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Research Article

Remarks on bounds for the Laplacian spread of a graph

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Abstract: The Laplacian spread of a simple graph is defined as the difference between the largest and the second-smallest eigenvalues of its Laplacian matrix. In this work, we derive bounds for the Laplacian spread, providing conditional refinements of Theorems 3.1 and 4.1 in [X. Chen and K.C. Das, Some results on the Laplacian spread of a graph, Linear Algebra Appl. 505 (2016), 245–260]. In addition, we present examples that illustrate the independence of the obtained bounds and show that, for these examples, each bound yields a sharper estimate than the corresponding result in [X. Chen and K.C. Das, Some results on the Laplacian spread of a graph, Linear Algebra Appl. 505 (2016), 245–260].

Keywords: simple graphs, Laplacian matrix, Laplacian eigenvalues, Laplacian spread.

AMS Subject classification: 05C50, 15A45

1. Introduction

Let $G = (V, E)$ be a simple graph with n vertices and m edges, where $V = \{v_1, v_2, \ldots, v_n\}$ denotes the set of vertices and $E \subseteq V \times V$ denotes the set of edges. Let d_i be the degree of vertex v_i. For a graph with n vertices, the adjacency matrix $A(G) = [a_{ij}]$ is an $n \times n$ matrix defined as

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge between } v_i \text{ and } v_j \\ 0, & \text{otherwise.} \end{cases}$$

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The Laplacian matrix $L(G)$ of a graph G is derived from the adjacency matrix $A(G)$ and the degree matrix $D(G)$, where $D(G) = \operatorname{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix containing the degrees of the vertices v_i. It is defined as $L(G) = D(G) - A(G)$. Further, $L(G)$ is symmetric and positive semidefinite, and hence possesses non-negative real eigenvalues. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the eigenvalues of $L(G)$. It is well known that $\mu_n = 0$, and its multiplicity equals the number of connected components of G. Therefore, $\mu_{n-1} > 0$ if and only if G is connected, and μ_{n-1} is commonly referred to as the algebraic connectivity of G. For more details, see [6, 11, 12, 14, 15, 17, 18]. Let $\Delta(G)$ and $\delta(G)$ be the largest and smallest degree of graph G, respectively. Then, a graph G is said to be k-regular if $\Delta(G) = \delta(G) = k$. Moreover, a graph G is complete if $\Delta(G) = \delta(G) = n - 1$.
The first Zagreb index [13] is defined as

$$M_1(G) = \sum_{i=1}^{n} d_i^2.$$

The term Laplacian spread of a graph was introduced by Fan et al. [10] and is defined as

$$S_L(G) = \mu_1 - \mu_{n-1},$$

that is, the difference between the largest and the second smallest Laplacian eigenvalues. Beginning with Fan et al. [10], several authors have obtained bounds for the Laplacian spread of a graph in terms of easily computable quantities such as $n, m, \Delta(G), \delta(G), M_1(G)$, etc.; see [1–5, 7–9] and the references therein.

The main goal of this paper is to establish new lower and upper bounds for the Laplacian spread of a simple graph. Section 1 provides background material and a brief overview of related work. In Section 2, we introduce the necessary notation and auxiliary lemmas used throughout the paper. The main results are presented and proved in Section 3.

2. Preliminaries

Here we present some notation and lemmas that will be helpful in deriving the main results in Section 3.
Let μ_i be the eigenvalues of $L(G)$. As one of the eigenvalues of $L(G)$ is zero, we can define the arithmetic mean of the remaining $n-1$ eigenvalues of $L(G)$ as

$$\bar{\mu} = \sum_{i=1}^{n-1} \frac{\mu_i}{n-1} = \frac{2m}{n-1}.$$

Let T and Q denote the total numbers of triangles and quadrilaterals in the graph G, respectively. Let $S_l = \sum_{i=1}^{n} {d_i}^l, C_{dd} = \sum_{v_i \sim v_j} d_i d_j, C_{dt} = \sum_{v_i \in V} d_i t_i$, where d_i

and t_i represent the degree and the number of triangles incident to vertex v_i in G, respectively.

