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Research Article

New generalizations and identities of Mersenne-Lucas numbers and polynomials with structural constraints

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Abstract: This paper introduces and investigates two new sequences, $\{R_n^{(k)}\}$ and $\{R_n^{(k)}(x)\}$, which provide a distinct generalization of the Mersenne–Lucas numbers and polynomials, respectively, where the index n is expressed in the form $n = sk + r$, with $0 \leq r < k$. We derive several identities for these sequences in relation to the classical Mersenne and Mersenne–Lucas numbers and polynomials. Furthermore, we examine their algebraic properties and establish connections with existing sequences and polynomial families. In addition, we obtain closed-form expressions, Cassini-type identities, partial sums, recurrence relations, and various combinatorial identities associated with these sequences.

Keywords: Mersenne-Lucas numbers, Mersenne polynomials, Mersenne-Lucas polynomials, recurrence relation.

AMS Subject classification: 11B37, 11B39, 11B83

1. Introduction

The Mersenne sequence $\{M_n\}_{n\geq 0}$ is given [1, 11] by recurrence relation $M_{n+2} = 3M_{n+1} - 2M_n$ with initial values $M_0 = 0,\ M_1 = 1$. The Mersenne-Lucas numbers $\{R_n\}_{n\geq 0}$ are defined by the same recurrence relation but with the different initial values $R_0 = 2,\ R_1 = 3$. The first few Mersenne-Lucas numbers are: $2,\ 3,\ 5,\ 9,\ 17,\ 33,\ 65,\ 129, \ldots$.

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In literature these numbers are termed as the Fermat numbers [21]. The Fermat numbers are special numbers that intrigued mathematicians for centuries. The Fermat numbers named after the French mathematician Marin Fermat, who made significant contributions to the study of prime numbers. These numbers have applications in coding theory, cryptography, prime number generation, and other areas of mathematics. These numbers nowadays famous as 'Mersenne-Lucas numbers' which continues to be an active area of research in mathematics.

The characteristic equation for the above sequence is $\tau^2 - 3\tau + 2 = 0$ which have two roots $\tau_1 = 2$ and $\tau_2 = 1$. So the Binet's formula for the Mersenne and Mersenne-Lucas numbers are given as

$$M_n = 2^n - 1 \quad \text{and} \quad R_n = 2^n + 1, \tag{1.1}$$

and they are related as $R_n = M_n + 2$ for all integers $n \geq 0$.
Using Binet form of Mersenne-Lucas numbers, the Cassini like identity for $n \geq 1$ is

$$R_n^2 - R_{n+1}R_{n-1} = -2^{n-1}. \tag{1.2}$$

In recent year, several papers appeared discussing the generalizations and new families of an existing sequence. A famous example is the Horadam sequence, which generalizes almost all the second order homogeneous named sequences like Fibonacci, Lucas, Mersenne, Balancing, Jacobsthal, Pell, etc. One different kind of generalization of Fibonacci numbers was discussed by Mikkawy and Sogabe in [5]. Considering [5] as reference many authors extended the work for other well-known sequences like Fibonacci polynomials, Jacobsthal numbers, balancing numbers, Leonardo numbers, etc. For instance, Catarino et al. [2] studied a new families of the Jacobsthal and the Jacobsthal-Lucas numbers. Ozkan et al. discussed Gauss Fibonacci and Lucas polynomials with an application in [13] while in [14] discussed d-Gaussian Fibonacci/Lucas polynomials and associated matrices. Prasad et al. presented a new family of the Leonardo numbers with Gaussian version in [20] while in [18] discussed about new families of k-balancing and k-Lucas balancing numbers with polynomial version. Recently, Kumari et al. [12] studied a new family of generalized k-Mersenne numbers (polynomials) and enumerated several identities. The generalized k-Mersenne numbers $\{M_n^{(k)}\}$ and k-Mersenne polynomials $\{M_n^{(k)}(x)\}$ for $n = sk + r$ are defined as

$$M_n^{(k)} = (\lambda_1^s - \lambda_2^s)^{k-r}(\lambda_1^{s+1} - \lambda_2^{s+1})^r, \quad \text{where } \lambda_1 = 2 \text{ and } \lambda_2 = 1, \tag{1.3}$$
$$M_n^{(k)}(x) = (\lambda_1^s(x) - \lambda_2^s(x))^{k-r}(\lambda_1^{s+1}(x) - \lambda_2^{s+1}(x))^r, \tag{1.4}$$

where $\lambda_1(x) = (3x + \sqrt{9x^2 - 8})/2$ and $\lambda_2(x) = (3x - \sqrt{9x^2 - 8})/2$.

For our study, we consider the form $n = sk + r$ similar to [12]. The motivation behind considering the form $sk + r$, where $0 \leq r < k$ is its nature that breaks n

with respect to modulo k and simplifies the larger terms of the sequence in terms of classic Mersenne-Lucas sequence. This form partition the number n into k distinct arithmetic progressions. The motivation is to create a more flexible framework where the standard Mersenne properties are preserved at every k-th step, while allowing the internal behavior (controlled by r) to follow a modified rule. Algebraically, this structure allows the definition of the generalized k-Mersenne-Lucas numbers $R_n^{(k)}$ using a modified Binet formula for new families. This work is also related to [3], where the authors studied the k-Mersenne-Lucas numbers. In both works, for $k = 1$, we get the Mersenne-Lucas numbers, but for $k \geq 2$, they give two different families. In [11], the authors examined a generalized form of the Mersenne numbers defined with arbitrary initial values. They further derived various algebraic properties of the Mersenne–Lucas numbers and polynomials, which exhibit significant connections with the present work. Furthermore, this sequence is a companion sequence to [12], similar to how the Mersenne-Lucas sequence is a companion to the Mersenne sequence.

Motivated by the above works on Mersenne-type sequences and [5], this article extends the study of Mersenne-Lucas numbers and polynomials in a generalized form, analogous to (1.3) and (1.4). In our case, we have initial conditions 2 and 3 (2 and $3x$ for polynomials), which are defined in the next sections.

Here we identify algebraic properties of proposed sequences of numbers (polynomials) and show their connections with existing numbers and polynomials. Furthermore, we give closed form formula, Cassini's identity, partial sums, recurrence relations and various combined combinatorial identities of these sequences. For recent developments on Mersenne like sequences and their applications, one can see [4, 6, 7, 9, 10, 15–17, 19, 23–25].

For Mersenne and Mersenne-Lucas numbers, consider the following lemma consisting useful identities, which can be easily proved using Binet's formula (1.1).

