle-free graph and without leaf, then G has neither a $(1, \bar{3})$ -dset nor a $(1, \bar{4})$ -dset.*

Therefore, it is interesting to study triangle-free graphs G with minimum degree one having $(1, \bar{k})$ -dsets for $k \in \{3, 4\}$. For this purpose, we begin by providing a necessary and sufficient condition for the existence of $(1, \bar{3})$ -dsets, for connected triangle-free graph with $\text{diam}(G) \geq 3$.

Theorem 6. *Let G be a connected triangle-free graph with $\text{diam}(G) \geq 3$. Then G has a $(1, \bar{3})$ -dset if and only if G contains a leaf.*

Proof. The necessary condition follows from Proposition 1. For the sufficient condition, let u be a leaf of G and let t its support vertex. Since $\text{diam}(G) \geq 3$, u and t satisfy the condition of Theorem 4. Hence G has a $(1, \bar{3})$ -dset. \square

Corollary 6. *Let G be a bipartite graph with $\text{diam}(G) \geq 3$. Then G has a $(1, \bar{3})$ -dset if and only if G contains a leaf.*

Corollary 7. *If T is a tree with $\text{diam}(T) \geq 3$, then T has a $(1, \bar{3})$ -dset.*

It is worth noting that a connected triangle-free graph G with $\text{diam}(G) \geq 4$ containing a leaf may not have a $(1, \bar{4})$ -dset. This can be seen by considering the tree T obtained from two copies of P_4 by adding a new vertex attached to exactly one support vertex of each copy of P_4 . It is easy to see that T does not have a $(1, \bar{4})$ -dset.

The next result provides a necessary and sufficient condition for the existence of $(1, \bar{4})$ -dsets, for trees T with $\text{diam}(T) \geq 4$.

Theorem 7. *Let T be a tree with $\text{diam}(T) \geq 4$. Then T has a $(1, \bar{4})$ -dset if and only if T contains a leaf u such that $d_T(u, y) \neq 3$ for every leaf y of T .*

Proof. Assume that T has a $(1, \bar{4})$ -dset. Suppose, for a contradiction, that for every leaf u there exists a leaf y such that $d_T(u, y) = 3$. If t denotes the support vertex adjacent to u , then u and t do not satisfy the condition of Theorem 5, since $y \in A_T^3(u)$, $N_T(y) \cap A_T^4(u) = \emptyset$ and $y \notin N_T(t)$, a contradiction.

Conversely, assume that T contains a leaf u such that $d_T(u, y) \neq 3$ for every leaf y of T , and let t be the support vertex of u . Clearly $N_T^2(u) \subset N_T[t]$ and for all $y \in A_T^3(u)$, $N_T(y) \cap A_T^4(u) \neq \emptyset$, which means that u and t satisfy the condition of Theorem 5. Hence T has a $(1, \bar{4})$ -dset. □

As an example of graph verifying the conditions of Theorem 6 (Theorem 7, respectively), we consider the tree T obtained from the star $K_{1,p}$ where $p \geq 2$, by subdividing all its edges once. Then T has a $(1, \bar{3})$ -dset (resp., a $(1, \bar{4})$ -dset). Even though any $(1, \bar{3})$ -dset is different from every $(1, \bar{4})$ -dset, we still have equality $\gamma_{1, \bar{3}}(T) = \gamma_{1, \bar{4}}(T) = p$.

3. Graphs G with $\gamma_{1,k}(G) = \gamma_{1,\bar{k}}(G)$

In this section we consider relationships between the proper $(1, k)$ -domination and $(1, k)$ -domination numbers of a graph G . As mentioned above, $\gamma_{1,k}(G) \leq \gamma_{1,\bar{k}}(G)$ for every connected graph with $\text{diam}(G) \geq k$. In [11], Michalski et al. showed that for $k = 2$ the equality holds in the previous inequality for any connected noncomplete graph.