Lemma 1. *[16] Let G be a simple graph with $n \geq 2$ vertices and $m \geq 1$ edges. Then*

$$\begin{aligned}
(i) \quad & \sum_{i=1}^{n-1} \mu_i \ = S_1 = 2m, \\
(ii) \quad & \sum_{i=1}^{n-1} {\mu_i}^2 = S_1 + S_2 = 2m + M_1(G) = \alpha \ \textit{(say)}, \\
(iii) \quad & \sum_{i=1}^{n-1} {\mu_i}^3 = 3S_2 + S_3 - 6T = \beta, \\
(iv) \quad & \sum_{i=1}^{n-1} {\mu_i}^4 = -S_1 + 2S_2 + 4S_3 + S_4 + 8Q + 4C_{dd} - 8C_{dt} = \gamma.
\end{aligned}$$

Lemma 2. *Let G be a simple graph with $n \geq 2$ vertices and $m \geq 1$ edges. Then*

$$\begin{aligned}
(i) \quad & \sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^2 = \alpha - \frac{4m^2}{n-1} = K_1 \ \textit{(say)}, \\
(ii) \quad & \sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^4 = \gamma - \frac{8\beta m}{n-1} + \frac{24m^2\alpha}{(n-1)^2} - \frac{48m^4}{(n-1)^3} = K_2.
\end{aligned}$$

Proof. We can write

$$\sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^2 = \sum_{i=1}^{n-1} \mu_i^2 - (n-1)\bar{\mu}^2, \tag{2.1}$$

and

$$\sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^4 = \sum_{i=1}^{n-1} \mu_i^4 - 4\bar{\mu} \sum_{i=1}^{n-1} \mu_i^3 + 6\bar{\mu}^2 \sum_{i=1}^{n-1} \mu_i^2 - 3(n-1)\bar{\mu}^4. \tag{2.2}$$

The proof of Lemma 2 now follows immediately by combining (2.1), (2.2), and Lemma 1. □

The following two results were given by Chen and Das [7].

Lemma 3. *Let G be a simple graph with $n \geq 5$ vertices and $m \geq 1$ edges. Then*

$$S_L(G) \leq \sqrt{2M_1(G) + 4m - \frac{8m^2}{n-1}} = \sqrt{2K_1}. \tag{2.3}$$

Lemma 4. *Let G be a simple connected graph with $n \geq 2$ vertices and $m \geq 1$ edges. Then*

$$S_L(G) \geq \frac{2}{n-1}\sqrt{(n-1)(2m + M_1(G)) - 4m^2} = 2\sqrt{\frac{K_1}{n-1}}. \tag{2.4}$$

3. Main results

We begin by presenting the following theorem, which provides an upper bound for the Laplacian spread of a simple connected graph under a structural dispersion condition.

Theorem 1. *Let G be a simple connected graph on $n \geq 4$ vertices with $m \geq 1$ edges. If $(n-1)\left(2m + M_1(G)\right) - 8m^2 \geq 0$, then*

$$S_L(G) \leq \sqrt{\frac{2(n-1)}{n-3}\left(2m + M_1(G) - \frac{4m^2}{n-1} - \frac{\left(2m + M_1(G)\right)^2}{8m^2}\right)}. \tag{3.1}$$

Proof. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the eigenvalues of $L(G)$. Then, for any real number c, we have

$$\sum_{i=1}^{n-1}(\mu_i - \bar{\mu})^2 = \sum_{i=1}^{n-1}(\mu_i - c)^2 - (n-1)(\bar{\mu} - c)^2. \tag{3.2}$$