Lemma 1. *For $n, m \in \mathbb{N}$, we have*

1. $M_n R_n = M_{2n}$.
2. $M_{m+n} + M_{m-n} = 2^{m-n} R_{2n} - 2$.
3. $M_{m+n} - M_{m-n} = 2^{m-n} M_{2n}$.
4. $R_{m+n} + R_{m-n} = 2^{m-n} R_{2n} + 2$.
5. $R_{m+n} - R_{m-n} = 2^{m-n} M_{2n}$.

The organization of this article is as follows: In Section 2, we define and study the generalized k-Mersenne-Lucas numbers $\{R_n^{(k)}\}$ and present their properties. In Section 3, we introduce the generalized k-Mersenne-Lucas polynomials $\{R_n^{(k)}(x)\}$ and study its new families. Section 4 is devoted to the combined combinatorial identities where we present numerous identities showing interrelation and recurrence relations among Mersenne and Mersenne-Lucas polynomials.

2. Generalized k-Mersenne-Lucas numbers

Definition 1. For $k,\ n \in \mathbb{N}$ there is always unique $s, r \in \mathbb{N} \cup \{0\}$ such that $n = ks + r$, $0 \leq r < k$. Using these parameters we define the generalized k-Mersenne-Lucas numbers $\{R_n^{(k)}\}$ as

$$R_n^{(k)} = (\tau_1^s + \tau_2^s)^{k-r}(\tau_1^{s+1} + \tau_2^{s+1})^r \quad \text{with} \quad R_0^{(k)} = 2^k. \tag{2.1}$$

Substituting $\tau_1 = 2$ and $\tau_2 = 1$ in the above formula, we can have a closed formula for $R_n^{(k)}$ as follows

$$R_n^{(k)} = (2^s + 1)^{k-r}(2^{s+1} + 1)^r; \quad \text{where} \ \ n = ks + r,\ 0 \leq r < k.$$

Using (1.1) and Definition 1, the generalized k-Mersenne-Lucas numbers and Mersenne-Lucas numbers can be associated as

$$R_n^{(k)} = R_s^{k-r} R_{s+1}^r, \quad n = ks + r. \tag{2.2}$$

Thus for different values of k, some identities showing connection between the generalized k-Mersenne-Lucas numbers and Mersenne-Lucas numbers are enumerated below:

1. $R_{2s}^{(2)} = R_s^2$.
2. $R_{2s+1}^{(2)} = R_s R_{s+1}$.
3. $R_{2s+1}^{(2)} = 3R_{2s}^{(2)} - 2R_{2s-1}^{(2)}$.
4. $R_{3s}^{(3)} = R_s^3$.
5. $R_{3s+1}^{(3)} = R_s^2 R_{s+1}$.
6. $R_{3s+1}^{(3)} = 3R_{3s}^{(3)} - 2R_{3s-1}^{(3)}$.
7. $R_{3s+2}^{(3)} = R_s R_{s+1}^2$.
8. $R_{4s}^{(4)} = R_s^4$.
9. $R_{4s+1}^{(4)} = R_s^3 R_{s+1}^1$.
10. $R_{4s+2}^{(4)} = R_s^2 R_{s+1}^2$.
11. $R_{4s+3}^{(4)} = R_s^1 R_{s+1}^3$.

From Definition 1, we should note that if $k = 1$ then $r = 0$ and hence $n = s$, so $R_n^{(1)} = R_s$. Thus for $k = 1$ the sequence $R_n^{(k)}$ gives the classic Mersenne-Lucas numbers. Also from (2.2), for $n = 1$, we have $s = 1,\ r = 0$ if $k = 1$ and $s = 0,\ r = 1$ if $k \geq 2$, thus $R_1^{(k)} = \begin{cases} 3 & \text{if } k = 1 \\ R_0^{k-1} R_1 = 2^{k-1}3 & \text{if } k \geq 2. \end{cases}$

Example 1. If n is of the form $n = ks$ then $r = 0$ hence from (2.2)

$$R_{ks}^{(k)} = R_s^{k-0} R_{s+1}^0 = R_s^k.$$

Example 2. For $n = 3$ and $k = 2$, we have $s = 1$ and $r = 1$, so $R_3^{(2)} = R_1^1 R_2^1 = 3 \times 5 = 15$.

Remark 1. If $k > n$, then $s = 0$ and $r = n$, thus in this case

$$R_n^{(k)} = (2)^{k-n}(3)^n.$$

Thus, Definition 1 can be explicitly given as

$$R_n^{(k)} = \begin{cases} (2^s+1)^{k-r}(2^{s+1}+1)^r, & n \geq k, \\ 2^{k-n}3^n, & n < k, \end{cases} \quad \text{where} \quad n = ks + r, \ 0 \leq r < k.$$

The following lemma shows interrelations between the generalized k-Mersenne-Lucas numbers and Mersenne-Lucas numbers:

Lemma 2. *For $k, s \in \mathbb{N}$, we have*

1. $R_{ks}^{(k)} = R_s^k = (2^s + 1)^k$
2. $R_{ks-1}^{(k)} = R_{s-1}R_s^{k-1}$.
3. $R_{ks+1}^{(k)} = R_s^{k-1}R_{s+1}$.

Note that the notation $R_n^{(k)}$ does not mean a kth power of R_n, it represents the nth generalized k-Mersenne-Lucas numbers and R_n^k is the kth power of R_n.
Using the above relations, the first few values of generalized k-Mersenne-Lucas numbers $\{R_n^{(k)}\}$ are given in Table 1:

$R_n^{(k)}$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$R_0^{(k)}$	2	4	8	16	32	64	128	256
$R_1^{(k)}$	3	6	12	24	48	96	192	384
$R_2^{(k)}$	5	9	18	36	72	144	288	576
$R_3^{(k)}$	9	15	27	54	108	216	432	864
$R_4^{(k)}$	17	25	45	81	162	324	648	1296
$R_5^{(k)}$	33	45	75	135	243	486	972	1944
$R_6^{(k)}$	65	81	125	225	405	729	1458	2916
$R_7^{(k)}$	129	145	207	351	621	1134	2187	4374

Table 1. **Generalized k-Mersenne-Lucas numbers $R_n^{(k)}$ for $k = 1$ to 8 and $n = 0$ to 7**

The graph Figure 1 is added to visualize the exponential growth and the smooth transition of the piecewise function $R_n^{(k)}$ across the different values of n and k.

