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Research Article

Homological properties of the edge ideals for trees with small diameter

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Abstract: Let T be a tree of diameter at most 5. We investigate homological invariants of its edge ideal, including the projective dimension, the Castelnuovo–Mumford regularity, and the graded Betti numbers. For trees of diameter at most 3, all nonzero Betti numbers lie on the linear strand. For trees of diameter 4 and 5, we determine the regularity and the projective dimension explicitly. In the case of caterpillar trees of diameter 4, we compute all graded Betti numbers and provide an explicit formula relating them to the f-vector of the independence complex. Our results refine the combinatorial description of Betti numbers for forests and highlight structural features of trees with small diameter.

Keywords: Betti numbers, Stanley-Reisner rings, homological invariants, trees.

AMS Subject classification: 13D02, 13F55, 05E45, 05E40

1. Introduction

Let $\mathbb{K}$ be an arbitrary field. Every simple graph G with vertex set $V_G = \{v_1, v_2, \ldots, v_n\}$ and edge set E_G determines the ideal $I(G)$ of the polynomial ring $R = \mathbb{K}[x_1, x_2, \ldots, x_n]$ generated by the set $\{x_i x_j \mid v_i v_j \in E_G\}$. The ideal $I(G)$ and the quotient $R/I(G)$ are called, respectively, the *edge ideal* and *the edge ring* of G [28]. By Hilbert's Syzygy Theorem, the edge ring of G has a minimal free resolution

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which is unique up to isomorphism. Since $I(G)$ is homogeneous, this resolution is bigraded and takes the following form

$$0 \to \bigoplus_{j=e+1}^{s_e} R(-j)^{\beta_{e,j}} \to \cdots \longrightarrow \bigoplus_{j=i+1}^{s_i} R(-j)^{\beta_{i,j}} \longrightarrow \cdots \to \bigoplus_{j=2}^{s_1} R(-j)^{\beta_{1,j}} \to R \to R/I(G) \to 0, \tag{1.1}$$

where $e \leqslant n$, and $R(-j)$ denotes the one-dimensional graded free R-module such that $1 \in R(-j)$ has (internal) degree j. The number e in (1.1) is called the *projective dimension* of $R/I(G)$, and denoted by $\mathrm{pd}^{\mathbb{K}}(G)$. By definition, the number $\beta_{i,j}$ in (1.1), denoted from now on by $\beta^{\mathbb{K}}_{i,j}(G)$, is the Betti number of $R/I(G)$ in homological degree i and internal degree j. In general, these numbers depend on G and $\mathbb{K}$, but there are cases where the field is irrelevant [16].

The *Castelnuovo-Mumford regularity* (or simply *regularity*) of $R/I(G)$ and $I(G)$ are given by the numbers

$$\mathrm{reg}^{\mathbb{K}}(R/I(G)) = \max\{j - i \mid \beta^{\mathbb{K}}_{i,j}(G) \neq 0\} \qquad \text{and} \qquad \mathrm{reg}^{\mathbb{K}}(I(G)) = \mathrm{reg}^{\mathbb{K}}(R/I(G)) + 1.$$

Computing all the nonzero Betti numbers in (1.1) is in general very difficult. One of the tools to work them out is Hochster's formula, which allows to retrieve the several $\beta^{\mathbb{K}}_{i,j}(G)$'s from the reduced homology of suitable simplicial complexes (see Theorem 5 in Section 2).

The literature on the resolutions of edge ideals of graphs is really extensive. The Betti numbers involved in the linear strand (i.e. the numbers $\beta^{\mathbb{K}}_{i,i+1}(G)$) and the *extremal* ones (the nonzero $\beta^{\mathbb{K}}_{i,j}(G)$'s such that $\beta^{\mathbb{K}}_{l,r}(G) = 0$, for all $l \geqslant i, r \geqslant j+1$ and $r - l \geqslant j - i$) received special attention [2, 11, 12, 20, 24, 25].

The inspection of the Betti numbers related to specific graphs is another active area of research; for instance, there exist results on Ferrer's graphs [5]; some split graphs [26]; ladder graphs [27]; Grimaldi graphs [1]; multiple complete split-like graphs, clique stars and their generalization [22]; threshold graphs [6, 19] and chain graphs [23]. Moreover, Jacques [15] gave a recursive formula for the Betti numbers of edge ideals of forests and Hang and Wu recently dealt with unicyclic graphs [9].

This paper focuses on trees with diameter at most 5 and is organized as follows: Section 2 contains some terminology on graphs and some basics on the independent simplicial complex associated to an edge ideal. In Section 3, we discuss the homological invariants of edge ideal of trees of diameter 3 and 4. We find their regularity, initial Betti numbers, that is, those lying on the linear strand, and projective dimension. Section 4 is devoted to the detection of all the nonzero Betti numbers of caterpillars with diameter 4, proposing in particular a formula relating the graded Betti numbers of their Stanley-Reisner rings with the numbers f_i counting the i-dimensional faces of the associated complex.