Substituting $c = \frac{\mu_1 + \mu_{n-1}}{2}$ in (3.2), we observe that

$$\begin{aligned} \sum_{i=1}^{n-1}\mu_i^2 - (n-1)\bar{\mu}^2 &= \sum_{i=1}^{n-1}(\mu_i - \bar{\mu})^2 \\ &= \frac{1}{2}(\mu_1 - \mu_{n-1})^2 + \sum_{i=2}^{n-2}\left(\mu_i - \frac{\mu_1 + \mu_{n-1}}{2}\right)^2 \\ &\quad - (n-1)\left(\bar{\mu} - \frac{\mu_1 + \mu_{n-1}}{2}\right)^2. \end{aligned} \tag{3.3}$$

Also, the Cauchy-Schwarz inequality for real numbers s_i and r_i; $i = 1, 2, \ldots, n$ is given by

$$\sum_{i=1}^{n}{s_i}^2\sum_{i=1}^{n}{r_i}^2 \geq \left(\sum_{i=1}^{n}s_ir_i\right)^2. \tag{3.4}$$

Applying (3.4) for $r_i = 1$ and $s_i = \mu_i - \frac{\mu_1 + \mu_{n-1}}{2}, i = 2, \ldots, n-2$, we find that

$$\frac{1}{n-3}\sum_{i=2}^{n-2}\left(\mu_i - \frac{\mu_1 + \mu_{n-1}}{2}\right)^2 \geq \left(\frac{1}{n-3}\sum_{i=2}^{n-2}\left(\mu_i - \frac{\mu_1 + \mu_{n-1}}{2}\right)\right)^2. \tag{3.5}$$

It is easy to verify that

$$\sum_{i=2}^{n-2}\left(\mu_i - \frac{\mu_1 + \mu_{n-1}}{2}\right) = (n-1)\left(\bar{\mu} - \frac{\mu_1 + \mu_{n-1}}{2}\right). \tag{3.6}$$

Combining (3.3), (3.5) and (3.6), we find that

$$\sum_{i=1}^{n-1} \mu_i^2 - (n-1)\bar{\mu}^2 \geq \frac{1}{2}(\mu_1 - \mu_{n-1})^2 + \frac{2(n-1)}{(n-3)}\left(\bar{\mu} - \frac{\mu_1 + \mu_{n-1}}{2}\right)^2. \tag{3.7}$$

Note that for $i = 1, 2, \ldots, n-1$, we have $(\mu_1 - \mu_i)(\mu_i - \mu_{n-1}) \geq 0$. Simplifying and then adding these $n-1$ inequalities, we find that

$$\sum_{i=1}^{n-1} \mu_i^2 \leq (n-1)\bar{\mu}(\mu_1 + \mu_{n-1}) - (n-1)\mu_1\mu_{n-1}, \tag{3.8}$$

which implies that

$$\begin{aligned} \frac{\mu_1 + \mu_{n-1}}{2} - \bar{\mu} &\geq \frac{\sum_{i=1}^{n-1} \mu_i^2 - 2(n-1)\bar{\mu}^2 + (n-1)\mu_1\mu_{n-1}}{2\bar{\mu}(n-1)} \\ &\geq \frac{\sum_{i=1}^{n-1} \mu_i^2 - 2(n-1)\bar{\mu}^2}{2\bar{\mu}(n-1)}. \end{aligned} \tag{3.9}$$

Using Lemma 1, the right side of the inequality (3.9) is non-negative. Combining (3.7) and (3.9), we obtain

$$\sum_{i=1}^{n-1} \mu_i^2 - (n-1)\bar{\mu}^2 \geq \frac{1}{2}(\mu_1 - \mu_{n-1})^2 + \frac{2(n-1)}{(n-3)}\left(\frac{\sum_{i=1}^{n-1} \mu_i^2 - 2(n-1)\bar{\mu}^2}{2\bar{\mu}(n-1)}\right)^2. \tag{3.10}$$

A little calculation in (3.10) directly leads to (3.1). ☐

The following theorem provides a lower bound for the Laplacian spread of a simple connected graph, provided that $(n-1)(2m + M_1(G)) - 8m^2 \geq 0$.