Theorem 1. *For $k, s \in \mathbb{N}$, we have $R_{ks+1}^{(s)} = 3R_{ks}^{(s)} - 2R_{ks-1}^{(s)}$.*

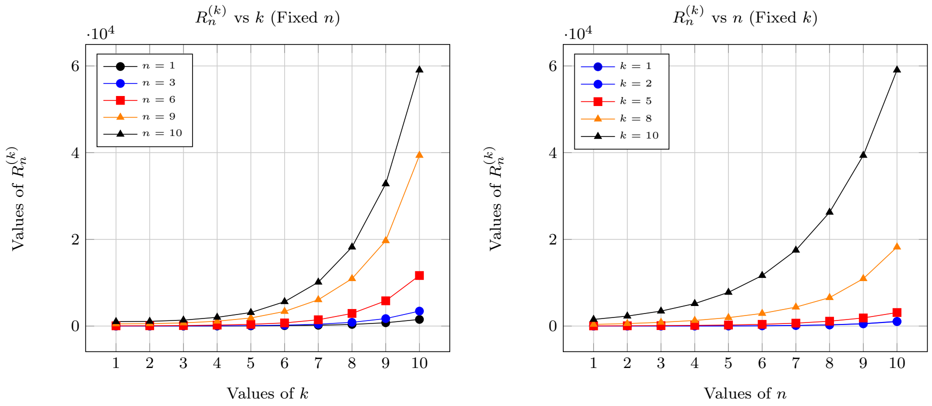

Figure 1. **Graph of generalized k-Mersenne-Lucas numbers $R_n^{(k)}$ with respect to k and n**

Proof. The result follows from Lemma 2, i.e.

$$\begin{aligned} 3R_{ks}^{(k)} - 2R_{ks-1}^{(k)} &= 3R_s^k - 2R_{s-1}R_s^{k-1} \\ &= R_s^{k-1}(3R_s - 2R_{s-1}) \\ &= R_s^{k-1}R_{s+1} = R_{ks+1}^{(k)}. \end{aligned}$$

□

Theorem 2. *For $k, s, m, n \in \mathbb{N}$, we have the following relations:*

1. $R_{ks+k}^{(k)} - R_{ks}^{(k)} = R_{s+1}^k - R_s^k$.
2. $R_{2(n+m-1)}^{(2)} - R_{n+m}R_{n+m-2} = -2^{n+m-2}$.

Proof. 1. Using (2.2), we have

$$R_{ks+k}^{(k)} - R_{ks}^{(k)} = [R_s^{k-d}R_{s+1}^k] - [R_s^{k-0}R_{s+1}^0] = R_{s+1}^k - R_s^k.$$

2. Using 1 of Lemma 2 and (1.2), we obtain

$$R_{2(n+m-1)}^{(2)} - R_{n+m}R_{n+m-2} = R_{(n+m-1)}^2 - R_{n+m}R_{n+m-2} = -2^{n+m-2}.$$

□

Theorem 3 (Cassini's identity). *For $n, k \geq 2$, we have*

$$R_{ks+a}^{(k)}R_{ks+a-2}^{(k)} - (R_{ks+a-1}^{(k)})^2 = \begin{cases} 2^{s-1}R_s^{2k-2} & : a = 1 \\ 0 & : a \neq 1. \end{cases}$$

Proof. If $a \neq 1$ then from (2.2), we substitute LHS as

$$\begin{aligned} R^{(k)}_{ks+a}R^{(k)}_{ks+a-2} - (R^{(k)}_{ks+a-1})^2 &= (R_s^{k-a}R_{s+1}^a)(R_s^{k-a+2}R_{s+1}^{a-2}) - (R_s^{k-a+1}R_{s+1}^{a-1})^2 \\ &= R_s^{2k-2a+2}[R_{s+1}^aR_{s+1}^{a-2} - (R_{s+1})^{2a-2}] \\ &= 0, \end{aligned}$$

and if $a = 1$, then

$$\begin{aligned} R^{(k)}_{ks+1}R^{(k)}_{ks-1} - (R^{(k)}_{ks})^2 &= (R_s^{k-1}R_{s+1})(R_{s-1}R_s^{k-1}) - (R_s^k)^2 \\ &= R_s^{2k-2}[R_{s+1}R_{s-1} - (R_s)^2] \\ &= 2^{s-1}R_s^{2k-2} \quad \text{(using (1.2))}. \end{aligned}$$

□

Theorem 4. *For $s, k \in \mathbb{N}$, the following summation identities hold true.*

1. $\displaystyle\sum_{a=0}^{k-1} R^{(k)}_{sk+a} = \Big(1 + \frac{1}{2^s}\Big)(R_{s+1}^k - R_s^k).$
2. $\displaystyle\sum_{a=0}^{k} aR^{(k)}_{sk+a} = \frac{R_s^{k+1}R_{s+1} - kR_s^2R_{s+1}^k + (k-1)R_sR_{s+1}^{k+1}}{2^{2s}}.$
3. $\displaystyle\sum_{a=0}^{k-1} \binom{k-1}{a} R^{(k)}_{sk+a} = (2^s+1)(2^s3+2)^{k-1}.$
4. $\displaystyle\sum_{a=0}^{k-1} (-1)^a \binom{k-1}{a} R^{(k)}_{sk+a} = (2^s+1)(-2^s)^{k-1}.$

Proof. 1. From (2.2) we can write $R^{(k)}_{sk+a} = R_s^{k-a}R_{s+1}^a = R_s^k(R_{s+1}/R_s)^a$. Thus

$$\begin{aligned} \sum_{a=0}^{k-1} R^{(k)}_{sk+a} &= R_s^k \sum_{a=0}^{k-1} \Big(\frac{R_{s+1}}{R_s}\Big)^a \\ &= R_s^k \frac{(R_{s+1}/R_s)^k - 1}{R_{s+1}/R_s - 1} \quad \text{(using arithmetic sum)} \\ &= R_s\Big(\frac{R_{s+1}^k - R_s^k}{R_{s+1} - R_s}\Big) \\ &= \Big(\frac{2^s+1}{2^s}\Big)(R_{s+1}^k - R_s^k). \end{aligned}$$

2. Since

$$\sum_{a=0}^{k} aR^{(k)}_{sk+a} = \sum_{a=0}^{k} aR_s^k(R_{s+1}/R_s)^a = R_s^{k-1}R_{s+1}\sum_{a=1}^{k} a\Big(\frac{R_{s+1}}{R_s}\Big)^{a-1}$$

and we should note that

$$\sum_{a=1}^{k} a x^{a-1} = \frac{(1 - k x^{k-1} + (k-1)x^{k})}{(1-x)^2}. \tag{2.3}$$