While general descriptions of Betti numbers for forests are available via Hochster's formula and Kimura's combinatorial characterization, explicit closed formulas are rare even for restricted classes of trees. The case of small diameter turns out to exhibit a rich interplay between induced matchings, combinatorial decompositions, and the topology of independence complexes. Our results provide a complete and explicit description of homological invariants for trees of diameter at most 5, and in particular yield a direct connection between graded Betti numbers and the f-vector of the independence complex for caterpillar trees of diameter 4.

2. Preliminaries on graphs and their independent simplicial complex

All graphs considered in this paper are simple. The complement of a graph $G = (V_G, E_G)$ is as usual denoted by $\overline{G}$.

For a given pair of vertices $u, v \in V_G$, we write $u \sim v$ if they are adjacent. In this case, we say that u is a neighbor of v and vice versa. We set $N_G(v) = \{u \in V_G \mid u \sim v\}$ and $N_G[v] = N_G(v) \cup \{v\}$ (the subscript will be omitted when the context makes it clear). The *vertex degree* of $v \in V_G$ is the number $\deg v = |N_G(v)|$, The symbol Δ_G denotes the maximum degree of G.

The complete graph, the path and the cycle with n vertices will be respectively denoted by K_n, P_n and C_n, whereas kG is the graph consisting of k disjoint copies of G. Clearly, $\overline{K}_n = nK_1$.

We write $H = (V_H, E_H) \subseteq G$ to mean that the graph H is an *induced* subgraph of G. This happens when $V_H \subseteq V_G$ and E_H is the set of all edges of G joining vertices in V_H. The induced subgraph H can be also denoted by $G[V_H]$ and called *the subgraph of G induced by V_H*.

A set S consisting of m vertices of G is *independent* (for G) if $G[S] \cong mK_1$. The complete bipartite graph with maximal independent sets of cardinality m and n is denoted by $K_{m,n}$.

We say that G is *H-free* if none of its induced subgraphs is isomorphic to H. A graph G is said to be *chordal* if every induced cycle has three vertices. In other words, a chordal graph is C_n-free for all $n \geqslant 4$. If instead both the graph G and $\overline{G}$ are C_n-free for all $n \geqslant 5$, we say that G is weakly chordal. It is easy to see that every chordal graph is weakly chordal.

We now state the characterization due to Fröberg of the graphs whose edge ideal has regularity 2.

Theorem 1. [7] *The edge ideal $I(G)$ of a graph G has regularity 2 if and only if the complement $\overline{G}$ is chordal.*

A matching M in a graph G is a subset of E_G containing vertex-disjoint edges. The size of a matching M is just its cardinality. If M is the edge set of an induced subgraph, then M is said to be an induced matching. The induced matching number

$\mathrm{ind}(G)$ of G is the largest size of an induced matching in G. In other words, $\mathrm{ind}(G)K_2$, but not $(\mathrm{ind}(G)+1)K_2$, is an induced subgraph of G.

The following result relates the regularity of edge ring of a weakly chordal G with its induced matching number, showing in particular the independence of the numbers $\mathrm{reg}^{\mathbb{K}}(R/I(G))$ and $\mathrm{reg}^{\mathbb{K}}(I(G)) = \mathrm{reg}^{\mathbb{K}}(R/I(G)) + 1$ from the field $\mathbb{K}$.

Theorem 2. [29] *If G is a weakly chordal graph, then $\mathrm{reg}^{\mathbb{K}}(R/I(G)) = \mathrm{ind}(G)$.*

A collection of complete bipartite subgraphs $\mathcal{B} = \{B_1, \dots, B_t\}$ of a graph G is said to be strongly disjoint if the B_i's are pairwise vertex-disjoint and there exists an induced matching $\{m_1, \dots, m_t\}$ such that m_i is an edge of subgraph B_i for $1 \leqslant i \leqslant t$. In [21], Nguyen and Vu defined the number

$$\mathcal{D}(G) = \max\left\{\sum_{i=1}^{t} |V_{B_i}| - t\right\}, \tag{2.1}$$

where V_{B_i} is the vertex set of the complete bipartite subgraph B_i, and the maximum is considered over all the strongly disjoint collections of complete bipartite subgraphs of G. The following theorem is an immediate consequence of [21, Theorem 7.7].