Theorem 2. *Let G be a simple connected graph on $n \geq 2$ vertices with $m \geq 1$ edges. If $(n-1)(2m + M_1(G)) - 8m^2 \geq 0$, then*

$$S_L(G) \geq \frac{2m + M_1(G)}{2m}. \tag{3.11}$$

Proof. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the eigenvalues of $L(G)$. Then, from (3.9), we have

$$\bar{\mu} \leq \frac{\mu_1 + \mu_{n-1}}{2} - \frac{\sum_{i=1}^{n-1} \mu_i^2 - 2(n-1)\bar{\mu}^2}{2\bar{\mu}(n-1)} = \tau \ (say). \tag{3.12}$$

Note that if $\sum_{i=1}^{n-1}\mu_i^2-2(n-1)\bar{\mu}^2\geq 0$, then $0\leq\bar{\mu}\leq\tau\leq\frac{\mu_1+\mu_{n-1}}{2}$.
Since the function $f(x)=(\mu_1-x)(x-\mu_{n-1})$ is monotonically increasing in the interval $\left[\mu_{n-1},\frac{\mu_1+\mu_{n-1}}{2}\right]$, therefore

$$\begin{aligned}(\mu_1-\bar{\mu})(\bar{\mu}-\mu_{n-1})&\leq(\mu_1-\tau)(\tau-\mu_{n-1})\\&=\frac{1}{4}(\mu_1-\mu_{n-1})^2-\left(\frac{\sum_{i=1}^{n-1}\mu_i^2-2(n-1)\bar{\mu}^2}{2\bar{\mu}(n-1)}\right)^2.\end{aligned}\tag{3.13}$$

Also from (3.8), we find that

$$\begin{aligned}\sum_{i=1}^{n-1}\mu_i^2-(n-1)\bar{\mu}^2&\leq-(n-1)\bar{\mu}^2+(n-1)\bar{\mu}\left(\mu_1+\mu_{n-1}\right)+(n-1)\mu_1\mu_{n-1}\\&=(n-1)(\mu_1-\bar{\mu})(\bar{\mu}-\mu_{n-1}).\end{aligned}\tag{3.14}$$

Combining (3.13) and (3.14), we get

$$\frac{1}{4}(\mu_1-\mu_{n-1})^2\geq\frac{\sum_{i=1}^{n-1}\mu_i^2-(n-1)\bar{\mu}^2}{n-1}+\left(\frac{\sum_{i=1}^{n-1}\mu_i^2-2(n-1)\bar{\mu}^2}{2\bar{\mu}(n-1)}\right)^2.\tag{3.15}$$

A little simplification in (3.15) directly leads to (3.11). □

Remark 1. Observe that the inequality (3.1) can be equivalently written as

$$S_L\left(G\right)\leq\sqrt{2K_1-\frac{4(n-1)}{n-3}\left(\frac{(n-1)K_1-4m^2}{4m(n-1)}\right)^2}.$$

Thus, the inequality (3.1) provides refinement of (2.3) under the hypothesis of Theorem 1.
Similarly, the inequality (3.11) can also be expressed as

$$S_L\left(G\right)\geq 2\sqrt{\frac{K_1}{n-1}+\left(\frac{(n-1)K_1-4m^2}{4m(n-1)}\right)^2}.$$

Therefore, the inequality (3.11) also gives refinement of (2.4) under the hypothesis of Theorem 2.

The following Theorems 3–5 establish lower bounds for the Laplacian spread of a simple graph involving K_1 and K_2, as defined in Lemma 2.

Theorem 3. *Let G be a simple connected graph on $n \geq 2$ vertices with $m \geq 1$ edges. Then*

$$S_L(G) \geq \left(\frac{8K_2}{n-1} + \frac{8{K_1}^2}{(n-1)^2} \right)^{1/4}. \tag{3.16}$$

Proof. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the eigenvalues of $L(G)$. Then, $a \leq x_i \leq b$ for $i = 1, 2, \ldots, n-1$, where $x_i = \mu_i - \bar{\mu}, a = \mu_{n-1} - \bar{\mu}$ and $b = \mu_1 - \bar{\mu}$. It is easy to see that

$$\left(x_i + \frac{a+b}{2} \right)^2 (x_i - a)(x_i - b) \leq 0, \qquad 1 \leq i \leq n-1. \tag{3.17}$$