So substituting $x = R_{s+1}/R_s$ in (2.3), we can write

$$\begin{aligned}
R_s^{k-1}R_{s+1}\sum_{a=1}^{k} a\Big(\frac{R_{s+1}}{R_s}\Big)^{a-1} &= R_s^{k-1}R_{s+1}\Big(\frac{1-k(R_{s+1}/R_s)^{k-1}+(k-1)(R_{s+1}/R_s)^{k}}{(1-R_{s+1}/R_s)^2}\Big)\\
&= \frac{R_s^{k-1}R_{s+1} - kR_{s+1}^{k} + (k-1)R_{s+1}^{k+1}/R_s}{(1-R_{s+1}/R_s)^2}\\
&= \frac{R_s^{k+1}R_{s+1} - kR_s^2R_{s+1}^{k} + (k-1)R_sR_{s+1}^{k+1}}{(R_s-R_{s+1})^2}\\
\sum_{a=0}^{k} aR_{sk+a}^{(k)} &= \frac{R_s^{k+1}R_{s+1} - kR_s^2R_{s+1}^{k} + (k-1)R_sR_{s+1}^{k+1}}{2^{2s}} \qquad \text{(using Eqn. (1.1))}.
\end{aligned}$$

3. Using relation (2.2) in LHS, we write

$$\begin{aligned}
\sum_{a=0}^{k-1}\binom{k-1}{a}R_{sk+a}^{(k)} &= \sum_{a=0}^{k-1}\binom{k-1}{a}R_s^{k-a}R_{s+1}^{a}\\
&= R_s\sum_{a=0}^{k-1}\binom{k-1}{a}R_s^{k-1-a}R_{s+1}^{a}\\
&= R_s(R_s+R_{s+1})^{k-1} \qquad \text{(using the Binomial theorem)}\\
&= (2^s+1)(2^s\times 3+2)^{k-1}.
\end{aligned}$$

4. The argument is very similar to identity 3, i.e. using (2.2) in LHS, we write

$$\begin{aligned}
\sum_{a=0}^{k-1}(-1)^a\binom{k-1}{a}R_{sk+a}^{(k)} &= \sum_{a=0}^{k-1}(-1)^a\binom{k-1}{a}R_s^{k-a}R_{s+1}^{a}\\
&= R_s\sum_{a=0}^{k-1}\binom{k-1}{a}R_s^{k-a}(-R_{s+1})^{a}\\
&= R_s(R_s-R_{s+1})^{k-1} \qquad \text{(using the Binomial theorem).}\\
&= (2^s+1)(-2^s)^{k-1}.
\end{aligned}$$

□

Theorem 5. *For integers s, a, b and $k \geq 1$, we have*

1. $R_{2sk}^{(k)} = (2^{2s}+1)^k$.
2. $(R_{a+b}+R_{a-b})^k = (2^{a-b}R_{2b}+2)^k$.

3. $(R_{a+b} - R_{a-b})^k = 2^{k(a-b)} M_{2b}^k$.
4. $R_{2a}^{(2)} - R_{2b}^{(2)} = 2^{2a} - 2^{2b} + 2^{a+1} - 2^{b+1}$.

Proof. 1. From Lemma 2 and Eqn. (1.1), we have $R_{2sk}^{(k)} = R_{2s}^k = (2^{2s} + 1)^k$.
2. From 4 of Lemma 1, we have $(R_{a+b} + R_{a-b})^k = (2^{a-b} R_{2b} + 2)^k$.
3. From 5 of Lemma 1, we have

$$(R_{a+b} - R_{a-b})^k = (2^{a-b} M_{2b})^k = 2^{k(a-b)} M_{2b}^k.$$

4. From Lemma 1, we write

$$\begin{aligned} R_{2a}^{(2)} - R_{2b}^{(2)} &= R_a^2 - R_b^2 = (R_a - R_b)(R_a + R_b) \\ &= (2^a - 2^b)(2^a + 2^b + 2) \\ &= 2^{2a} - 2^{2b} + 2^{a+1} - 2^{b+1}. \end{aligned}$$

□

3. Generalized Mersenne-Lucas polynomials and new families

The Mersenne polynomial $\{M_n(x)\}$ are given by $M_{n+2}(x) = 3xM_{n+1}(x) - 2M_n(x)$ with $M_0 = 0, \quad M_1(x) = 1$ which generalizes the Mersenne numbers . Similarly, a generalizations of the Mersenne-Lucas numbers are the Mersenne-Lucas polynomials defined as follows [11].

Definition 2. For $n \geq 0$, the Mersenne-Lucas polynomials $\{R_n(x)\}$ are given by

$$R_{n+2}(x) = 3xR_{n+1}(x) - 2R_n(x) \quad \textit{with} \;\; R_0 = 2, \;\; R_1(x) = 3x. \tag{3.1}$$

For $x = 1$, the above definition gives the Mersenne-Lucas numbers and for $x = k$ it gives the k-Mersenne-Lucas numbers[3]. The characteristic equation for the above recurrence relation (3.1) is $\tau^2 - 3x\tau + 2\tau = 0$ which have two roots $\tau_1(x) = (3x + \sqrt{9x^2 - 8})/2$ and $\tau_2(x) = (3x - \sqrt{9x^2 - 8})/2$. These roots satisfy

$$\tau_1(x) + \tau_2(x) = 3x, \quad \tau_1(x)\tau_2(x) = 2 \quad \text{and} \quad \tau_1(x) - \tau_2(x) = \sqrt{9x^2 - 8}. \tag{3.2}$$

Hence for $n \geq 0$, the Binet type formula for the nth Mersenne-Lucas polynomials is

$$R_n(x) = \tau_1^n(x) + \tau_2^n(x), \tag{3.3}$$

whereas the Binet type formula for Mersenne polynomials is

$$M_n(x) = \frac{\tau_1^n(x) - \tau_2^n(x)}{\sqrt{9x^2 - 8}}. \tag{3.4}$$

The first few Mersenne-Lucas polynomials are

$$R_0(x)=2,\ R_1(x)=3x,\ R_2(x)=9x^2-4,\ R_3(x)=27x^3-18x,\ R_4(x)=81x^4-72x^2+8,$$
$$R_5(x)=243x^5-270x^3+60x,\ R_6(x)=729x^6-972x^4+324x^2-16,\dots.$$

Note: As per study of Horadam [8], the above polynomial sequence (3.1) is named as the Fermat-Lucas polynomial while the well-known Mersenne polynomial is named as Fermat polynomial.
For algebraic properties of Mersenne like polynomial sequences, one can see [3, 11, 12, 22]. By virtue of [11], we have the following identities for $R_n(x)$ where $n,m\in\mathbb{N}$:

1. (Cassini's identity) $R_{n+1}(x)R_{n-1}(x)-R_n^2(x)=2^{n-1}(9x^2-8)$.
2. (Catlan's identity) $R_{n+m}(x)R_{n-m}(x)-R_n^2(x)=\dfrac{2^{n-m}[3xR_m(x)-2R_{m+1}(x)]^2}{9x^2-8}$.

