

Extremal Sombor index of chemical trees

Hafiza Munam Chand

National University of Science and Technology, Islamabad, Pakistan
munamchand20@gmail.com

Received: 2 August 2025; Accepted: 5 May 2026

Published Online: 11 May 2026

Abstract: A novel invariant based on vertex degree was recently proposed by Gutman, called the Sombor index, and defined as

$$SO(\mathcal{G}) = \sum_{v_1 v_2 \in \mathcal{E}(\mathcal{G})} \sqrt{d(v_1)^2 + d(v_2)^2},$$

where $d(v_1)$ is the degree of vertex v_1 in G . This paper investigates the extremal values of the Sombor index of chemical trees with exactly one vertex of degree 4. We characterize the tree attaining the maximum value of the Sombor index and provide the expression for their Sombor indices. Furthermore, we identify the minimum trees and demonstrate that these yield unique tree structures. These results contribute to the structural understanding of degree-based invariants in chemical graph theory.

Keywords: topological index, degree-based topological indices, Sombor index.

AMS Subject classification: 05C35, 05C05, 05C09, 05C92

1. Introduction

Graph theory is an area of mathematics focused on the study of graphs, which model relationships between objects. A graph $G = (V(G), E(G))$ is comprised of vertex set $\mathcal{V}(\mathcal{G})$, an edge set $\mathcal{E}(\mathcal{G})$, and an incidence function, denoted as $\psi(\mathcal{G})$, that links every edge with pair of vertices. The graph's size is dependent on the number of edges in $\mathcal{E}(\mathcal{G})$, while the number of vertices determines its order. A graph with n vertices is said to be of order n . The degree of a vertex v_1 is represented by $d_{\mathcal{G}}(v_1)$. A 0 degree vertex is known as an isolated vertex. A vertex with a degree of one is known as a leaf or pendant vertex, while an edge that is connected to a leaf is referred to as a pendant edge [16].

A simple connected acyclic graph is categorized as a tree. This mathematical structure was initially studied by Cayley [5] in 1857. An n -vertex tree always has $n - 1$ edges,

and each edge serves as a bridge, since there is only one distinct path that connects any two vertices.

Chemical graph theory (CGT) is an interdisciplinary field that combines the principles of graph theory and chemistry. It employs graph theory to assess molecules and reveal the physical properties of synthesized compounds.

The molecular topology of graphs is estimated by a topological index that is used to analyze the graphs using chemical and physical attributes such as evaporation, heat production, enthalpy, boiling point, and entropy. A topological index is a numerical representation of the topology of a molecular compound and remains invariant under graph transformations. This has been extensively deployed in various fields such as molecular topology, chemical graph theory, and mathematical chemistry [3]. Topological indices have had a notable impact on the development of QSPR/QSAR, in which chemical structures are associated with biological or other properties of molecules. The degree-based indices holds a special place among various types of topological indices due to their relevance and applications. Chemist Harold Wiener invented the first index in 1947 [23], called Wiener index. Over the years, a wide variety of topological indices has been proposed. For historical background and mathematical properties of topological indices, see [2, 10, 12, 14]. The Sombor index is a degree-based invariant introduced by Gutman [13] and defined as

$$SO(\mathcal{G}) = \sum_{v_1 v_2 \in \mathcal{E}(\mathcal{G})} \sqrt{d(v_1)^2 + d(v_2)^2},$$

where $d(v_1)$ is the degree of vertex v_1 in G .

Some results on the Sombor index can be found here [1, 6–9, 11, 13, 15, 17–21, 24]. Furthermore, the extremal Sombor index of chemical trees with a 0-vertex of degree 4 studied in this paper [4]. Inspired by the above-mentioned literature, this paper contributes to chemical graph theory with the following problem.

Problem statement

Let $\tau_{n,1}$ be the family of n -vertex chemical trees, with exactly one vertex of degree 4, that is, $n_4(T) = 1$. We characterize the extremal trees from a family of $\tau_{n,1}$.

- Identify the trees that maximize the Sombor index in the class $\tau_{n,1}$ and find the expression for the Sombor index.
- Find the minimum tree in the class of $\tau_{n,1}$ and find the general expression for different values of n .

The structure of this paper is as follows: Section 2 contains preliminaries and defines graph operations. Section 3 presents the main results of the stated problem, and the paper is concluded in Section 4.

2. Preliminaries and definitions

If $d(v) \leq 4$ for all $v \in \mathcal{V}(T)$, then T is said to be a chemical tree. The set of vertices and set of edges of the chemical tree are represented by $\mathcal{V}(T)$ and $\mathcal{E}(T)$, respectively. Let n be used to represent the total number of vertices in T , and $n_i = n_i(T)$ is used to represent the total number of vertices of T of degree i . Let $m_{i,j} = m_{i,j}(T)$ represent the total number of edges of T connecting a vertex of degree i with a vertex of degree j . Let $\mathcal{P} = \{(x, y) \in \mathbb{N} \times \mathbb{N} : 1 \leq x \leq y \leq 4\}$. Then the Sombor index is also equivalent to

$$SO(\mathcal{G}) = \sum_{(x,y) \in \mathcal{P}} \sqrt{d(x)^2 + d(y)^2} m_{x,y}. \quad (2.1)$$

Roberto et al. [6] defined some important relations between vertices and edges of chemical graphs.

$$n = n_1(T) + n_2(T) + n_3(T) + n_4(T), \quad (2.2)$$

$$2m_{1,1}(T) + m_{1,2}(T) + m_{1,3}(T) + m_{1,4}(T) = n_1(T), \quad (2.3)$$

$$m_{1,2}(T) + 2m_{2,2}(T) + m_{2,3}(T) + m_{2,4}(T) = 2n_2(T), \quad (2.4)$$

$$m_{1,3}(T) + m_{2,3}(T) + 2m_{3,3}(T) + m_{3,4}(T) = 3n_3(T), \quad (2.5)$$

$$m_{1,4}(T) + m_{2,4}(T) + m_{3,4}(T) + 2m_{4,4}(T) = 4n_4(T). \quad (2.6)$$

We will use these relations later to find the values of unknowns.

There are two fundamental graph operations: edge contraction and vertex identification. Edge contraction is the process of eliminating an edge from a graph while also merging the two endpoints of the edge. The limitation that the contraction must take place over vertices sharing an incident edge is eliminated by vertex identification, also known as vertex contraction. Vertex identification is a less restrictive form of edge contraction.