Simplifying and adding the resulting $n-1$ inequalities in (3.17), we get

$$\sum_{i=1}^{n-1} x_i^4 \leq \left(\frac{3}{4}(a+b)^2 - ab \right) \sum_{i=1}^{n-1} x_i^2 - \frac{1}{4}(n-1)(a+b)^2 ab. \tag{3.18}$$

Similarly, $(x_i - a)(x_i - b) \leq 0$, for $a \leq x_i \leq b$. Therefore

$$\sum_{i=1}^{n-1} x_i^2 \leq -(n-1)ab. \tag{3.19}$$

Combining (3.18) and (3.19), we get

$$\sum_{i=1}^{n-1} {x_i}^4 \leq -(n-1)ab(a^2 + b^2 + ab). \tag{3.20}$$

Therefore, we have

$$\sum_{i=1}^{n-1} {x_i}^4 + (n-1)a^2b^2 \leq -(n-1)ab(a^2 + b^2). \tag{3.21}$$

Applying the AM-GM inequality to the right side of (3.21) and then combining with (3.19), we find that

$$\sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^4 + \frac{\left(\sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^2\right)^2}{n-1} \leq (n-1)\frac{(\mu_1 - \mu_{n-1})^4}{8}. \tag{3.22}$$

The proof of Theorem 3 follows from (3.22) using Lemma 2. □

Theorem 4. *Let G be a simple connected graph on $n \geq 2$ vertices with $m \geq 1$ edges. Then*

$$S_L(G) \geq \left(\frac{12K_2}{n-1}\right)^{1/4}. \tag{3.23}$$

Proof. The proof of the theorem follows directly from (3.20) by applying the AM-GM inequality to the right-hand side of (3.20) and then using Lemma 2. □

Theorem 5. *Let G be a simple connected graph on $n \geq 2$ vertices with $m \geq 1$ edges. Then*

$$S_L\left(G\right) \geq \left(\frac{3^5 K_1 K_2}{4(n-1)^2}\right)^{1/6}. \tag{3.24}$$

Proof. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} \geq \mu_n = 0$ be the eigenvalues of $L\left(G\right)$. Then, $a \leq x_i \leq b$ for $i = 1, 2, \dots, n-1$, where $x_i = \mu_i - \bar{\mu}, a = \mu_{n-1} - \bar{\mu}$ and $b = \mu_1 - \bar{\mu}$. Multiplying $\sum_{i=1}^{n-1} x_i^2$ on both sides of (3.18), we have

$$\sum_{i=1}^{n-1} x_i^4 \sum_{i=1}^{n-1} x_i^2 \leq \left[\left(\frac{3}{4}(b-a)^2 + 2ab\right) \sum_{i=1}^{n-1} x_i^2 - (n-1)\left(\frac{1}{4}(b-a)^2 + ab\right) ab\right] \sum_{i=1}^{n-1} x_i^2. \tag{3.25}$$

Combining (3.19) and (3.25), we get

$$\begin{aligned}\sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^4 \sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^2 \leq & (n-1)^2 (\mu_1 - \bar{\mu})^2 (\mu_{n-1} - \bar{\mu})^2 ((\mu_1 - \mu_{n-1})^2 \\ & + 3(\mu_1 - \bar{\mu})(\mu_{n-1} - \bar{\mu})).\end{aligned} \tag{3.26}$$

Consider $f(x) = (\mu_1 - x)^2 (\mu_{n-1} - x)^2 \left((\mu_1 - \mu_{n-1})^2 + 3(\mu_1 - x)(\mu_{n-1} - x)\right)$. Thus, $f(x)$ achieves its maximum value in the interval $[\mu_{n-1}, \mu_1]$ at $x = \frac{\mu_1 + 2\mu_{n-1}}{3}$ and $x = \frac{2\mu_1 + \mu_{n-1}}{3}$, which is $\frac{4}{3^5}(\mu_1 - \mu_{n-1})^6$. Hence, from (3.26), we have

$$\sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^4 \sum_{i=1}^{n-1} (\mu_i - \bar{\mu})^2 \leq \frac{4}{3^5}(n-1)^2 (\mu_1 - \mu_{n-1})^6. \tag{3.27}$$