3.1. Generalized k-Mersenne-Lucas polynomial

Now, we define a new family of the Mersenne-Lucas polynomial called generalized k-Mersenne-Lucas polynomial $\{R_n^{(k)}(x)\}$ and give their properties in context of existing polynomial sequences.

Definition 3. Let $n,k\in\mathbb{N}$, then there exist unique $s,r\in\mathbb{N}\cup\{0\}$ such that $n=sk+r$, $0\le r<k$. Then the generalized k-Mersenne-Lucas polynomial $\{R_n^{(k)}(x)\}$ is defined as

$$R_n^{(k)}(x)=(\tau_1^s(x)+\tau_2^s(x))^{k-r}\left(\tau_1^{s+1}(x)+\tau_2^{s+1}(x)\right)^r \quad \text{with} \quad R_0^{(k)}(x)=2^k, \tag{3.5}$$

where $\tau_1(x)=(3x+\sqrt{9x^2-8})/2$ and $\tau_2(x)=(3x-\sqrt{9x^2-8})/2$.

Note that for $x=1$, (3.5) gives the generalized k-Mersenne-Lucas numbers as defined in the previous section. From (3.3) and Definition 3, the relation between the generalized k-Mersenne-Lucas polynomials and Mersenne-Lucas polynomials is

$$R_n^{(k)}(x)=R_s^{k-r}(x)R_{s+1}^r(x), \quad n=ks+r \text{ and } 0\le r<k. \tag{3.6}$$

If $k=1$ then $r=0$ and hence $n=s$. So, from (3.6) $R_s^{(1)}(x)=R_n(x)$.

Example 3. For $k=2$ and $n=3$, we have $s=1$ and $r=1$, thus

$$R_3^{(2)}(x)=R_1^1(x)R_2^1(x)=3x(9x^2-4)=27x^3-12x.$$

From (3.6), we have the following relations between the generalized k-Mersenne-Lucas polynomials and Mersenne-Lucas polynomials for $k=2,3,4$:

1. $R^{(2)}_{2s}(x) = R^2_s(x)$.
2. $R^{(2)}_{2s+1}(x) = R_s(x)R_{s+1}(x)$.
3. $R^{(2)}_{2s+1}(x) = 3R^{(2)}_{2s}(x) - 2R^{(2)}_{2s-1}(x)$.
4. $R^{(3)}_{3s}(x) = R^3_s(x)$.
5. $R^{(3)}_{3s+1}(x) = R^2_s(x)R_{s+1}(x)$.
6. $R^{(3)}_{3s+2}(x) = R_s(x)R^2_{s+1}(x)$.
7. $R^{(4)}_{4s}(x) = R^4_s(x)$.
8. $R^{(4)}_{4s+1}(x) = R^3_s(x)R_{s+1}(x)$.
9. $R^{(4)}_{4s+2}(x) = R^2_s(x)R^2_{s+1}(x)$.
10. $R^{(4)}_{4s+3}(x) = R^1_s(x)R^3_{s+1}(x)$.

Similar to the generalized k-Mersenne-Lucas numbers, the generalized k-Mersenne-Lucas polynomials posses the following properties:

Theorem 6. *For $k, s \in \mathbb{N}$, we have*

1. $R^{(k)}_{ks-1}(x) = R_{s-1}(x)R^{k-1}_s(x)$.

2. $R^{(k)}_{ks}(x) = R^k_s(x)$.

3. $R^{(k)}_{ks+k}(x) = R^k_{s+1}(x)$.

4. $R^{(k)}_{ks+1}(x) = 3xR^{(k)}_{ks}(x) - 2R^{(k)}_{ks-1}(x)$.

5. $R^{(k)}_{sk+k}(x) - R^{(k)}_{sk}(x) = R^k_{s+1}(x) - R^k_s(x)$.

Proof. 1. $R^{(k)}_{ks-1}(x) = R^{(k)}_{k(s-1)+(k-1)}(x) = R_{s-1}(x)R^{k-1}_s(x)$ using (3.6).
2. & 3. The identity follows from (3.6) by taking $r = 0$ and $r = k$, respectively.
4. Using (3.6) and the first identity in RHS, we have

$$\begin{aligned}
3xR^{(k)}_{ks}(x) - 2R^{(k)}_{ks-1}(x) &= 3xR^k_s(x) - 2R_{s-1}(x)R^{k-1}_s(x)\\
&= R^{k-1}_s(x)[3xR_s(x) - 2R_{s-1}(x)]\\
&= R^{k-1}_s(x)R_{s+1}(x) \quad \text{(using (3.1))}\\
&= R^{(s)}_{ks+1}(x).
\end{aligned}$$

5. The result follows from the second and third identities. □

Thus, the first few values of the generalized k-Mersenne-Lucas polynomials are as follows. Note that $R^t_n(x)$ means t times multiplication of $R_n(x)$.