Theorem 3. *Let G be a weakly chordal graph. Then, the projective dimension of $R/I(G)$ is independent from the field $\mathbb{K}$ and is equal to $\mathcal{D}(G)$.*

It is easily seen that every tree T is weakly chordal. In fact, $\overline{C}_5 = C_5$ and $C_3 \subseteq \overline{C}_n$ for all $n \geqslant 6$. Thus, $\overline{T}$ is C_n-free for all $n > 4$, since its complement, i.e. the tree T, does not contain cycles. It follows that Theorems 2 and 3 apply to every tree T and we are entitled to write $\mathrm{reg}(R/I(T))$ and $\mathrm{pd}(T)$ instead of $\mathrm{reg}^{\mathbb{K}}(R/I(T))$ and $\mathrm{pd}^{\mathbb{K}}(T)$. Let now F be a forest, i.e. a disjoint union of trees. Every complete bipartite subgraph of F is necessarily a star. A strongly disjoint collection of star subgraphs

$$\mathcal{S} = \{K_{1,q_1}, \dots, K_{1,q_t}\}, \qquad \text{where } \sum_{i=1}^{t} q_i = q,$$

is said to be of type (q,t). Kimura found in [17] a very nice characterization for the Betti numbers of the edge ideal of forests. Next theorem is essentially a rewording of [17, Theorem 4.1].

Theorem 4. *Let F be a forest. The Betti number $\beta^{\mathbb{K}}_{q,q+t}(F)$ is equal to the number of subsets $W \subseteq V_F$ with $q+t$ vertices such that $G[W]$ contains a strongly disjoint collection of star subgraphs of type (q,t).*

A simplicial complex Δ on a finite set $V = \{v_1, v_2, \ldots, v_n\}$ is a collection of nonempty subsets of V containing each singleton of V and satisfying the following property: if $\sigma \in \Delta$, then every nonempty subset of σ also belongs to Δ. Thus, Δ is closed under set inclusion.

Each finite set belonging to Δ is equivalently called *simplex* or *face* of Δ. The maximal simplexes under inclusion are called *facets*. A face $\sigma \in \Delta$ is said to be ℓ-dimensional (or an ℓ-face) if $|\sigma| = \ell + 1$. The dimension of Δ is the number $\dim \Delta := \max\{|\sigma| \mid \sigma \in \Delta\} - 1$.

A simplicial complex Δ' is a *subcomplex of* Δ if $\Delta' \subseteq \Delta$. The subcomplex Δ_S induced by $S \subseteq V$ is by definition the simplicial complex $\Delta_S := \{\sigma \in \Delta \mid \sigma \subseteq S\}$.

The simplicial complex Δ determines the Stanley-Reisner ideal I_Δ of the ring $R = \mathbb{K}[x_1, x_2, \ldots, x_{|V|}]$ [28]. The squarefree monomial ideal I_Δ is generated by all the squarefree monomials $x_{i_1} x_{i_2} \cdots x_{i_p}$ such that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ is not a face of Δ. The quotient ring $\mathbb{K}[\Delta] = R/I_\Delta$ is called the Stanley-Reisner ring of Δ. The following theorem, due to Hochster, provides a formula that relates the graded Betti numbers of R/I_Δ to the reduced homology of suitable induced complexes of Δ.

Theorem 5. [14] *The graded Betti numbers of the Stanley-Reisner ring* $\mathbb{K}[\Delta] = R/I_\Delta$ *satisfy the equality*

$$\beta_{i,j}(\mathbb{K}[\Delta]) = \sum_{\substack{S \subseteq V \\ |S| = j}} dim_{\mathbb{K}} \widetilde{H}_{j-i-1}(\Delta_S; \mathbb{K}),$$

for each $i, j > 0$.

Suppose now $V = V_G$, the vertex set of a graph G. The simplicial complex

$$\Delta(G) = \big\{\sigma \mid \sigma \text{ is an independent subset of } V_G\big\},$$

on V_G is known as the *independent complex* of G. It is straightforward to check that $I(G) = I_{\Delta(G)}$; therefore, the Stanley-Reisner ring $\mathbb{K}[\Delta(G)]$ is isomorphic to the edge ring $R/I(G)$. That is why the Hochster formula allows to compute the Betti numbers $\beta^{\mathbb{K}}_{i,j}(G)$ by looking at the simplicial homology of induced subcomplexes of $\Delta(G)$. For basic background on simplicial homology the reader may usefully consult [10].

Another useful tool to compute the Betti numbers of the Stanley-Reisner ring $\mathbb{K}[\Delta(T)]$ is its Hilbert series expressed in terms of the numbers of the faces of $\Delta(T)$ with fixed dimension.