Theorem 8 ([11]). *For an arbitrary connected noncomplete graph G , we have*

$$\gamma_{1,\bar{2}}(G) = \gamma_{1,2}(G).$$

However, for $k = \{3, 4\}$, inequality $\gamma_{1,k}(G) \leq \gamma_{1,\bar{k}}(G)$ can be strict, as can be seen by the graphs in Figure 1. Indeed, for the graph G_1 , the set $\{t, a_2\}$ is a $\gamma_{1,3}$ -set which is

not a $\gamma_{1,3}$ -set, while the set $\{t, a'_3, a''_3\}$ is a $\gamma_{1,3}$ -set. Also, for the graph G_2 , the set $\{t, a_3\}$ is a $\gamma_{1,4}$ -set which is not a $\gamma_{1,4}$ -set, while the set $\{t, a'_4, a''_4\}$ is a $\gamma_{1,4}$ -set.

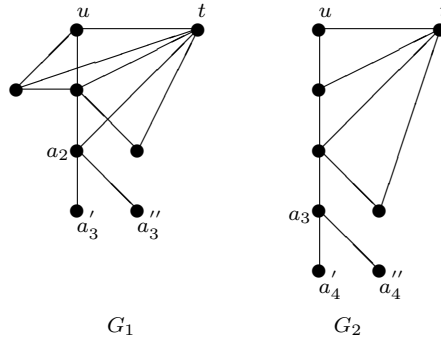


Figure 1. Graphs with $\gamma_{1,3}(G_1) < \gamma_{1,3}(G_1)$ and $\gamma_{1,4}(G_2) < \gamma_{1,4}(G_2)$.

As we have seen, by Corollary 5, connected triangle-free graphs with no leaf have no $\gamma_{1,k}$ -set for $k \in \{3, 4\}$. Accordingly, it is interesting to study equality between the proper $(1, k)$ -domination and $(1, k)$ -domination numbers for connected triangle-free graphs having a leaf. For this purpose, we begin by paths to show that $\gamma_{1,k}(P_n) = \gamma_{1,k}(P_n)$ for $k \in \{3, 4\}$. It is well-known that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. On the other hand, Raczek [13] proved the following result, for $\gamma_{1,2}(P_n)$.

Theorem 9 ([13]). For any integer $n \geq 2$, we have $\gamma_{1,2}(P_n) = \lceil \frac{n+2}{3} \rceil$.

The next result provides the exact value of the $(1, 3)$ -domination number for paths.

Theorem 10. For any integer $n \geq 2$, we have $\gamma_{1,3}(P_n) = \lceil \frac{n+2}{3} \rceil$.

Proof. Let P_n be a path of order $n \geq 2$, with vertices labeled in order x_1, x_2, \dots, x_n . Since the result is obvious for $n = 2$, let $n \geq 3$ and consider the following cases.

Case 1. $n = 3p$ for some integer $p \geq 1$. It is easy to see that the only minimum dominating set is $D = \{x_2, x_5, \dots, x_{3p-1}\}$. But D is not $(1, 3)$ -dset, since x_1 is not $(1, 3)$ -dominated by D . It means that $\gamma_{1,3}(P_n) > \gamma(P_n) = \frac{n}{3} = p$. Moreover, the set $D_1 = \{x_1, x_4, \dots, x_{3p-5}\} \cup \{x_{3p-3}, x_{3p-1}\}$ is a $(1, 3)$ -dset of cardinality $p + 1 = \frac{n+3}{3}$. Hence $\gamma_{1,3}(P_n) = \frac{n+3}{3} = \lceil \frac{n+2}{3} \rceil$.

Case 2. $n = 3p + 1$ for some integer $p \geq 1$. It is easy to see that the set $D_2 = \{x_1, x_4, \dots, x_{3p-2}\} \cup \{x_{3p}\}$ is a $(1, 3)$ -dset of cardinality $p + 1 = \frac{n+2}{3}$. Since $\gamma_{1,3}(P_n) \geq \gamma(P_n) = \frac{n+2}{3}$, we have $\gamma_{1,3}(P_n) = \frac{n+2}{3} = \lceil \frac{n+2}{3} \rceil$.

Case 3. $n = 3p + 2$ for some integer $p \geq 1$. It is easy to check that every minimum dominating set of P_n is not a $(1, 3)$ -dset leading to $\gamma_{1,3}(P_n) > \gamma(P_n) = \frac{n+1}{3} = p + 1$.