Theorem 7 (Cassini's identity). *For $s, k \geq 0$, we have*

$$R^{(k)}_{ks+a}(x)R^{(k)}_{ks+a-2}(x) - (R^{(k)}_{ks+a-1}(x))^2 = \begin{cases} 2^{n-1}(9x^2-8)R^{2k-2}_s(x) & : a = 1,\\ 0 & : a \neq 1. \end{cases}$$

$R_n^{(k)}(x)$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$
$R_0^{(k)}(x)$	2	2^2	2^3	2^4	2^5
$R_1^{(k)}(x)$	$3x$	$2(3x)$	$2^2(3x)$	$2^3(3x)$	$2^4(3x)$
$R_2^{(k)}(x)$	$9x^2-4$	$(3x)^2$	$2(3x)^2$	$2^2(3x)^2$	$2^3(3x)^2$
$R_3^{(k)}(x)$	$27x^3-18x$	$(3x)R_2(x)$	$(3x)^3$	$2(3x)^3$	$2^2(3x)^3$
$R_4^{(k)}(x)$	$81x^4-72x^2+8$	$R_2^2(x)$	$(3x)^2R_2(x)$	$(3x)^4$	$2(3x)^4$
$R_5^{(k)}(x)$	$243x^5-270x^3+60x$	$R_2(x)R_3(x)$	$(3x)R_2^2(x)$	$(3x)^3R_2(x)$	$(3x)^5$
$R_6^{(k)}(x)$	$729x^6-972x^4+324x^2-16$	$R_3^2(x)$	$R_2^3(x)$	$(3x)^2R_2^2(x)$	$(3x)^4R_2(x)$
$R_7^{(k)}(x)$	$2187x^7-3402x^5$ $+1512x^3-168x$	$R_3(x)R_4(x)$	$R_2^2(x)R_3(x)$	$(3x)R_2^3(x)$	$(3x)^3R_2^2(x)$
$R_8^{(k)}(x)$	$6561x^8-11664x^6$ $+6480x^4-1152x^2+32$	$R_4^2(x)$	$R_2(x)R_3^2(x)$	$R_2^4(x)$	$(3x)^2R_2^3(x)$

Table 2. **Generalized k-Mersenne-Lucas polynomials $R_n^{(k)}(x)$ for $k=1,...,5$**

Proof. Let $a \neq 1$, so from (3.6), we have

$$\begin{aligned}
&R_{ks+a}^{(k)}(x)R_{ks+a-2}^{(k)}(x)-(R_{ks+a-1}^{(k)}(x))^2\\
&\qquad=(R_s^{k-a}(x)R_{s+1}^{a}(x))(R_s^{k-a+2}(x)R_{s+1}^{a-2}(x))-(R_s^{k-a+1}(x)R_{s+1}^{a-1}(x))^2\\
&\qquad=R_s^{2k-2a+2}(x)[(R_{s+1}(x))^{2a-2}-(R_{s+1}(x))^{2a-2}]\\
&\qquad=0
\end{aligned}$$

and if $a=1$ then using Theorem 6, we have

$$\begin{aligned}
R_{ks+1}^{(k)}(x)R_{ks-1}^{(k)}(x)-(R_{ks}^{(k)}(x))^2&=(R_s^{k-1}(x)R_{s+1}(x))(R_{s-1}R_s^{k-1}(x))-(R_s^{k}(x))^2\\
&=R_s^{2k-2}(x)[R_{s+1}(x)R_{s-1}(x)-(R_s(x))^2]\\
&=2^{n-1}(9x^2-8)R_s^{2k-2}(x)\quad\text{(using Cassini's idenity 1).}
\end{aligned}$$

□

4. Combined combinatorial identities

In this section, we provide some combined combinatorial identities of Mersenne and Mersenne-Lucas polynomials. We give some results showing interrelation between Mersenne and Mersenne-Lucas polynomials which we used later to establish many new results.

Throughout the article we adopted the notation $M_n(x)$ for the nth Mersenne polynomials, $R_n(x)$ for the nth Mersenne-Lucas polynomials, and $M_n^r(x)$ for the rth power of $M_n(x)$. Similarly for the Mersenne-Lucas numbers and polynomials.

Note that the notations of type $M_n^{(k)}(x)$ does mean a kth power of $M_n(x)$, it represents the k-Mersenne polynomials and $M_n^{(k)}$ for the k-Mersenne numbers.

Theorem 8. *For $m, n \in \mathbb{N}$, we have the following combined interrelations:*

1. $M_{2n}(x) = M_n(x)R_n(x)$.
2. $R_{2n}(x) = (9x^2 - 8)M_n^2(x) + 2^{n+1}$.
3. $R_n^2(x) = (9x^2 - 8)M_n^2(x) + 2^{n+2}$.
4. $R_{2n}(x) = R_n^2(x) - 2^{n+1}$.
5. $R_{n+m}(x)R_{n-m}(x) = \dfrac{2^{n-m}[3xR_m(x) - 2R_{m+1}(x)]^2}{9x^2 - 8} + R_{2n}(x) + 2^{n+1}$.
6. $M_{2n+1}(x) = \begin{cases} M_n(x)R_{n+1}(x) + 2^n, & \text{and} \\ M_{n+1}(x)R_n(x) - 2^n. \end{cases}$
7. $M_{2n+1}(x) \pm M_{2n}(x) = \begin{cases} M_n(x)[R_{n+1}(x) \pm R_n(x)] + 2^n, & \text{and} \\ R_n(x)[M_{n+1}(x) \pm M_n(x)] - 2^n. \end{cases}$

Proof. 1. Using Binet type formulas (3.3) and (3.4) in RHS, we can write

$$\begin{aligned} M_n(x)R_n(x) &= \Big(\frac{\tau_1^n(x) - \tau_2^n(x)}{\sqrt{9x^2-8}}\Big)\big(\tau_1^n(x) + \tau_2^n(x)\big) \\ &= \frac{\tau_1^{2n}(x) - \tau_2^{2n}(x)}{\sqrt{9x^2-8}} = M_{2n}(x). \end{aligned}$$

2. Since

$$\begin{aligned} M_n^2(x) &= \Big(\frac{\tau_1^n(x) - \tau_2^n(x)}{\sqrt{9x^2-8}}\Big)^2 \\ &= \Big(\frac{\tau_1^{2n}(x) + \tau_2^{2n}(x) - 2\tau_1^n(x)\tau_2^n(x)}{9x^2-8}\Big) \\ &= \frac{R_{2n}(x) - 2 \times 2^n}{9x^2-8} \quad \text{(using (3.3) and } \tau_1(x)\tau_2(x) = 2). \end{aligned}$$

Thus, on rearranging the above equation, we get the required result.
3. From 2, we have

$$\begin{aligned} M_n^2(x) &= \Big(\frac{\tau_1^{2n}(x) + \tau_2^{2n}(x) - 2\tau_1^n(x)\tau_2^n(x)}{9x^2-8}\Big) \\ &= \Big(\frac{[\tau_1^n(x) + \tau_2^n(x)]^2 - 4\tau_1^n(x)\tau_2^n(x)}{9x^2-8}\Big) \\ &= \Big(\frac{R_n^2(x) - 4 \times 2^n}{9x^2-8}\Big) \quad \text{(using (3.3) and } \tau_1(x)\tau_2(x) = 2) \\ (9x^2-8)M_n^2(x) &= R_n^2(x) - 2^{n+2}, \end{aligned}$$

which on rearrangement gives the required result.
4. Substituting 2 from 3 gives

$$R_n^2(x) - R_{2n}(x) = 2^{n+2} - 2^{n+1} = 2^{n+1}.$$

5. Since the Catalan's identity is

$$R_{n+m}(x)R_{n-m}(x) - R_n^2(x) = \frac{2^{n-m}[3xR_m(x) - 2R_{m+1}(x)]^2}{9x^2-8}.$$