For a simplicial complex Δ with $\dim \Delta = d - 1$, its face vector (f-vector from now on) is $(f_{-1}, f_0, \ldots, f_{d-1})$, where f_i is the number of i-dimensional faces of Δ. The first component f_{-1} is always equal to 1, and it refers to the single empty face of Δ. If we replace t with $t - 1$ in the polynomial $p(t) = \sum_{i=0}^{d} f_{i-1} t^{d-i}$, we arrive at

$$\sum_{i=0}^{d} f_{i-1}(t-1)^{d-i} = \sum_{i=0}^{d} h_i t^{d-i}.$$

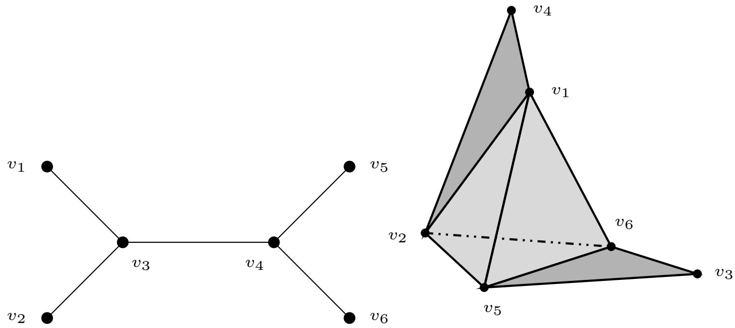


Figure 1. **The double star $D_{2,2}$ and its independent simplicial complex**

The vector $h = (h_0, \dots, h_b)$ where $b = \max\{i \mid h_i \neq 0\}$ is called the h-vector of Δ, Clearly, $b \leqslant d$. The h-polynomial (see, [4, Lemma 5.1.8]) can also be obtained as

$$\sum_i h_i x^i = \sum_{i=0}^{d} f_{i-1} x^i (1-x)^{d-i}.$$

The f-vector and h-vector are related as follows:

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}, \quad \text{and} \quad f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{i-j} h_j, \tag{2.2}$$

where $0 \leqslant i \leqslant d$. Obviously, $f_{d-1} = \sum_{i=0}^{d} h_i$.

For the simplicial complex Δ on n vertices with f-vector $(f_{-1}, f_0, \dots, f_{d-1})$ and h-vector $(h_0, h_1, \dots, h_d)$, the Hilbert series (Hilbert-Poincaré) of Stanley–Reisner ring $\mathbb{K}[\Delta]$ of Δ is

$$\mathcal{H}(\mathbb{K}[\Delta], t) = \sum_{i=0}^{d} \frac{f_{i-1} t^i}{(1-t)^i} = \frac{h_0 + h_1 t + \cdots + h_d t^d}{(1-t)^d}.$$

When G is a graph with n vertices and we are considering the independent complex $\Delta(G)$, it is useful to rewrite the above series as

$$\mathcal{H}(\mathbb{K}[\Delta], t) = \sum_{i=0}^{d} \frac{f_{i-1} t^i}{(1-t)^i} = \frac{(1-t)^{n-d}(h_0 + h_1 t + \cdots + h_d t^d)}{(1-t)^n}.$$

After setting $h'(t) = (1-t)^{n-d}(h_0 + h_1 t + \cdots + h_d t^d) = \sum_{i=0}^{n} h'_i t^i$ we obtain

$$h'_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{n-d}{i-j} h_j, \tag{2.3}$$

where $0 \leqslant i \leqslant n$ and the h_j's are given by (2.2). Thus, the Hilbert series of $\mathbb{K}[\Delta]$ can be written as

$$\mathcal{H}(\mathbb{K}[\Delta], t) = \frac{1}{(1-t)^n} \sum_{i=0}^{n} h_i' t^i. \tag{2.4}$$

Let M be a graded finite $\mathbb{K}[x_1, x_2, \ldots, x_n]$-module with finite projective dimension and graded free resolution (see [4, Lemma 4.1.13]). The relation between the Hilbert series and graded Betti numbers can be retrieved by

$$\mathcal{H}(M, t) = \frac{\sum_{i,j} (-1)^i \beta_{i,i+j} t^j}{(1-t)^n}. \tag{2.5}$$

3. Homological invariants of trees

The double star graph $D_{a,b}$ is a tree formed by joining the central vertices of $K_{1,a}$ and $K_{1,b}$ by an edge (see Fig. 2). We assume that $a \geqslant b$ and note that $D_{a,b}$ has $n = a + b + 2$ vertices. The double star $D_{2,2}$ along with its independent simplicial complex is shown in Fig. 1.