Definition 1. [22] Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph containing an edge $e = uv$ with $u \neq v$. Let

$$f(x) = \begin{cases} x, & \text{if } x \in \mathcal{V}(\mathcal{G}) \setminus \{u, v\}, \\ h, & \text{otherwise.} \end{cases}$$

The contraction of e results a new graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}')$, where $\mathcal{V}' = (\mathcal{V} \setminus \{u, v\} \cup \{h\})$ and $\mathcal{E}' = \mathcal{E} \setminus \{e\}$.

3. Extremal Sombor index of chemical trees $\tau_{n,1}$

3.1. Maximum Sombor index of family of chemical trees $\tau_{n,1}$

We will determine the maximum Sombor index of a chemical tree from a family of $\tau_{n,1}$ in this section.

Lemma 1. *Let $T \in \tau_{n,1}$ be the tree with the maximum Sombor index. Then $n_2(T) \leq 1$.*

Proof. On contrary, assume that $n_2(T) > 1$. Let two vertices u and v have degree 2 in T . We have two cases:

Case (i). When u and v are adjacent.

Let $uv \in \mathcal{E}(T)$ with $d_T(u) = d_T(v) = 2$. Then $\sqrt{d_T(u)^2 + d_T(v)^2} = \sqrt{8}$. We contract the edge uv and replace the vertices u and v with a new vertex v_1 . Furthermore, we attach a new pendant vertex u_1 to v_1 . Let us call the new tree T' . Then $d_{T'}(u_1) = 1$, $d_{T'}(v_1) = 3$. The adjacent vertices of v_1 are u' and v' other than u_1 . Thus $\sqrt{d_{T'}(u_1)^2 + d_{T'}(v_1)^2} = \sqrt{10}$.

$$\begin{aligned} \mathcal{SO}(T') - \mathcal{SO}(T) &= \sum_{xy \in F'} \sqrt{d_{T'}(x)^2 + d_{T'}(y)^2} + \sqrt{d_{T'}(u_1)^2 + d_{T'}(v')^2} + \sqrt{d_{T'}(u_1)^2 + d_{T'}(u')^2} \\ &\quad + \sqrt{d_{T'}(u_1)^2 + d_{T'}(v_1)^2} - \sum_{xy \in F} \sqrt{d_T(x)^2 + d_T(y)^2} - \sqrt{d_T(u)^2 + d_T(v)^2} \\ &\quad - \sqrt{d_T(u)^2 + d_T(u')^2} - \sqrt{d_T(v)^2 + d_T(v')^2}. \end{aligned}$$

Here $F' = \mathcal{E}(T') \setminus \{u_1v_1, u_1u', u_1v'\}$ and $F = \mathcal{E}(T) \setminus \{uv, uu', vv'\}$. After simplification and putting the values of $d_T(u)$, $d_T(v)$, $d_{T'}(u_1)$ and $d_{T'}(v_1)$, we obtain

$$\mathcal{SO}(T') - \mathcal{SO}(T) = \sqrt{9 + d_{T'}(v')^2} + \sqrt{9 + d_{T'}(u')^2} + \sqrt{10} - \sqrt{8} - \left[4 + d_T(u')^2\right]^{\frac{1}{2}} - \sqrt{4 + d_T(v')^2}.$$

Thus,

$$\mathcal{SO}(T') - \mathcal{SO}(T) > 0.$$

This contradicts that T has largest Sombor index in $\tau_{n,1}$.

Case (ii). When u and v are non-adjacent.

Let u_1, u_2 are adjacent to u and let v_1, v_2 are adjacent to v . Without loss of generality, there exists a path from u to v , and v_2 lies on path while v_1 does not. Now construct a new tree by removing v_1 from v and attach it at u .

$$T' = \{T \setminus \{vv_1\}\} \cup \{uv_1\}.$$

Clearly, $d_{T'}(u) = d_T(u) + 1$, $d_{T'}(v) = d_T(v) - 1$, and $d_{T'}(r_i) = d_T(r_i)$ for all $r_i \in \mathcal{V}(T) \setminus \{u, v\}$. After simplification and putting the values of $d_T(u)$ and $d_T(v)$, we get

$$\begin{aligned}
\mathcal{SO}(T') - \mathcal{SO}(T) &= \sqrt{9 + d_T(u_1)^2} + \sqrt{9 + d_T(u_2)^2} + \sqrt{9 + d_T(v_1)^2} + \sqrt{1 + d_T(v_2)^2} \\
&\quad - \sqrt{4 + d_T(u_1)^2} - \sqrt{4 + d_T(u_2)^2} - \sqrt{4 + d_T(v_1)^2} - \sqrt{4 + d_T(v_2)^2} \\
&= 5 \left[\frac{1}{\sqrt{9 + d_T(u_1)^2} + \sqrt{4 + d_T(u_1)^2}} + \frac{1}{\sqrt{9 + d_T(u_2)^2} + \sqrt{4 + d_T(u_2)^2}} \right. \\
&\quad \left. + \frac{1}{\sqrt{9 + d_T(v_1)^2} + \sqrt{4 + d_T(v_1)^2}} \right] - 3 \left[\frac{1}{\sqrt{1 + d_T(v_2)^2} + \sqrt{4 + d_T(v_2)^2}} \right].
\end{aligned}$$

The first term gets the lowest value by putting the highest values of $d_T(u_1)$, $d_T(u_2)$ and $d_T(v_1)$. Moreover, the latter term gets the highest value when we put the lowest value of $d_T(v_2)$. This implies that,

$$\mathcal{SO}(T') - \mathcal{SO}(T) > 0,$$

which is a contradiction to our assumption. So, $n_2(T) \leq 1$. ■

Lemma 2. *Let $T \in \tau_{n,1}$ and has maximum Sombor index in $\tau_{n,1}$. Then*

$$n_2(T) = \begin{cases} 1, & \text{when } n \text{ is even,} \\ 0, & \text{when } n \text{ is odd.} \end{cases}$$

Proof. From Lemma 1, we know that $n_2(T) \leq 1$. So, we have two possibilities: $n_2(T) = 0$ or $n_2(T) = 1$.

Case 1. On contrary, assume that $n_2(T) = 0$ and n is even, that is, $n = 2i$, for some positive integer i . This gives

$$2i = n_1(T) + n_2(T) + n_3(T) + n_4(T).$$

That is,

$$2i = n_1(T) + 0 + n_3(T) + 1.$$

Here, $n_1(T) + n_3(T) = 2j$, for some positive integer j . Thus $2i \neq 2j + 1$, which is contradiction to our assumption. So, $n_2(T) = 1$, when n is even.