So using fourth identity, substituting $R_n^2(x)$ gives

$$R_{n+m}(x)R_{n-m}(x) = \frac{2^{n-m}[3xR_m(x) - 2R_{m+1}(x)]^2}{9x^2-8} + R_{2n}(x) + 2^{n+1}.$$

6. Using Binet formulas (3.3) and (3.4) in RHS, we can write

$$\begin{aligned} M_{n+1}(x)R_n(x) &= \Big(\frac{\tau_1^{n+1}(x) - \tau_2^{n+1}(x)}{\sqrt{9x^2-8}}\Big)\big(\tau_1^n(x) + \tau_2^n(x)\big) \\ &= \frac{\tau_1^{2n+1}(x) - \tau_2^{2n+1}(x) + \tau_1^n(x)\tau_2^n(x)(\tau_1(x) - \tau_2(x))}{\sqrt{9x^2-8}} \\ &= M_{2n+1}(x) + 2^n \quad (\text{using } \tau_1(x)\tau_2(x) = 2). \end{aligned}$$

Similarly $M_n(x)R_{n+1}(x) = M_{2n+1}(x) - 2^n$ holds.
7. The result is established by taking sum (difference) of first and sixth identity. ☐

Theorem 9. *For positive integers n, m with $m \geq n$, we have*

1. $R_{m+n}(x) + 2^n R_{m-n}(x) = R_m(x)R_n(x)$.
2. $R_{m+n}(x) - 2^n R_{m-n}(x) = (9x^2-8)M_m(x)M_n(x)$.
3. $R_{m+n}^2(x) - 2^{2n}R_{m-n}^2(x) = R_{2(m+n)}(x) - 2^n R_{2(m-n)}(x)$.
4. $2R_{m+n}(x) = R_m(x)R_n(x) + (9x^2-8)M_m(x)M_n(x)$.
5. $2^{n+1}R_{m-n}(x) = R_m(x)R_n(x) - (9x^2-8)M_m(x)M_n(x)$.
6. $R_{m+n}^2(x) - 2^{2n}R_{m-n}^2(x) = (9x^2-8)M_{2m}(x)M_{2n}(x)$.
7. $R_{2n+1}(x) = \begin{cases} R_n(x)R_{n+1}(x) - 2^n(3x), & \textit{and} \\ (9x^2-8)M_n(x)M_{n+1}(x) + 2^n(3x). \end{cases}$
8. $R_{2n+1}(x) \pm R_{2n}(x) = R_n(x)[R_{n+1}(x) \pm R_n(x)] - 2^n(3x) \mp 2^{n+1}$.

Proof. 1. Let us assume that $m \geq n$, then using Binet type formula (3.3), we have

$$\begin{aligned} R_m(x)R_n(x) &= (\tau_1^m(x) + \tau_2^m(x))(\tau_1^n(x) + \tau_2^n(x)) \\ &= \tau_1^{m+n}(x) + \tau_2^{m+n}(x) + \tau_1^m(x)\tau_2^n(x) + \tau_1^n(x)\tau_2^m(x) \\ &= R_{n+m}(x) + \tau_1^n(x)\tau_2^n(x)[\tau_1^{m-n}(x) + \tau_2^{m-n}(x)] \\ &= R_{n+m}(x) + 2^n R_{m-n}(x) \quad \text{(using (3.2))}. \end{aligned}$$

2. Proceeding similar to 1 using (3.4), we have

$$\begin{aligned} M_m(x)M_n(x) &= \Big(\frac{\tau_1^m(x) - \tau_2^m(x)}{\sqrt{9x^2-8}}\Big)\Big(\frac{\tau_1^n(x) - \tau_2^n(x)}{\sqrt{9x^2-8}}\Big) \\ &= \frac{\tau_1^{m+n}(x) + \tau_2^{m+n}(x) - \tau_1^m(x)\tau_2^n(x) - \tau_1^n(x)\tau_2^m(x)}{9x^2-8} \\ &= \frac{R_{m+n}(x) - \tau_1^n(x)\tau_2^n(x)[\tau_1^{m-n}(x) + \tau_2^{m-n}(x)]}{9x^2-8} \\ &= \frac{R_{m+n}(x) - 2^n R_{m-n}(x)}{9x^2-8} \quad \text{(Since } \tau_1(x)\tau_2(x) = 2\text{)}. \end{aligned}$$

Rearranging the above equation gives the required identity.
3. On doing product of respective sides of first and second identity, in LHS we get $R^2_{m+n}(x) - 2^{2n}R^2_{m-n}(x)$ and the RHS is

$$\begin{aligned} R^2_{m+n}(x) - 2^{2n}R^2_{m-n}(x) &= (9x^2-8)\big(R_m(x)R_n(x)M_m(x)M_n(x)\big) \\ &= (9x^2-8)M_{2m}(x)M_{2n}(x) \quad \text{(using 1 of Theorem 8)} \\ &= R_{2(m+n)}(x) - 2^n R_{2(m-n)}(x) \quad \text{(using second identity)}. \end{aligned}$$

4, 5, & 6. The fourth identity is obtained from addition of first and second while fifth identity is obtained from difference of first and second. The sixth identity is derived from identity 3.
7. Substituting $m = n+1$ in the first and second identity proves it.
8. The result is established by taking sum (difference) of the seventh identity of Theorem 9 and fourth identity of Theorem 8. □

Theorem 10. *Let m and n be any positive integer, then we have*

1. $R_{2n}(x) = (9x^2-8)M_n^2(x) + 2^{n+1}$.
2. $R^2_{2n}(x) = R_{4n}(x) + 2^{n+1}M_{n+1}$.
3. $2R_{2n}(x) = (9x^2-8)M_n^2(x) + R_n^2(x)$.
4. $R_n^2(x) = (9x^2-8)M_n^2(x) + 2^{n+2}$.
5. $R_{m+1}(x) + 2R_{m-1}(x) = 3xR_m(x)$.