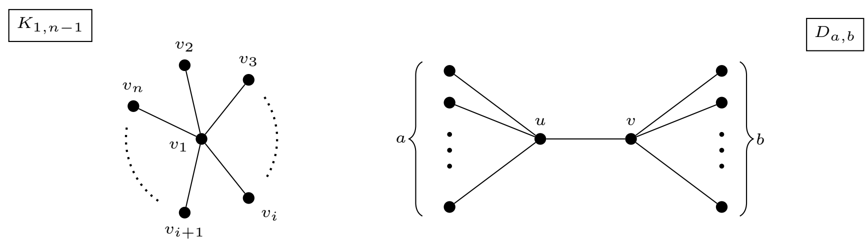


Figure 2. **Trees with diameter at most** 3

The only tree with diameter 1 is $P_2 = K_{1,1}$. The trees with diameter 2 are stars of type $K_{1,n-1}$ with $n \geqslant 3$, whereas each tree of diameter 3 is isomorphic to a double star $D_{a,b}$ for suitable positive integers $a \geqslant b$. An arbitrary tree of diameter 4 can be represented as in Fig. 3, where $r \geqslant 2$, $s_0 \geqslant 0$ and $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_r > 0$. Fig. 4 depicts a tree with diameter 5, where $h, k \geqslant 1$, $s_0, t_0 \geqslant 0$, $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_h > 0$ and $t_1 \geqslant t_2 \geqslant \cdots \geqslant t_k > 0$.

We emphasize that the parametrizations adopted in this section are exhaustive. In particular, every tree of diameter at most 5 is isomorphic to exactly one of the types considered above, and all results proved in Section 3 hold for *all* trees with the given diameter, without any genericity or non–degeneracy assumption.

The nonzero Betti numbers of trees belonging to the linear strand can be computed according to the following proposition which is a direct consequence of Corollary 2.6 in [25]. Nowadays, Proposition 1 can also be deduced from Theorem 4.

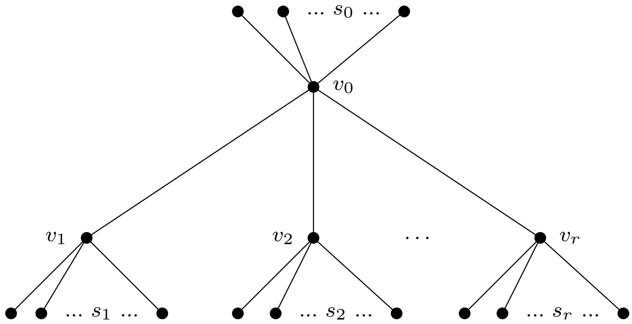


Figure 3. **The tree** $T(s_0; s_1, \dots, s_r)$

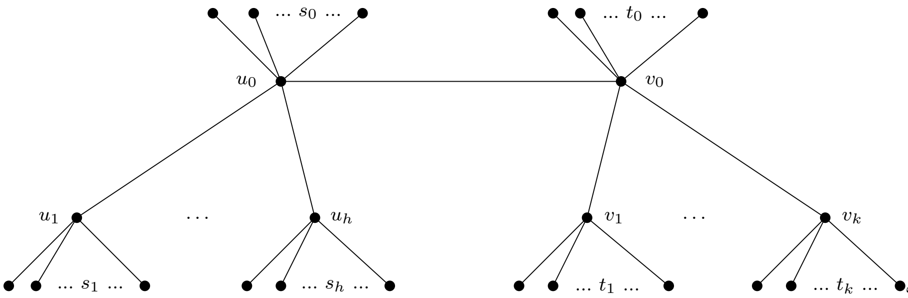


Figure 4. **The tree** $T(s_0, t_0; s_1, \dots, s_h; t_1, \dots, t_k)$

Proposition 1. *Let T be a tree. For each field $\mathbb{K}$, the Betti numbers of $R/I(T)$ belonging to the linear strand are*

$$\beta^{\mathbb{K}}_{i,i+1}(T) = \begin{cases} |E(T)| & \text{if } i = 1; \\ \displaystyle\sum_{v \in V_G} \binom{\deg v}{i} & \text{if } i \geqslant 2. \end{cases} \tag{3.1}$$

From (3.1) we immediately see that $\beta^{\mathbb{K}}_{i,i+1}(T) = 0$ for all $i > \Delta_G$ and

$$\beta^{\mathbb{K}}_{\Delta_G, \Delta_G+1} = \big|\{v \in v_G \mid \deg v = \Delta_G\}\big|.$$

Proposition 1 and Fig. 2–4 allow to compute the Betti numbers in the linear strands for all trees with diameter up to 5. We give the explicit formulæ in Corollary 1. Observe that only non-pendant vertices give a non-trivial contribution in the summation of the formula (3.1).