Case 2. On contrary, assume that $n_2(T) = 1$ and n is odd, that is, $n = 2j + 1$, for some positive integer j .

This gives,

$$2j + 1 = n_1(T) + 1 + n_3(T) + 1.$$

Here, $n_1(T) + n_3(T) = 2k$, for some positive integer k . Thus

$$2j \neq 2k + 1,$$

which gives contradiction. So, $n_2(T) = 0$, when n is odd. ■

Lemma 3. *Let $T \in \tau_{n,1}$ with maximum Sombor index among the trees in $\tau_{n,1}$. Then*
 (i) $n_1(T) = \frac{n+3}{2}$, and $n_3(T) = \frac{n-5}{2}$. when $n \geq 7$ is odd.
 (ii) $n_1(T) = \frac{n}{2} + 1$, and $n_3(T) = \frac{n}{2} - 3$, $n \geq 8$ is even.

Proof. (i) Recall equation (2.2)

$$n = n_1(T) + n_2(T) + n_3(T) + n_4(T). \quad (3.1)$$

and from the degree-sum formula, we have

$$2(n-1) = n_1(T) + 2n_2(T) + 3n_3(T) + 4n_4(T). \quad (3.2)$$

By putting the values of $n_2(T) = 0$ from Lemma 2 and $n_4(T) = 1$ in equations (3.1) and (2), we get

$$n = n_1(T) + 0 + n_3(T) + 1. \quad (3.3)$$

$$2(n-1) = n_1(T) + 2(0) + 3n_3(T) + 4(1). \quad (3.4)$$

By solving equations (3.3) and (3.4) simultaneously, we get $n_1(T) = \frac{n+3}{2}$, and $n_3(T) = \frac{n-5}{2}$.

(ii) By putting the values $n_2(T) = 1$ from Lemma 2 and $n_4(T) = 1$ in equations (3.1) and (2), we have,

$$n = n_1(T) + 1 + n_3(T) + 1, \quad (3.5)$$

and

$$2(n-1) = n_1(T) + 2(1) + 3n_3(T) + 4(1). \quad (3.6)$$

By solving equations (3.5) and (3.6) simultaneously, we attain $n_1(T) = \frac{n}{2} + 1$, and $n_3(T) = \frac{n}{2} - 3$. This concludes the proof. \blacksquare

Lemma 4. *Let $T \in \tau_{n,1}$ and has maximum Sombor index. Then the following holds true:*
 (i) $m_{2,2}(T) = m_{3,4}(T) = 0$, if n is even.
 (ii) $m_{2,2}(T) = m_{2,3}(T) = m_{2,4}(T) = 0$, if n is odd.

Proof. (i) When n is even.

In Lemma 1, we proved that $n_2(T) \leq 1$, and from Lemma 2, we obtain $n_2(T) = 1$, when n is even. So, it is quite obvious that $m_{2,2}(T) = 0$.

Next, we claim that $m_{3,4}(T) = 0$. On contrary, assume that there exists an edge $uv \in \mathcal{E}(T)$ such that $d_T(u) = 3$ and $d_T(v) = 4$. Let u_1, u_2 are adjacent to u other than v and v_1, v_2, v_3 are adjacent to v other than u . Since $T \in \tau_{n,1}$ and n is even, therefore there must be at least one vertex of degree 2, say z , such that, $d_T(z) = 2$, and z_1, z_2 are adjacent to z . Without loss of generality, there exists a path from u to z , and u_1 does not lie on that path. Construct a new tree T' as follows:

$$T' = \{T \setminus \{uu_1\}\} \cup \{zu_1\}.$$

Now, $d_{T'}(u) = d_T(u) - 1$ and $d_{T'}(z) = d_T(z) + 1$ and $d_{T'}(r_i) = d_T(r_i)$ for all $r_i \in \mathcal{V}(T) \setminus \{u, z\}$. After simplification and putting the values of $d_T(u)$, $d_T(v)$, and $d_T(z)$. We obtain

$$\begin{aligned} \mathcal{SO}(T') - \mathcal{SO}(T) &= \sqrt{20} + \sqrt{2^2 + d_T(u_2)^2} + \sqrt{3^2 + d_T(u_1)^2} + \sqrt{3^2 + d_T(z_1)^2} + \sqrt{3^2 + d_T(z_2)^2} \\ &\quad - \sqrt{25} - \sqrt{3^2 + d_T(u_1)^2} - \sqrt{3^2 + d_T(u_2)^2} - \sqrt{2^2 + d_T(z_1)^2} - \sqrt{2^2 + d_T(z_2)^2} \\ &= 5 \left[\frac{1}{\sqrt{9 + d_T(z_1)^2} + \sqrt{4 + d_T(z_1)^2}} + \frac{1}{\sqrt{9 + d_T(z_2)^2} + \sqrt{4 + d_T(z_2)^2}} \right] \\ &\quad - 5 \left[\frac{1}{\sqrt{9 + d_T(u_2)^2} + \sqrt{4 + d_T(u_2)^2}} \right] + \sqrt{20} - \sqrt{25}. \end{aligned}$$

The first term attains maximum value when we put the lowest possible value of $d_T(z_1)$ and $d_T(z_2)$. Also, the second term attains its minimum value by putting the highest possible value of $d_T(u_2)$. By the construction of T' the lowest value of $d_T(z_1)$ and $d_T(z_2)$ is 1 and 2. Without loss of generality, we can assume that $d_T(z_1) = 1$, $d_T(z_2) = 2$ and $d_T(u_2) = 3$.

$$\begin{aligned} \mathcal{SO}(T') - \mathcal{SO}(T) &= 5 \left[\frac{1}{\sqrt{10} + \sqrt{5}} + \frac{1}{\sqrt{13} + \sqrt{8}} \right] - 5 \left[\frac{1}{\sqrt{18} + \sqrt{13}} \right] + \sqrt{20} - \sqrt{25}. \\ &\approx 0.542. \end{aligned}$$

Thus,

$$\mathcal{SO}(T') - \mathcal{SO}(T) > 0.$$

This gives contradiction to our assumption that T has the largest Sombor index among $\tau_{n,1}$. This gives $m_{3,4}(T) = 0$.