6. $R_{m+1}(x) - 2R_{m-1}(x) = (9x^2 - 8)M_m(x)$.
7. $2R_{m+1}(x) - 3xR_m(x) = (9x^2 - 8)M_m(x)$.
8. $3xR_m(x) - 4R_{m-1}(x) = (9x^2 - 8)M_m(x)$.

Proof. For identities 1 to 4, substituting $m = n$ in identities 2 to 5, respectively, of Theorem 9 and some manipulations proves them. For identities 5 & 6, substituting $n = 1$ in identities 1 to 2 of Theorem 9 gives the required result. Identity 7 is the sum of 5 & 6 and 8 is the difference of 5 & 6. □

Theorem 11. *Let n and t be any positive integers such that $n \geq t$, then we have*

1. $R_n(x)R_{nt}(x) = R_{n(t+1)}(x) + 2^n R_{n(t-1)}(x)$.
2. $M_n(x)M_{nt}(x) = \dfrac{R_{n(t+1)}(x) - 2^n R_{n(t-1)}(x)}{(9x^2 - 8)}$.
3. $R_n(x)R_{nt}(x) + (9x^2 - 8)M_n(x)M_{nt}(x) = 2R_{n(t+1)}(x)$.

Theorem 12. *Let $k, s, s_1, s_2 \in \mathbb{N}$, then the following identities are provided:*

1. $M^{(k)}_{2sk}(x) = M^{(k)}_{sk}(x)R^{(k)}_{sk}(x)$.
2. $R^{(k)}_{2sk}(x) = [(9x^2 - 8)M_s^2(x) + 2^{s+1}]^k$.
3. $2^{nk}R^{(k)}_{k(n-m)}(x) = [R_m(x)R_n(x) - R_{n+m}(x)]^k$.
4. $R^{(k)}_{s_1k}(x)R^{(k)}_{s_2k}(x) = [R_{s_1+s_2}(x) + 2^n R_{s_1-s_2}(x)]^k$.

Proof. 1. Using Theorem 6 and identity 1 of Theorem 8, we have

$$M^{(k)}_{2sk}(x) = M^k_{2s}(x) = [M_s(x)R_s(x)]^k = M^k_s(x)R^k_s(x) = M^{(k)}_{sk}(x)R^{(k)}_{sk}(x).$$

2. Result follows from identity 2 of Theorem 8.
3. Result follows from identity 6 of Theorem 8.
4. Result follows from identity 1 of Theorem 9 and 2 of Theorem 6. □

4.1. Combined combinatorial identities for the Mersenne-Lucas numbers

For positive integers m and n, here we provide several combined combinatorial identities for the Mersenne and Mersenne-Lucas numbers. Since for $x = 1$, $M_n(x)$ and $R_n(x)$ gives M_n and R_n, respectively, so all the results given below can be proved easily by substituting $x = 1$ and some manipulations in the above given identities for $M_n(x)$ and $R_n(x)$. Hence we omitted the proof of below theorems.

Theorem 13. *Let M_n and R_n be the nth Mersenne and Mersenne-Lucas numbers, respectively. Then for $m, n \in \mathbb{N}$, we have the following combined results:*

1. $R_{2n} = M_n^2 + 2^{n+1}$.
2. $R_{2n} = R_n^2 - 2^{n+1}$.
3. $R_n^2(x) = M_n^2 + 2^{n+2}$.
4. $R_m R_n = R_{m+n} + 2^n R_{m-n}$.
5. $M_m M_n = R_{m+n} - 2^n R_{m-n}$.
6. $M_{2m} M_{2n} = R_{m+n}^2 - 2^{2n} R_{m-n}^2$.
7. $2R_{m+n} = R_m R_n + M_m M_n$.
8. $2^{n+1} R_{m-n} = R_m R_n - M_m M_n$.

Theorem 14. *Let m and n be any positive integer, then we have*

1. $R_{2n} = M_n^2 + 2^{n+1}$.
2. $R_{2n}^2 = R_{4n} + 2^{n+1} M_{n+1}$.
3. $2R_{2n} = M_n^2 + R_n^2$.
4. $R_n^2 = M_n^2 + 2^{n+2}$.
5. $R_n R_{nt} = R_{n(t+1)} + 2^n R_{n(t-1)}$.
6. $M_n M_{nt} = R_{n(t+1)} - 2^n R_{n(t-1)}$.
7. $R_{m+1} + 2R_{m-1} = 3R_m$.
8. $R_{m+1} - 2R_{m-1} = M_m$.
9. $2R_{m+1} - 3R_m = M_m$.
10. $3R_m - 4R_{m-1} = M_m$.
11. $R_n R_{nt} + M_n M_{nt} = 2R_{n(t+1)}$.

Theorem 15. *The following combined results holds true for the Mersenne and Mersenne-Lucas numbers.*

1. $R_{m+n}^2 - 2^{2n} R_{m-n}^2 = R_{2(m+n)} - 2^n R_{2(m-n)}$.
2. $R_{2n+1} + R_{2n} = R_n[R_{n+1} + R_n] - 2^n(3) - 2^{n+1}$.
3. $R_{2n+1} - R_{2n} = R_n[R_{n+1} - R_n] - 2^n(3) + 2^{n+1}$.
4. $R_{n+m} R_{n-m} = 2^{n-m}[3xR_m - 2R_{m+1}]^2 + R_{2n} + 2^{n+1}$.
5. $R_{2n+1} = \begin{cases} R_n(x) R_{n+1} - 2^n(3), & \textit{and} \\ M_n M_{n+1} + 2^n(3). \end{cases}$
6. $M_{2n+1} = \begin{cases} M_n R_{n+1} + 2^n, & \textit{and} \\ M_{n+1} R_n - 2^n. \end{cases}$
7. $M_{2n+1} \pm M_{2n} = \begin{cases} M_n(x)[R_{n+1} \pm R_n] + 2^n, & \textit{and} \\ R_n(x)[M_{n+1} \pm M_n] - 2^n. \end{cases}$

5. Conclusion

This article presents a study on two new sequences $\{R_n^{(k)}\}$ and $\{R_n^{(k)}(x)\}$ claiming a new generalization of the Mersenne-Lucas numbers with restrictions n have the structure $n = sk + r, \quad 0 \leq r < k$. The study gives several new identities for these newly sequences in terms of existing Mersenne and Mersenne-Lucas numbers and polynomials. Also, we identified their algebraic properties in connections with existing numbers and polynomials. Moreover, the closed form formula, Cassini's identity, series sums, recurrence relations, etc. are obtained. This study provides several key identities uncovering previously unexplored patterns and relationships.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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