Moreover, the set $D_3 = \{x_1, x_4, \dots, x_{3p+1}\} \cup \{x_{3p-1}\}$ is a $(1, 3)$ -dset of cardinality $p + 2$. Hence $\gamma_{1,3}(P_n) = \frac{n+4}{3} = \lceil \frac{n+2}{3} \rceil$. \square

Now, we are ready to show that $\gamma_{1,k}(P_n) = \gamma_{1,\bar{k}}(P_n)$ for $k \in \{3, 4\}$.

Proposition 2. *For any integer $n \geq 4$, we have $\gamma_{1,3}(P_n) = \gamma_{1,\bar{3}}(P_n)$.*

Proof. Let P_n be a path of order $n \geq 4$. Since $\gamma_{1,3}(P_n) \leq \gamma_{1,\bar{3}}(P_n)$, it is enough to show that P_n contains a $\gamma_{1,3}$ -set which is a $(1, \bar{3})$ -dset. For that, it is easy to see, according to the values of n , that the sets D_1, D_2 and D_3 given in the proof of Theorem 10 are also $(1, \bar{3})$ -dsets of P_n , which completes the proof. \square

Proposition 3. *For any integer $n \geq 5$, we have $\gamma_{1,4}(P_n) = \gamma_{1,\bar{4}}(P_n)$.*

Proof. Let P_n be a path of order $n \geq 5$. Since $\gamma(G) = \gamma_{1,4}(P_n) \leq \gamma_{1,\bar{4}}(P_n)$, it is enough to show that P_n contains a γ -set which is a $(1, \bar{4})$ -dset. It is easy to see that $D_1 = \{x_2, x_5, \dots, x_{3p-1}\}$, $D_2 = \{x_2, x_5, \dots, x_{3p-1}\} \cup \{x_{3p+1}\}$ and $D_3 = \{x_2, x_5, \dots, x_{3p+2}\}$ are minimum dominating sets of P_n for $n = 3p$, $n = 3p + 1$ and $n = 3p + 2$, respectively. Moreover, one can see that x_1 is $(1, \bar{4})$ -dominated by each of these three sets. Therefore, D_i is a $(1, \bar{4})$ -dset of P_n for each $i \in \{1, 2, 3\}$, leading to the desired equality. \square

From Theorems 9, 10, and Propositions 2 and 3, we deduce the following.

Corollary 8. *For any integer $n \geq 5$, we have:*

1. $\gamma_{1,\bar{3}}(P_n) = \gamma_{1,3}(P_n) = \gamma_{1,2}(P_n) = \lceil \frac{n+2}{3} \rceil$.
2. $\gamma_{1,\bar{4}}(P_n) = \gamma_{1,4}(P_n) = \gamma(P_n) = \lceil \frac{n}{3} \rceil$.
3. $\gamma_{1,\bar{3}}(P_n) = \gamma_{1,\bar{4}}(P_n)$ if and only if $n \equiv 1 \pmod{3}$.

Restricted to the class of trees T , the equality $\gamma_{1,k}(T) = \gamma_{1,\bar{k}}(T)$ is possible, and can be seen by considering the trees $S_{p,q}^*$ and $S_{p,q}^{**}$ both obtained from a double star $S_{p,q}$ by subdividing the edge joining the support vertices once and twice, respectively. Then $\gamma_{1,3}(S_{p,q}^*) = \gamma_{1,\bar{3}}(S_{p,q}^*) = 2$ and $\gamma_{1,4}(S_{p,q}^{**}) = \gamma_{1,\bar{4}}(S_{p,q}^{**}) = 2$. However, the difference $\gamma_{1,\bar{k}}(T) - \gamma_{1,k}(T)$ can be arbitrary large as can be seen by $S_{p,q}$ and $S_{p,q}^*$. Indeed, $\gamma_{1,3}(S_{p,q}) = \gamma_{1,4}(S_{p,q}^*) = 2$, while $\gamma_{1,\bar{3}}(S_{p,q}) = \gamma_{1,\bar{4}}(S_{p,q}^*) = p + 1$.

The next result provides a necessary and sufficient condition for connected triangle-free graphs G such that $\gamma_{1,k}(G) = \gamma_{1,\bar{k}}(G)$ for $k \in \{3, 4\}$.