(ii) When n is odd.

From Lemma 2, we have $n_2(T) = 0$, when n is odd. So, all the edges whose endpoint degree is 2 are zero. This implies that, $m_{2,2}(T) = m_{2,3}(T) = m_{2,4}(T) = 0$. ■

Lemma 5. *Let $T \in \tau_{n,1}$ be a chemical tree with maximum Sombor index. Then for $n > 6$,*

$$m_{1,4}(T) = 3.$$

Proof. On contrary, suppose that, $m_{1,4}(T) < 3$ i.e., $m_{1,4}(T) = k \in \{0, 1, 2\}$. Let u be a vertex with $d_T(u) = 4$ and $N_T(u) = u_i$ here $i = \{1, \dots, 4\}$ According to the k , without loss of generality, we can assume that,

- If $k = 0$, then $\sum_{i=1}^4 d_T(u_i) \geq 2$.
- If $k = 1$, then $\sum_i^{i-1} d_T(u_i) \geq 2$.
- If $k = 2$, then $\sum_i^{i-2} d_T(u_i) \geq 2$.

Let us recall Lemma 1, $n_2(T) \leq 1$ that is $n_2(T) = 0$ or $n_2(T) = 1$. We now treat all the possible values of k in a unified way, using two cases based on the values of $n_2(T)$.

Case 1. When $n_2(T) = 0$. Then there exists one vertex u_j such that $d_T(u_i) = 3$. Let its neighbor other than u be u'_j, u''_j . Construct a new tree:

$$T' = (T \setminus \{u_j u'_j, u_j u''_j\}) \cup \{z u'_j, z u''_j\}.$$

We now compute the change in the Sombor index

$$\begin{aligned} \mathcal{SO}(T') - \mathcal{SO}(T) &= \sqrt{17} + \sqrt{9 + d_{T'}(z'_1)^2} + \sqrt{9 + d_{T'}(u'_j)^2} + \sqrt{9 + d_{T'}(u''_j)^2} \\ &\quad - \sqrt{25} - \sqrt{9 + d_T(u'_j)^2} - \sqrt{9 + d_T(u''_j)^2} - \sqrt{1 + d_T(z_1)^2}. \end{aligned}$$

The difference is positive for all $d_T(z_1) \in \{2, 3\}$. So,

$$\mathcal{SO}(T') > \mathcal{SO}(T)$$

contracting the condition of maximum of T .

Case 2. When $n_2(T) = 1$. Then there exists one vertex u_i such that $d_T(u_i) = 2$. Let its neighbour other than u be u'_i . Construct a new tree:

$$T' = (T \setminus \{u u_i\}) \cup \{z u'_i\}.$$

We now compute the change in the Sombor index

$$\begin{aligned} \mathcal{SO}(T') - \mathcal{SO}(T) &= \sqrt{17} + \sqrt{4 + d_T(u'_i)^2} + \sqrt{4 + d_T(z'_1)^2} \\ &\quad - \sqrt{20} - \sqrt{4 + d_T(u_i)^2} - \sqrt{1 + d_T(z_1)^2}. \end{aligned}$$

With $d_T(z_1) \in \{2, 3\}$ this, it is also positive. So,

$$\mathcal{SO}(T') - \mathcal{SO}(T) > 0$$

again, a contradiction. ■

Lemma 6. *Let $T \in \tau_{n,1}$ and has the maximum Sombor index. Then $m_{1,2}(T) = 0$.*

Proof. To prove this, we have two cases.

Case 1. When n is odd.

From Lemma 2, $n_2(T) = 0$, when n is odd. In this case $m_{1,2}(T) = 0$.

Case 2. When n is even.

From Lemma 2, $n_2(T) = 1$, when n is even. To prove $m_{1,2}(T) = 0$. On contrary, suppose that $m_{1,2}(T) > 0$. Let $v \in \mathcal{V}(T)$ and $d_T(v) = 2$, and v_1, v_2 are adjacent vertices to v and $d_T(v_1) = 1$. We know that $n_4(T) = 1$. Let $u \in \mathcal{V}(T)$ such that $d_T(u) = 4$ and adjacent vertices of u are u_1, u_2, u_3, u_4 . From Lemma 1, $m_{1,4}(T) = 3$. We can assume that $d_T(u_1), d_T(u_2), d_T(u_3) = 1$. There is only one possibility, that is, $d_T(u_4) = 3$ for $n \geq 8$, and $N_T(u_4) = \{u_4', u_4''\}$. There exists a path from u to v and u_4' does not lie on that path. Let us construct a new tree T' as follows:

$$T' = \{T \setminus \{uu_4'\}\} \cup \{vu_4'\}.$$

Clearly, $d_{T'}(u_4) = d_T(u_4) - 1$, $d_{T'}(v) = d_T(v) + 1$, and $d_{T'}(r_i) = d_T(r_i)$ for all $r_i \in \mathcal{V}(T) \setminus \{u_4, v\}$.

$$\begin{aligned} \mathcal{SO}(T') - \mathcal{SO}(T) &= \sqrt{20} + \sqrt{4 + d_T(u_4'')} + \sqrt{10} + \sqrt{9 + d_T(u_4')^2} + \sqrt{9 + d_T(v_2)^2} \\ &\quad - \sqrt{25} - \sqrt{9 + d_T(u_4'')} - \sqrt{9 + d_T(u_4')^2} - \sqrt{5} - \sqrt{4 + d_T(v_2)^2}. \end{aligned}$$

Clearly, $d_T(v_2) = d_T(u_4'') \in \{1, 3\}$. In both cases:

$$\mathcal{SO}(T') - \mathcal{SO}(T) > 0.$$

This is a contradiction to our assumption. So, $m_{1,2}(T) \not> 0$. This implies, $m_{1,2}(T) = 0$. ■

Lemma 7. *Let $T \in \tau_{n,1}$ and has the maximum Sombor index, where $n \geq 8$ is even. Then $m_{1,3}(T) = \frac{n}{3} - 2$, $m_{2,4}(T) = m_{2,3}(T) = 1$, and $m_{3,3}(T) = \frac{n-8}{2}$.*