Corollary 1. *Let $r \geqslant 2$. The following equalities hold.*

$$\beta^{\mathbb{K}}_{i,i+1}(K_{1,n-1}) = \begin{cases} n-1 & \text{if } i=1; \\ \binom{n-1}{i} & \text{if } i \geqslant 2. \end{cases} \tag{3.2}$$

$$\beta^{\mathbb{K}}_{i,i+1}(D_{a,b}) = \begin{cases} a+b+1 & \text{if } i=1; \\ \binom{a+1}{i} + \binom{b+1}{i} & \text{if } i \geqslant 2. \end{cases} \tag{3.3}$$

$$\beta_{i,i+1}(T(s_0;s_1,\dots,s_r)) = \begin{cases} s_0+s_1+\cdots+s_r+r & \text{if } i=1; \\ \binom{s_0+r}{i} + \sum_{j=1}^{r}\binom{s_j+1}{i} & \text{if } i \geqslant 2. \end{cases} \tag{3.4}$$

$$\beta^{\mathbb{K}}_{i,i+1}(T) = \begin{cases} 1+h+k+\sum_{i=0}^{h} s_i + \sum_{i=0}^{k} t_i & \text{if } i=1; \\ \binom{s_0+h+1}{i} + \binom{t_0+k+1}{i} + \sum_{j=1}^{h}\binom{s_j+1}{i} + \sum_{j=1}^{k}\binom{t_j+1}{i} & \text{if } i \geqslant 2, \end{cases}$$

where $T := T(s_0,t_0;s_1,\dots,s_h;t_1,\dots,t_k)$.

We point out that (3.1) and (3.2) are known (they appeared, for instance, in [23, 25]). Let T be any tree. As explained in Section 2, the regularity of $R/(I(T))$ does not depend on $\mathbb{K}$, The next theorem computes $\operatorname{reg}(I(T)) = \operatorname{reg}(R/I(T)) + 1$ whenever $\operatorname{diam}(T) \leqslant 5$.

Theorem 6. *The following equalities hold.*

$$\operatorname{reg}(I(K_{1,n-1})) = \operatorname{reg}(I(D_{a,b})) = 2 \qquad \text{for all } n \geqslant 2 \text{ and } a \geqslant b \geqslant 1; \tag{3.5}$$

$$\operatorname{reg}(I(T_4)) = r+1 \qquad \text{and} \qquad \operatorname{reg}(I(T_5)) = h+k+1, \tag{3.6}$$

where $T_4 := T(s_0;s_1,\dots,s_r)$ (resp. $T_5 = T(s_0,t_0;s_1,\dots,s_h;t_1,\dots,t_k)$) is the tree depicted in Fig. 3 (resp. Fig. 4).

Proof. By Theorem 2, $\operatorname{reg}^{\mathbb{K}}(I(T)) = \operatorname{ind}(G) + 1$ for every tree T. Thus, Equation (3.5) holds since stars and double stars are $2K_2$-free (see Fig. 2).
Let now $T_4 := T(s_0;s_1,\dots,s_r)$ be the tree depicted in Fig. 3. If an induced matching contains an edge incident with v_0, then its cardinality is necessarily 1. It follows that every maximal induced matching is of type $\{e_1,\dots,e_r\}$, where e_i is an edge connecting v_i with one of its pendant neighbors. Hence, $\operatorname{reg}(I(T_4)) = r+1$. The

argument to prove the other equality of (3.6) is similar. A maximal induced matching is one of the following three types:

$$\{e_0, e'_1, \dots, e'_k\}, \quad \{e'_0, e_1, \dots, e_h\} \quad \text{and} \quad \{e_1, \dots, e_h, e'_1, \dots e'_k\},$$

where e_i for $0 \leqslant i \leqslant h$) (resp. e'_i for $0 \leqslant i \leqslant k$) is an edge connecting u_i (resp. v_i) with one of its pendant neighbors. □

From Theorem 6 one sees that the nonzero Betti numbers for nonempty trees of diameter $\leqslant 3$ all belong to the linear strand. Therefore, they can be all obtained from (3.1) and (3.2).
The projective dimension for trees of small diameter will be now obtained with the aid of Theorem 3.

Theorem 7. *The following equalities hold.*

$$\operatorname{pd}(K_{1,n-1}) = n - 1 \qquad \text{for all } n \geqslant 2; \tag{3.7}$$

$$\operatorname{pd}(D_{a,b}) = a + 1 \qquad \text{for all } a \geqslant b \geqslant 1; \tag{3.8}$$

$$\operatorname{pd}(T(s_0; s_1, \dots, s_r)) = \max\{s_0 + r, n - (s_0 + r)\} \qquad \text{(see Fig. 3)}, \tag{3.9}$$

$$\operatorname{pd}(T_5) = \max\{p_{T_5}, q_{T_5}, r_{T_5}\}, \tag{3.10}$$

where $T_5 = T(s_0, t_0; s_1, \dots, s_h; t_1, \dots, t_k)$ *is the tree depicted Fig. 4, and*

$$p_{T_5} = s_0 + h + 1 + \sum_{i=1}^{k} t_i, \quad q_{T_5} = t_0 + k + 1 + \sum_{i=1}^{h} s_i \quad \text{and} \quad r_{T_5} = 2 + \sum_{i=1}^{h} s_i + \sum_{i=1}^{k} t_i. \tag{3.11}$$