Theorem 11. *Let G be a connected triangle-free graph with $\text{diam}(G) \geq 3$. Then $\gamma_{1,3}(G) = \gamma_{1,\bar{3}}(G)$ if and only if G has a $\gamma_{1,3}$ -set D containing a support vertex which is isolated in $G[D]$.*

Proof. Assume that $\gamma_{1,3}(G) = \gamma_{1,\overline{3}}(G)$. Then G has a $(1, 3)$ -dset D which is also a $(1, \overline{3})$ -dset. By Proposition 1 there is a leaf $u \notin D$ whose support vertex, say t , belongs to D and $d_G(u, z) = 3$ for some $z \in D$. Since there is no vertex $y \in D - \{t\}$ such that $d_G(u, y) = 2$, we deduce that t is isolated in $G[D]$.

Conversely, assume that G has a $\gamma_{1,3}$ -set D containing a support vertex t which is isolated in $G[D]$, and let u be a leaf neighbor of t . Since t is isolated in $G[D]$ and $\text{diam}(G) \geq 3$, there is a vertex $z \in D$ such that $d_G(u, z) = 3$. Therefore u is adjacent to t and $d_G(u, z) = 3$, which means that D is a $(1, \overline{3})$ -dset of G . Hence $\gamma_{1,\overline{3}}(G) \leq |D| = \gamma_{1,3}(G)$, and the equality follows from $\gamma_{1,3}(G) \leq \gamma_{1,\overline{3}}(G)$. \square

Theorem 12. *Let G be a connected triangle-free graph with $\text{diam}(G) \geq 4$. Then $\gamma_{1,4}(G) = \gamma_{1,\overline{4}}(G)$ if and only if G has a $\gamma_{1,4}(G)$ -set D containing a support vertex t such that $d_G(t, y) \geq 3$ for every $y \in D - \{t\}$.*

Proof. Assume that $\gamma_{1,4}(G) = \gamma_{1,\overline{4}}(G)$. Thus G has a $(1, 4)$ -dset D which is also a $(1, \overline{4})$ -dset. By Proposition 1, there is a leaf u that is adjacent to a support vertex t and $d_G(u, z) = 4$ for some vertex z in D . Therefore $d_G(t, y) \geq 3$ for every $y \in D - \{t\}$. Conversely, assume that G has a $\gamma_{1,4}(G)$ -set D containing a support vertex t such that $d_G(t, y) \geq 3$ for every $y \in D - \{t\}$, and let u be a leaf adjacent to t . Clearly $u \in V - D$, $N_G(u) \cap D = \{t\}$ and $d_G(u, y) \geq 4$ for every $y \in D - \{t\}$. Then there exists a vertex z in D such that $d_G(u, z) = 4$ and there is no vertex $y \in D - \{t\}$ with $d_G(u, y) \leq 3$. Therefore u is adjacent to t and $d_G(u, z) = 4$, meaning that D is $(1, \overline{4})$ -set of G . Hence $\gamma_{1,\overline{4}}(G) \leq |D| = \gamma_{1,4}(G)$, and the equality follows from $\gamma_{1,4}(G) \leq \gamma_{1,\overline{4}}(G)$. \square

The following problem can be raised for trees.

Problem. Give a constructive characterization of trees T with $\gamma_{1,k}(G) = \gamma_{1,\overline{k}}(G)$, for $k \in \{3, 4\}$.

4. Complexity results

In [13], Raczek showed that the decision problem corresponding to the problem of computing the $(1, 2)$ -dominating number is \mathcal{NP} -complete for bipartite and split graphs. We recall that for $k = 4$ (respectively, $k = 1$), $(1, k)$ -dset coincide with dominating sets (respectively, 2-dominating sets) for which the corresponding decision problems are \mathcal{NP} -complete for bipartite and chordal graphs (see [1–5]).

Our aim in this section is to study the complexity of the $(1, 3)$ -domination number, the proper $(1, 3)$ -domination number and the proper $(1, 4)$ -domination number to which we shall refer as $(1, 3)$ -DOMINATING SET, $(1, \overline{3})$ -DOMINATING SET and $(1, \overline{4})$ -DOMINATING SET, respectively.

$(1, 3)$ -DOMINATING SET

Instance: A graph G and a positive integer p .

Question: Does G have a $(1, 3)$ -dset of size at most p ?

(1, $\bar{3}$)-DOMINATING SET

Instance: A graph G and a positive integer p .