Proof. From Lemmas 6–7, we know that $m_{1,4}(T) = 3$ and $m_{1,2}(T) = 0$. Also, Lemma 3 shows that, $n_1(T) = \frac{n}{2} + 1$. Putting these values in equation (2.3) we get,

$$m_{1,3}(T) = \frac{n}{2} - 2. \tag{3.7}$$

Solving equation (2.6) by using Lemmas 4 and 5, we have

$$m_{2,4}(T) = 1. \tag{3.8}$$

Solving equation (2.4) by using equation (3.8) and Lemmas 2, 4, and 6, we get

$$m_{2,3}(T) = 1. \quad (3.9)$$

Solving equation (2.5) by using the above equations (3.7), (3.9), and Lemmas 3 and 4, we have

$$m_{3,3}(T) = \frac{n-8}{2}.$$

This completes the proof. ■

Theorem 1. *Let $T \in \tau_{n,1}$ and has the maximum Sombor index, where $n \geq 8$ is even. Then*

$$SO(T) = \left(\frac{n}{2} - 2\right)\sqrt{10} + 3\sqrt{17} + \sqrt{13} + 2\sqrt{5} + (n-8)\left(\frac{3}{\sqrt{2}}\right).$$

Proof. The Sombor index of a tree is defined as

$$\begin{aligned} SO(T) &= \sum_{(x,y) \in P} \sqrt{d(x)^2 + d(y)^2} m_{x,y}. \\ SO(T) &= \sqrt{5}m_{1,2} + \sqrt{10}m_{1,3} + \sqrt{17}m_{1,4} + \sqrt{8}m_{2,2} + \sqrt{13}m_{2,3} \\ &\quad + \sqrt{20}m_{2,4} + \sqrt{18}m_{3,3} + \sqrt{20}m_{3,4} + \sqrt{25}m_{4,4}. \end{aligned}$$

By using Lemmas 4 and 7, we get

$$\begin{aligned} SO(T) &= \left(\frac{n}{2} - 2\right)\sqrt{10} + 3\sqrt{17} + \sqrt{13} + \sqrt{20} + \left(\frac{n-8}{2}\right)\sqrt{18} \\ &= \left(\frac{n}{2} - 2\right)\sqrt{10} + 3\sqrt{17} + \sqrt{13} + 2\sqrt{5} + (n-8)\frac{3}{\sqrt{2}}. \end{aligned}$$

■



Figure 1. Maximum trees when $n = 8$, and $n = 14$.

Lemma 8. *Let $T \in \tau_{n,1}$ and has the maximum Sombor index, where $n \geq 7$ is odd. Then $m_{1,3}(T) = \frac{n-3}{2}$, $m_{3,4}(T) = 1$, and $m_{3,3}(T) = \frac{n-7}{2}$.*

Proof. From Lemma 5, we have $m_{1,4}(T) = 3$ and from Lemma 6, $m_{1,2}(T) = 0$. Also from Lemma 2, and Lemma 3, we attain $n_2(T) = 0$ and $n_1(T) = \frac{n+3}{2}$, respectively. Solving equation (2.3) using the above results, we obtain

$$m_{1,3}(T) = \frac{n-3}{2}. \quad (3.10)$$

As $n_4(T) = 1$, so $m_{4,4}(T) = 0$. From Lemma 4, $m_{2,4}(T) = 0$ and from Lemma 5, $m_{1,4}(T) = 3$. Solving equation (2.6) we get

$$m_{3,4}(T) = 1. \quad (3.11)$$

By using Lemmas 3 and 4, equations (3.10), and (3.11) in equation (2.5), we obtain

$$m_{3,3}(T) = \frac{n-7}{2}. \quad (3.12)$$

The proof is completed. ■

Theorem 2. *Let $T \in \tau_{n,1}$ and has the maximum Sombor index, where $n \geq 7$ is odd. Then*

$$SO(T) = \left(\frac{n-3}{2}\right)\sqrt{10} + 3\sqrt{17} + (n-7)\frac{3}{\sqrt{2}} + 5.$$

Proof. From equation (2.1),

$$\begin{aligned} SO(T) &= \sum_{(x,y) \in P} \sqrt{d(x)^2 + d(y)^2} m_{x,y} \\ &= \sqrt{5}m_{1,2} + \sqrt{10}m_{1,3} + \sqrt{17}m_{1,4} + \sqrt{8}m_{2,2} + \sqrt{13}m_{2,3} \\ &\quad + \sqrt{20}m_{2,4} + \sqrt{18}m_{3,3} + \sqrt{20}m_{3,4} + \sqrt{25}m_{4,4}. \end{aligned}$$

By using Lemmas 4-6, and Lemma 8, we get

$$\begin{aligned} SO(T) &= \left(\frac{n-3}{2}\right)\sqrt{10} + 3\sqrt{17} + \left(\frac{n-7}{2}\right)\sqrt{18} + \sqrt{25} \\ SO(T) &= \left(\frac{n-3}{2}\right)\sqrt{10} + 3\sqrt{17} + (n-7)\frac{3}{\sqrt{2}} + 5. \end{aligned}$$

■

Corollary 1. *The Sombor index of any tree in $\tau_{n,1}$ is*

$$SO(P_n) \leq SO(T) \leq SO(T'),$$

where P_n is the path, T is any tree in $\tau_{n,1}$ and T' is the maximum tree in the class of chemical trees $\tau_{n,1}$.



Figure 2. Maximum trees when $n = 7, 9$.

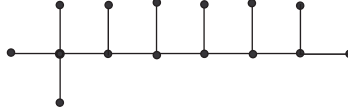


Figure 3. Maximum tree when $n = 15$.

4. Minimum Sombor index of family of chemical trees $\tau_{n,1}$

This section aims to determine the minimum Sombor index of a family of $\tau_{n,1}$.

Lemma 9. *Let $T \in \tau_{n,1}$ and has the minimum Sombor index. Then*

(i) $n_3(T) = 0$.

(ii) $n_1(T) = 4$, and $n_2(T) = n - 5$.

Proof.