The inequality (3.24) follows from (3.27) and using Lemma 2. □

Corollary 1. *Under the hypothesis of Lemma 2, we have $K_2 \geq \frac{1}{n-1} K_1^2$.*

Proof. The proof follows by applying (3.4) for $r_i = 1$ and $s_i = (\mu_i - \bar{\mu})^2$, where $1 \leq i \leq n-1$. □

We now show the effectiveness of our results by means of examples.

Example 1. Let G_1, G_2, G_3, and G_4 be simple connected graphs as given below.

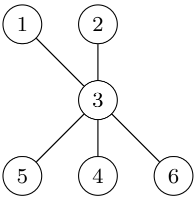

Figure 1. G_1

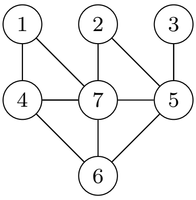

Figure 2. G_2

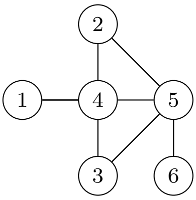

Figure 3. G_3

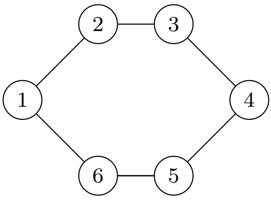

Figure 4. G_4

We now compare our results for these graphs to the bounds given by Chen and Das [7]. Table 1 clearly demonstrates that Theorems 3–5 are independent of each other and provide better approximations than (2.4).

Table 1. **Comparison Table**

	K_1	K_2	$S_L(G)$	(2.4)	(3.16)	(3.23)	(3.24)
G_1	20	260	5	4	4.8295	4.9980	4.8262
G_2	21.3333	117.7778	5.2784	3.7712	4.0085	3.9176	4.0231
G_3	16.8000	72.2560	4.4721	3.6661	3.7882	3.6288	3.7870
G_4	7.2	14.4960	3	2.4000	2.5114	2.4286	2.5159

We now establish a theorem that conditionally sharpens (2.4) for simple connected graphs, and is optimal among Theorems 3-5.

Theorem 6. *Let G be a simple connected graph on $n \geq 2$ vertices and $m \geq 1$ edges. If $K_2 \geq \frac{3^7 K_1^2}{4^5(n-1)}$, then*

$$S_L(G) \geq \left(\frac{12K_2}{n-1}\right)^{1/4}. \tag{3.28}$$

Moreover, if $K_2 \leq \frac{3^7 K_1^2}{4^5(n-1)}$*, then*

$$S_L\left(G\right) \geq \left(\frac{3^5 K_1 K_2}{4(n-1)^2}\right)^{1/6} \quad \text{for} \quad \left(\frac{K_2}{n-1} + \frac{K_1^2}{(n-1)^2}\right)^3 - \frac{3^{10} K_1^2 K_2^2}{2^{13}(n-1)^4} \leq 0 \tag{3.29}$$

and

$$S_L\left(G\right) \geq \left(\frac{8K_2}{n-1} + \frac{8{K_1}^2}{(n-1)^2}\right)^{1/4} \quad \text{for} \quad \left(\frac{K_2}{n-1} + \frac{K_1^2}{(n-1)^2}\right)^3 - \frac{3^{10} K_1^2 K_2^2}{2^{13}(n-1)^4} \geq 0. \tag{3.30}$$

Proof. The proof of Theorem 6 directly follows by combining Theorems 3-5. ☐

Remark 2. Notice that Theorem 3 always provides a better estimate than (2.4) using Corollary 1; hence, Theorem 6 also gives a refinement of (2.4) for simple connected graphs.

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