Question: Does G have a $(1, \bar{3})$ -dset of size at most p ?

(1, $\bar{4}$)-DOMINATING SET

Instance: A graph G and a positive integer p .

Question: Does G have a $(1, \bar{4})$ -dset of size at most p ?

We will show that the three decision problems mentioned above are \mathcal{NP} -complete for bipartite graphs by reducing the well-known \mathcal{NP} -complete problem Exact-3-Cover (X3C) to them.

EXACT COVER BY 3-SETS (X3C)

Instance: Given a set of elements X with $|X| = 3q$ and a collection C of 3-element subsets of X .

Question: Does C contain an exact cover for X , i.e. Can we find a subcollection C' of C such that every element of X occurs in exactly one member of C' ?

Theorem 13. *(1, 3)-DOMINATING SET is \mathcal{NP} -complete for bipartite graphs.*

Proof. (1, 3)-DOMINATING SET is members of \mathcal{NP} , since we can check in polynomial time whether a set S has cardinality at most p and is a $(1, 3)$ -dset.

Next, we show how to construct a bipartite graph $G = (V, E)$ and a positive integer p from any instance X and C of X3C so that, X3C has a solution if and only if G has a $(1, 3)$ -dset of cardinality at most p .

We are given an instance $X = \{x_1, \dots, x_{3q}\}$ and $C = \{C_1, \dots, C_m\}$ of X3C, where C_j are subsets of X of size $|C_j| = 3$ for $1 \leq j \leq m$, with $m \geq 2$. For each $x_i \in X$, we create a vertex x_i . For each $C_j \in C$, we create a vertex c_j and add a path $P_4 : u-t-y-z$. To complete the construction, we add all edges $x_i c_j$ if $x_i \in C_j$ and all edges $z c_j$ for $j \in \{1, 2, \dots, m\}$. Clearly G is bipartite (see Figure 2). Set $p = q + 2$.

Suppose that the instance X and C of X3C has a solution C' . We construct a $(1, 3)$ -dset D as follows: put in D , the vertices t, z and c_i for each $C_i \in C'$. Clearly y and u are $(1, 3)$ -dominated by D . Now, observe that since C' is a solution for X3C, each x_i has exactly one neighbor in D and exactly q vertices of $Y = \{c_1, c_2, \dots, c_m\}$ are in D . Moreover, each x_i is at distance two from z . The remaining vertices of Y are adjacent to z and are $(1, 3)$ -dominated by D . Therefore, D is a $(1, 3)$ -dset for G of cardinality $q + 2 = p$.

Conversely, assume that G has a $(1, 3)$ -dset D , of cardinality at most $p = q + 2$. Clearly for the path $u-t-y-z$ at least two vertices are in D . Now, let $n_1 = |D \cap X|$ and $q_1 = |D \cap Y|$. Then $|D| \geq n_1 + q_1 + 2$. Since the vertices of $X - D$ have to be dominated by $D \cap Y$, $|X - D| \leq 3|D \cap Y|$. So, we have $3q - n_1 \leq 3q_1$, leading to $n_1 \geq 3q - 3q_1$. Therefore $|D| \geq n_1 + q_1 + 2 \geq 3q - 2q_1 + 2$. Combining this with the fact that $|D| \leq p = q + 2$, we deduce that $q_1 \geq q$. It follows that $|D \cap Y| = q$, and hence every vertex x_i of X has exactly one neighbor in $D \cap Y$, meaning that $C' = \{C_i : c_i \in D \cap Y\}$ is an exact cover for X . □

Looking at the proof of Theorem 13, we can observe that D is also a $(1, \bar{3})$ -dset, since u is $(1, \bar{3})$ -dominated by D . Therefore we can state the following.

Corollary 9. $(1, \bar{3})$ -DOMINATING SET is \mathcal{NP} -complete for bipartite graphs.

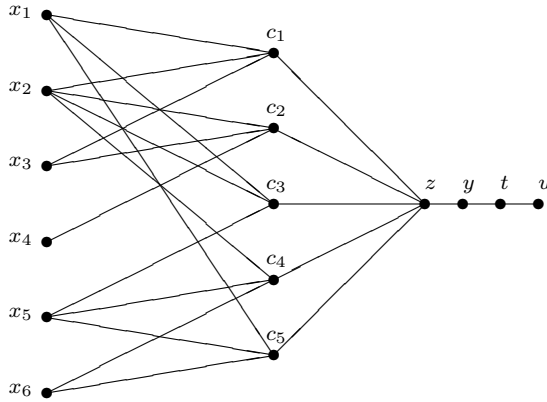


Figure 2. A construction of bipartite graph G for $q = 2$.