(i) On contrary, suppose that $n_3(T) > 0$. Let $v \in \mathcal{V}(T)$ and $d_T(v) = 3$. Let v_1, v_2 and v_3 are adjacent vertices of v . Let there exist a pendant vertex z and $N_T(z) = \{z_1\}$. Without loss of generality, there exists a path from v to z . Let v_1 not lie on that path. Let us construct a new tree T' by removing v_1 from v and attach it at z .

$$T' = \{T \setminus \{vv_1\}\} \cup \{v_1z\}.$$

So, $d_{T'}(v) = d_T(v) - 1$, $d_{T'}(z) = d_T(z) + 1$, and $d_{T'}(r_i) = d_T(r_i)$ for all $r_i \in \mathcal{V}(T) \setminus \{v, z\}$.

$$\begin{aligned} \mathcal{SO}(T') - \mathcal{SO}(T) &= \sqrt{4 + d_T(v_2)^2} + \sqrt{4 + d_T(v_3)^2} + \sqrt{4 + d_T(v_1)^2} + \sqrt{4 + d_T(z_1)^2} \\ &\quad - \sqrt{9 + d_T(v_2)^2} - \sqrt{9 + d_T(v_3)^2} - \sqrt{9 + d_T(v_1)^2} - \sqrt{1 + d_T(z_1)^2}. \\ &= -5 \left[\frac{1}{\sqrt{4 + d_T(v_1)^2} + \sqrt{9 + d_T(v_1)^2}} + \frac{1}{\sqrt{4 + d_T(v_2)^2} + \sqrt{9 + d_T(v_2)^2}} \right. \\ &\quad \left. + \frac{1}{\sqrt{4 + d_T(v_3)^2} + \sqrt{9 + d_T(v_3)^2}} \right] + 3 \left[\frac{1}{\sqrt{4 + d_T(z_1)^2} + \sqrt{1 + d_T(z_1)^2}} \right]. \end{aligned}$$

The first term attains the minimum value while putting the highest possible values of $d_T(v_1)$, $d_T(v_2)$, and $d_T(v_3)$. By the construction of T' highest values of $d_T(v_1) = d_T(v_2) = 2$, and $d_T(v_3) = 4$. Similarly, the last term attains the maximum value while putting lowest possible value of $d_T(z_1)$. By the construction of T' the lowest value of $d_T(z_1) = 2$.

$$\mathcal{SO}(T') - \mathcal{SO}(T) = -1.70.$$

This implies that,

$$\mathcal{SO}(T') - \mathcal{SO}(T) < 0.$$

Which is a contradiction. So, $n_3(T) = 0$.

(ii) As $n_4(T) = 1$, and from (i), $n_3(T) = 0$. Putting the values in equation (2.1) and in the degree-sum formula, we get

$$n = n_1(T) + n_2(T) + 0 + 1, \quad (4.1)$$

$$2(n - 1) = n_1(T) + 2n_2(T) + 0 + 4. \quad (4.2)$$

By solving equations (4.1) and (4.2) simultaneously, we obtain $n_1(T) = 4$, and $n_2(T) = n - 5$. ■

Remark . If the tree T has exactly one vertex of degree 4, that is, $n_4(T) = 1$, then the tree contains exactly 4 leaves in the case of minimum trees.

Lemma 10. *Let $T \in \tau_{n,1}$ be a tree with a minimum Sombor index. Then $m_{1,3}(T) = m_{2,3}(T) = m_{3,3}(T) = m_{3,4}(T) = 0$.*

Proof. From Lemma 9, $n_3(T) = 0$, so all the edges whose end vertex degree is 3 zero. ■

Lemma 11. *Let $T \in \tau_{n,1}$, $3n_4 + 2 \leq n < 5n_4 + 4$ be a tree with minimum Sombor index. Then $m_{2,2}(T) = 0$.*

Proof. On contrary, assume that $m_{2,2}(T) \neq 0$. Let $v, v_1 \in \mathcal{V}(\mathcal{G})$ with $d_T(v) = d_T(v_1) = 2$. Let the adjacent vertices of v and v_1 are v' and v'_1 , respectively. As $T \in \tau_{n,1}$ there must be a vertex of degree 4, such that $d_T(u) = 4$ and $N_T(u) = \{u_1\}$. There exists a path from u_1 to v_1 and v'_1 does not lie on that path. Let us contrive a new tree T' as follows:

$$T' = \{T \setminus \{v_1 v'_1\}\} \cup \{u_1 v'_1\}.$$

Clearly, $d_{T'}(v_1)=d_T(v_1) - 1$, $d_{T'}(u_1)=d_T(u_1) + 1$, and $d_{T'}(r_i)=d_T(r_i)$ for all $r_i \in \mathcal{V}(T)\setminus\{u_1, z_1\}$. Then

$$\mathcal{SO}(T') - \mathcal{SO}(T) = \sqrt{4 + d_T(v_1)'} + \sqrt{20} + \sqrt{5} - \sqrt{17} - \sqrt{8} - \sqrt{4 + d_T(v_1)'}.$$

This implies,

$$\mathcal{SO}(T') - \mathcal{SO}(T) < 0.$$

This completes the proof. ■

Lemma 12. *Let $T \in \tau_{n,1}$, $n \geq 5n_4 + 4$ be the tree with minimum Sombor index. Then $m_{1,4}(T) = 0$.*

Proof. On Contrary, assume that $m_{1,4}(T) > 0$. Let $d_T(u) = 4$ and u_1, u_2, u_3, u_4 are adjacent vertices of u and $d_T(u_1) = 1$. Let there exist a leaf z and adjacent vertex to z is z_1 and $N_T(z_1) = \{z_1'\}$. Without loss of generality, there exists a path from u_1 to z_1 . Construct a new tree T' as follows:

$$T' = \{T \setminus \{zz_1\}\} \cup \{zu_1\}.$$

Clearly, $d_{T'}(z_1) = d_T(z_1) - 1$, $d_{T'}(u_1) = d_T(u_1) + 1$, and $d_{T'}(r_i) = d_T(r_i)$ for all $r_i \in \mathcal{V}(T)\setminus\{u_1, z_1\}$. After simplification and putting the values of $d_T(u)$, $d_T(u_1)$ and $d_T(z_1)$, we obtain

$$\mathcal{SO}(T') - \mathcal{SO}(T) = \sqrt{4 + d_T(z)'} + \sqrt{20} + \sqrt{1 + d_T(z_1)'} - \sqrt{17} - [4 + d_T(z)']^{\frac{1}{2}} - \sqrt{4 + d_T(z_1)'}.$$

Clearly, $d_T(z_1') \neq 1$. From Lemma 9, $n_3(T) = 0$. So, the only possibility is $d_T(z_1') = 2$, which gives

$$\mathcal{SO}(T') - \mathcal{SO}(T) < 0.$$

This gives contradiction to our assumption. So, $m_{1,4}(T) = 0$. ■

Lemma 13. *Let $T \in \tau_{n,1}$ has the minimum Sombor index. Then*

(i) $m_{1,2}(T) = m_{2,4}(T) = n - 5$, and $m_{1,4}(T) = 9 - n$, when $3n_4 + 2 \leq n < 5n_4 + 4$