Proof. Equations (3.7) and (3.8) are immediately deduced from (3.2), (3.3) and (3.5). Let now $T_4 = T(s_0; s_1, \dots, s_r)$ be a tree of diameter 4. Since T_4 is a tree, the complete bipartite subgraphs of T_4 are all stars. By Theorem 3, we have the equality $\operatorname{pd}(T_4) = \mathcal{D}(T_4)$; and this number is achieved by one of the following two collections of strongly disjoint complete bipartite subgraphs of T_4:

$$\mathcal{B}(T_4) = \{B_1\} \quad \text{and} \quad \mathcal{B}'(T_4) = \{B'_1, \dots, B'_r\},$$

where $B_1 = T_4[N[v_0]] \cong K_{1,s_0+r}$, $B'_1 = T_4[N[v_1]] \cong K_{1,s_1+1}$, and

$$B'_i = T_4[N[v_i] - \{v_0\}] \cong K_{1,s_i} \qquad \text{for } 2 \leqslant i \leqslant r.$$

Now, Equation (3.9) comes from (2.1) once you note that

$$\sum_{i=1}^{r} |V_{B'_i}| - r = 1 + \sum_{i=1}^{r} s_i = n - (s_0 + r).$$

The argument to prove (3.10) is similar. The number $\mathrm{pd}(T_5) = \mathcal{D}(T_5)$ is achieved by one of the following three collections of strongly disjoint complete bipartite subgraphs of T_4: $\mathcal{B}(T_5) = \{B_0, B_1, \dots, B_k\}$, $\mathcal{B}(T_5) = \{B'_0, B'_1, \dots, B'_h\}$ and $\mathcal{B}''(T_5) = \{B''_1, \dots, B''_{h+k}\}$, where

$$
\begin{aligned}
B_0 &= T_5[N[u_0] - \{v_0\}] \cong K_{1,s_0+h} & B'_0 &= T_5[N[v_0] - \{u_0\}] \cong K_{1,s_0+k}\\
B_1 &= T_5[N[v_1]] \cong K_{1,t_1+1} & B'_1 &= T_5[N[u_1]] \cong K_{1,s_1+1}\\
B_i &= T_5[N[v_i] - \{v_0\}] \cong K_{1,t_i} \ \ (2 \leqslant i \leqslant k) & B'_i &= T_5[N[u_1] - \{u_0\}] \cong K_{1,s_i} \ \ (2 \leqslant i \leqslant h)\\
B''_i &= B_i \ \ (1 \leqslant i \leqslant k) & B''_{k+i} &= B'_i \ \ (1 \leqslant i \leqslant h).
\end{aligned}
$$

The proof ends by taking into account (2.1) and the equalities

$$\sum_{i=0}^{k} |V_{B_i}| - (k+1) = p_{T_5}, \qquad \sum_{i=0}^{h} |V_{B'_i}| - (h+1) = q_{T_5}, \quad \text{and} \quad \sum_{i=1}^{h+k} |V_{B''_i}| - (h+k) = r_{T_5}.$$

□

Example 1. Consider the following three trees with diameter 5:

$$T' := T(0,0;1;1) \cong P_6; \quad T'' := T(2,0;1;1); \quad \text{and} \quad T''' := T(1,2;2;1).$$

By looking at (3.11) and Fig. 5, one computes

$$(p_{T'}, q_{T'}, r_{T'}) = (3,3,4), \quad (p_{T''}, q_{T''}, r_{T''}) = (5,3,4); \quad \text{and} \quad (p_{T'''}, q_{T'''}, r_{T'''}) = (5,6,5).$$

Consistently with Theorem 7, the software Macaulay2 [8] confirms that $\mathrm{pd}(T') = r_{T'} = 4$, $\mathrm{pd}(T'') = p_{T''} = 5$ and $\mathrm{pd}(T''') = q_{T'''} = 6$; furthermore, as predicted by (3.6), $\mathrm{reg}(I(T'))$, $\mathrm{reg}(I(T''))$ and $\mathrm{reg}(I(T'''))$ are all equal to 3. □

It is natural to ask whether, for a given tree T, the maximal homological dimension for a nonzero Betti number in the linear strand can be equal to $\mathrm{pd}(T)$. A quick look at Theorems 6 and 7 suffices to realize that this never happens for trees of diameter 5. On the contrary, for a tree $T := T(s_0; s_1, \dots, s_r)$ of diameter 4 and n vertices,

$$\max\{i \mid \beta_{i,i+1}(T) \neq 0\} = \mathrm{pd}(T) \iff s_0 + r \geqslant \frac{n}{2}.$$

As the regularity of a tree T increases, it becomes quite hard to find *all* its Betti numbers, in spite of Theorems 4 and 5. These two achievements allow us to prove our Theorem 8 in two different ways. Before showing the homological proof of it (our preference is due to its cross-disciplinary flavour) we insert next Lemma 1 which is in any case considered folklore among scholars.