Finally to show the \mathcal{NP} -completeness of $(1, \bar{4})$ -DOMINATING SET, we construct a bipartite graph G' in the same way as G in the proof of Theorem 13, but instead of adding the path $u-t-y-z$, we add the path $u-t-y_1-y_2-z$. It is not difficult to see that $\{t, z\} \cup \{c_i : C_i \in C'\}$ is $(1, \bar{4})$ -dset of G' , since u is $(1, \bar{4})$ -dominated by D . Therefore we have the following result.

Theorem 14. $(1, \bar{4})$ -DOMINATING SET is \mathcal{NP} -complete for bipartite graphs.

Acknowledgment: We are grateful to the referee for her/his careful reading of the manuscript and useful suggestions.

Conflict of Interest: The authors declare no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] T.J. Bean, M.A. Henning, and H.C. Swart, *On the integrity of distance domination in graphs*, Australas. J. Combin. **10** (1994), 29–43.

- [2] A.A. Bertossi, *Dominating sets for split and bipartite graphs*, Inform. Process. Lett. **19** (1984), no. 1, 37–40.
[https://doi.org/10.1016/0020-0190\(84\)90126-1](https://doi.org/10.1016/0020-0190(84)90126-1).
- [3] K.S. Booth and J.H. Johnson, *Dominating sets in chordal graphs*, SIAM J. Comput. **11** (1982), no. 1, 191–199.
<https://doi.org/10.1137/0211015>.
- [4] G.J. Chang and G.L. Nemhauser, *The k -domination and k -stability problems in sun-free chordal graphs*, SIAM J. Algebraic Discrete Methods **5** (1984), no. 3, 332–345.
<https://doi.org/10.1137/0605034>.
- [5] A.K. Dewdney, *Fast turning reductions between problems in \mathcal{NP} ; chapter 4; reductions between \mathcal{NP} -complete problems*, Tech. Report 71, Dep. Computer Science, Univ. Western Ontario, 1981.
- [6] J.F. Fink and M.S. Jacobson, *On n -domination, n -dependence and forbidden subgraphs*, Graph Theory with Applications to Algorithms and Computer Science (1985), 301–311.
<https://doi.org/10.5555/21936.25447>.
- [7] S.M. Hedetniemi, S.T. Hedetniemi, J. Knisely, D. F. Rall, and T.W. Haynes, *Secondary domination in graphs*, AKCE Int. J. Graphs Comb. **5** (2008), no. 2, 103–115.
<https://doi.org/10.1080/09728600.2008.12088856>.
- [8] A. Kosiorowska, A. Michalski, and I. Włoch, *On minimum intersections of certain secondary dominating sets in graphs*, Opuscula Math. **43** (2023), no. 6, 813–827.
<https://doi.org/10.7494/OpMath.2023.43.6.813>.
- [9] A. Michalski, *Secondary dominating sets in graphs and their modification*, Book of Abstracts, The 7th Gdańsk Workshop on Graph Theory, Gdańsk, Poland, 2019.
- [10] A. Michalski and I. Włoch, *On the existence and the number of independent $(1, 2)$ -dominating sets in the G -join of graphs*, Appl. Math. Comput. **377** (2020), 125155.
<https://doi.org/10.1016/j.amc.2020.125155>.
- [11] A. Michalski, I. Włoch, M. Dettlaff, and M. Lemańska, *On proper $(1, 2)$ -dominating sets in graphs*, Math. Methods Appl. Sci. **45** (2022), no. 11, 7050–7057.
- [12] J. Raczek, *Polynomial algorithm for minimal $(1, 2)$ -dominating set in networks*, Electronics **11** (2022), no. 3, 300.
<https://doi.org/10.3390/electronics11030300>.
- [13] ———, *Complexity issues on of secondary domination number*, Algorithmica **86** (2024), 1163–1172.
<https://doi.org/10.1007/s00453-023-01192-2>.