(ii) $m_{1,2}(T) = m_{2,4}(T) = 4$, and $m_{2,2}(T) = n - 9$, when $n \geq 5n_4 + 4$

Proof. (i) By using Lemma 10. As $n(T) \geq 5$ and $n_4(T) = 1$ so $m_{1,1}(T) = m_{4,4} = 0$ then equations (2.3)–(2.6) become

$$m_{1,2}(T) + m_{1,4}(T) = n_1(T), \tag{4.3}$$

$$m_{1,2}(T) + 2m_{2,2}(T) + m_{2,4}(T) = 2n_2(T), \tag{4.4}$$

$$m_{1,4}(T) + m_{2,4}(T) = 4n_4(T). \tag{4.5}$$

From Lemma 4.4, $m_{2,2}(T) = 0$. Then rewriting the equation (4.4) as

$$m_{1,2}(T) + m_{2,4}(T) = 2n_2(T). \quad (4.6)$$

By subtracting equation (4.6) from equation (4.3), we get

$$m_{1,4}(T) - m_{2,4}(T) = n_1(T) - 2n_2(T) \quad (4.7)$$

Adding equations (4.5) and (4.7), we have

$$2m_{1,4} = n_1 - 2n_2 - 4n_4.$$

As $n_4 = 1$ and putting the value of n_1 and n_2 from Lemma 9, we get

$$m_{1,4}(T) = 9 - n$$

Putting the value of $m_{1,4}(T)$ in equations (4.3) and (4.5), we get $m_{2,4}(T) = n - 5$ and $m_{1,2}(T) = n - 5$.

(ii) From Lemma 12, $m_{1,4}(T) = 0$ and from Lemma 9, $n_1 = 4$ So, equation (4.3) becomes,

$$m_{1,2}(T) = 4. \quad (4.8)$$

Here $n_4(T) = 1$. By using Lemma 12 in (4.5), we get

$$m_{2,4}(T) = 4. \quad (4.9)$$

Using Lemma 9 and equations (4.8) and (4.9) in (4.4),

$$\begin{aligned} 2m_{2,2}(T) + 8 &= 2(n - 5), \\ 2m_{2,2}(T) &= 2n - 18, \\ m_{2,2}(T) &= n - 9. \end{aligned}$$

This concludes the proof. ■

Theorem 3. *Let $T \in \tau_{n,1}$ be the tree with minimum Sombor index among the trees in $\tau_{n,1}$. Then*

$$SO(T) = \begin{cases} (3\sqrt{5} - \sqrt{17})n + 9\sqrt{17} - 15\sqrt{5}, & 3n_4 + 2 \leq n < 5n_4 + 4 \\ 2n\sqrt{2} + 12\sqrt{5} - 18\sqrt{2}, & n \geq 5n_4 + 4. \end{cases}$$

Proof. Rewrite equation (2.1),

$$\begin{aligned} \mathcal{SO}(T) &= \sqrt{5}m_{1,2} + \sqrt{10}m_{1,3} + \sqrt{17}m_{1,4} + \sqrt{8}m_{2,2} + \sqrt{13}m_{2,3} \\ &\quad + \sqrt{20}m_{2,4} + \sqrt{18}m_{3,3} + \sqrt{20}m_{3,4} + \sqrt{25}m_{4,4}. \end{aligned}$$

By using Lemma 10, the above equation can be rewritten as

$$\mathcal{SO}(T) = \sqrt{5}m_{1,2} + \sqrt{17}m_{1,4} + \sqrt{8}m_{2,2} + \sqrt{20}m_{2,4} + \sqrt{25}m_{4,4}. \quad (4.10)$$

- For $3n_4 + 2 \leq n < 5n_4 + 4$.

By using Lemmas 11 and 13, equation (4.10) becomes

$$\begin{aligned} \mathcal{SO}(T) &= \sqrt{5}m_{1,2} + \sqrt{17}m_{1,4} + \sqrt{20}m_{2,4} \\ &= (n-5)\sqrt{5} + (9-n)\sqrt{17} + (n-5)\sqrt{20} \\ &= (n-5)(\sqrt{5} + \sqrt{20}) + (9-n)\sqrt{17}. \end{aligned}$$

By simplifying the above term, we get $\mathcal{SO}(T) = (3\sqrt{5} - \sqrt{17})n + 9\sqrt{17} - 15\sqrt{5}$.

- For $n \geq 5n_4 + 4$, equation (4.10) can be written as by using Lemma 12,

$$\mathcal{SO}(T) = \sqrt{5}m_{1,2} + \sqrt{8}m_{2,2} + \sqrt{20}m_{2,4}.$$

By using Lemma 13, the above equation becomes $\mathcal{SO}(T) = 4\sqrt{5} + (n-9)\sqrt{8} + 4\sqrt{20} = 2n\sqrt{2} + 12\sqrt{5} - 18\sqrt{2}$. ■



Figure 4. Unique trees when $n = 5$, and $n = 6$.

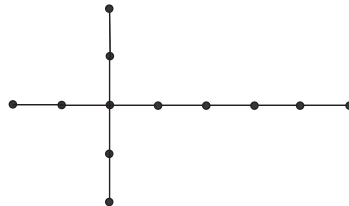


Figure 5. Tree with minimum Sombor index when $n = 12$.

Corollary 2. In the class of $\tau_{n,1}$ the trees are unique for $n = 5$, and $n = 6$.

5. Conclusion

Within this work, we determine the trees that attain the maximum and minimum Sombor index in the class $\tau_{n,1}$. For the maximum Sombor index, two distinct cases arise, even and odd, based on the value of n . When n is even, $n_2(T) = 1$; for odd values of n , $n_2(T) = 0$. However, the values of $n_1(T)$ and $n_2(T)$ depend on the specific value of n . In the minimum Sombor index, two structural regimes appear: one for $n < 9$ and another for $n \geq 9$. The following table summarizes the results for the extremal Sombor index of this class.

Case	Condition on n	Sombor Index
Maximum	Even $n \geq 8$	$\mathcal{SO}(T) = \left(\frac{n}{2} - 2\right)\sqrt{10} + 3\sqrt{17} + \sqrt{13} + 2\sqrt{5} + (n - 8)\left(\frac{3}{\sqrt{2}}\right)$.
	Odd $n \geq 7$	$\mathcal{SO}(T) = \left(\frac{n-3}{2}\right)\sqrt{10} + 3\sqrt{17} + (n - 7)\frac{3}{\sqrt{2}} + 5$.
Minimum	$5 \leq n < 9$	$\mathcal{SO}(T) = (3\sqrt{5} - \sqrt{17})n + 9\sqrt{17} - 15\sqrt{5}$
	$n \geq 9$	$\mathcal{SO}(T) = 2n\sqrt{2} + 12\sqrt{5} - 18\sqrt{2}$

Table 1. Sombor index of maximum and minimum trees for n

It is noteworthy that the extremal trees are unique for $n = 5$ and $n = 6$. These unique cases are listed under the minimum category, as their structural configuration more closely resembles the behavior observed in other minimal Sombor index trees. Furthermore, the study of k -vertices of degree 4, i.e., $\tau_{n,k}$, remains an interesting direction for future work.

Conflict of Interest: The author declares that he has no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

- [1] F. Asif, Z. Zahid, M.N. Husin, M. Cancan, Z. Taş, M. Alaeiyan, and M.R. Farahani, *On Sombor indices of line graph of silicate carbide $Si_2C_3-I[p, q]$* , J. Discrete Math. Sci. Cryptogr. **25** (2022), no. 1, 301–310.
<https://doi.org/10.1080/09720510.2022.2043621>.
- [2] A.T. Balaban and O. Ivanciuc, *Historical development of topological indices*, Topological indices and related descriptors in QSAR and QSPR, CRC Press, 2000, pp. 31–68.
- [3] D. Bonchev, *Information Theoretic Indices for Characterization of Chemical Structures*, Research Studies Press, 1983.

- [4] S. Bonte, G. Devillez, V. Dusollier, A. Hertz, and H. Mélot, *Extremal chemical graphs of maximum degree at most 3 for 33 degree-based topological indices*, 2025. <https://arxiv.org/10.48550/arXiv.2501.02246>.
- [5] A. Cayley, *On the theory of the analytical forms called trees*, The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **13** (1890), no. 85, 172–176. <https://doi.org/10.1080/14786445708642275>.
- [6] R. Cruz, I. Gutman, and J. Rada, *Sombor index of chemical graphs*, Appl. Math. Comput. **399** (2021), 126018. <https://doi.org/10.1016/j.amc.2021.126018>.
- [7] K. Das and I. Gutman, *On Sombor index of trees*, Appl. Math. Comput. **412** (2022), 126575. <https://doi.org/10.1016/j.amc.2021.126575>.
- [8] K.C. Das, A. Ghalavand, and A.R. Ashrafi, *On a conjecture about the Sombor index of graphs*, Symmetry **13** (2021), no. 10, 1830. <https://doi.org/10.1080/09720510.2022.2043621>.
- [9] K.C. Das, M. Imran, and T. Vetrik, *General Sombor index of graphs and trees*, J. Discrete Math. Sci. Cryptogr. **28** (2025), 101–111. <https://doi.org/10.47974/JDMSC-1918>.
- [10] E. Estrada, *Generalization of topological indices*, Chemical Physics Letters **336** (2001), no. 3–4, 248–252. [https://doi.org/10.1016/S0009-2614\(01\)00127-0](https://doi.org/10.1016/S0009-2614(01)00127-0).
- [11] N. Ghanbari and S. Alikhani, *Sombor index of certain graphs*, Iranian J. Math. Chem. **12** (2021), no. 1, 27–37. <https://doi.org/10.22052/ijmc.2021.242106.1547>.
- [12] I. Gutman, *Degree-based topological indices*, Croatica Chemica Acta **86** (2013), no. 4, 351–361.
- [13] ———, *Geometric approach to degree-based topological indices: Sombor indices*, MATCH Commun. Math. Comput. Chem. **86** (2021), no. 1, 11–16.
- [14] I. Gutman and J. Tošović, *Testing the quality of molecular structure descriptors. Vertex-degree-based topological indices*, J. Serbian Chem. Soc. **78** (2013), no. 6, 805–810. <https://doi.org/10.2298/JSC121002134G>.
- [15] A.E. Hamzaa, Z. Raza, A. Ali, and Z. Alsheekhussaina, *On Sombor indices of tricyclic graphs*, MATCH Commun. Math. Comput. Chem. **90** (2023), no. 1, 223–234. <https://doi.org/10.46793/match.90-1.223H>.
- [16] M. Kano, G.Y. Katona, and Z. Király, *Packing paths of length at least two*, Discrete Math. **283** (2004), no. 1–3, 129–135. <https://doi.org/10.1016/j.disc.2004.01.016>.
- [17] T. Li and B. Li, *Ordering unicyclic graphs with a fixed girth by p -Sombor indices*, Journal of Advances in Mathematics **24** (2025), 12–23. <https://doi.org/10.24297/jam.v24i.9714>.
- [18] B.A. Rather, M. Imran, and S. Pirzada, *Sombor index and eigenvalues of comar-*

- imal graphs of commutative rings*, J. Algebra Appl. **23** (2024), no. 06, 2450115.
<https://doi.org/10.1142/S0219498824501159>.
- [19] Z. Tang, Q. Li, and H. Deng, *Trees with extremal values of the Sombor-index-like graph invariants*, MATCH Commun. Math. Comput. Chem. **90** (2023), no. 1, 203–222.
<https://doi.org/10.46793/match.90-1.203T>.
- [20] Ž. Vukićević, *On the Sombor index of chemical trees*, Math. Montisnigri **L** (2021), 5–14.
<https://doi.org/10.20948/mathmontis-2021-50-1>.
- [21] P. Wei and M. Liu, *Note on Sombor index of connected graphs with given degree sequence*, Discrete Appl. Math. **330** (2023), 51–55.
<https://doi.org/10.1016/j.dam.2023.01.002>.
- [22] D.B. West, *Introduction to Graph Theory (2nd edition)*, 2000.
- [23] H. Wiener, *Structural determination of paraffin boiling points*, J. American Chem. Soc. **69** (1947), no. 1, 17–20.
<https://doi.org/10.1021/ja01193a005>.
- [24] T. Zhou, Z. Lin, and L. Miao, *The extremal Sombor index of trees and unicyclic graphs with given matching number*, J. Discrete Math. Sci. Cryptogr. **26** (2022), 1–12.
<https://doi.org/10.1080/09720529.2021.2015090>.