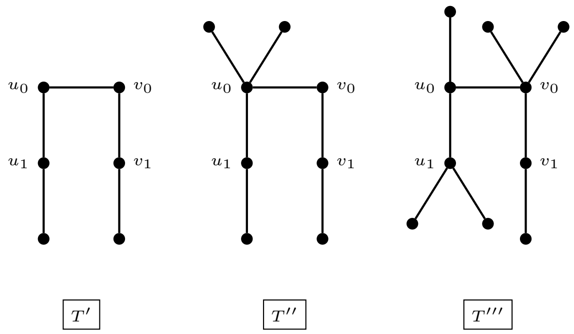


Figure 5. **The trees considered in Example 1.**

Lemma 1. *Let G be a graph and let $(\tau, \rho) := (\mathrm{pd}^{\mathbb{K}}(G), \mathrm{reg}^{\mathbb{K}}(R/I(G))$. The graph G has a unique extremal ($\mathbb{K}$)-Betti number if and only if $\beta^{\mathbb{K}}_{\tau,\tau+\rho}(G) \neq 0$.*

Proof. By definition of projective dimension and regularity, the integers

$$a = \max\{i \mid \beta^{\mathbb{K}}_{i,i+\rho}(G) \neq 0\} \quad \text{and} \quad b = \max\{j \mid \beta^{\mathbb{K}}_{\tau,\tau+j}(G) \neq 0\}$$

are well-defined. Moreover, $\beta^{\mathbb{K}}_{a,a+\rho}(G)$ and $\beta^{\mathbb{K}}_{\tau,\tau+b}(G)$ are both extremal. Thus, if G has only one extremal Betti number, then $a = \tau$ and $b = \rho$. Assume now $\beta^{\mathbb{K}}_{\tau,\tau+\rho}(G) \neq 0$. It is clear that any other nonzero Betti number $\beta^{\mathbb{K}}_{i,j}(G) \neq 0$ cannot be extremal, since surely $i \leqslant \mathrm{pd}^{\mathbb{K}}(G) = \tau$, and if $i = \tau$ then $j < \mathrm{reg}^{\mathbb{K}}(R/I(G)) = \rho$. $\square$

Theorem 8. *Let $T := T(s_0; s_1, \ldots, s_r)$ be a tree of diameter 4 and n vertices. Then $\beta_{n-(s_0+r),n-s_0}(T) = 1$. Moreover, if $(s_0 + r) \leqslant \frac{n}{2}$, then $\beta_{n-(s_0+r),n-s_0}(T)$ is the only extremal Betti number of T.*

Proof. For each $S \subset V_T$ we denote by $|\Delta_S|$ the geometric realization of the complex Δ_S. As usual, we denote by $\mathbb{S}^n$ the n-dimensional (real) sphere in $\mathbb{R}^{n+1}$, and, for each topological space X, by CX and ΣX the cone and the suspension of X respectively. It is well known that the space CX, being contractible, has trivial homology in positive degrees and $\Sigma\mathbb{S}^n$ is homeomorphic to $\mathbb{S}^{n+1}$. Furthermore, it is straightforward to show that $|\Delta_S|$ is homotopy equivalent to the space obtained, for every fixed $i \in \{0, \ldots, r\}$, by collapsing to a single point, say N_i, all the subcomplex generated by the vertices in $N(v_i) \cap S$ which are pendant (in T). By writing $X \simeq Y$ we mean that the two topological spaces X and Y are homotopy equivalent.
Recall that, in our assumptions, $r \geqslant 2$. By Theorem 5,

$$\beta_{n-(s_0+r),n-s_0}(T) = \sum_{\substack{S \subseteq V \\ |S| = n-s_0}} \dim_{\mathbb{K}} \widetilde{H}_{r-1}(\Delta_S; \mathbb{K}).$$

We claim that

$$\widetilde{H}_{r-1}(\Delta_S;\mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } S = S_r := \{v_0,\dots,v_r\} \cup \bigcup_{i=1}^r N(v_i), \\ 0 & \text{otherwise.} \end{cases}$$

The equality $\widetilde{H}_{r-1}(\Delta_{S_r};\mathbb{K}) = \mathbb{K}$ comes from $|\Delta_{S_r}| \simeq \mathbb{S}^{r-1}$. The inductive argument to prove this consists in showing that $|\Delta_{S_{r+1}}| \simeq \Sigma|\Delta_{S_r}|$ for all $r \geqslant 2$.
We illustrate the situation for $r \in \{2,3\}$ in Fig. 6, where $\check{\Delta}_{S_2} \simeq \Delta_{S_2}$ and $\check{\Delta}_{S_3} \simeq \Delta_{S_3}$ are depicted. Note that the shaded simplex $v_0N_1N_2$ in $\check{\Delta}_{S_2}$ (resp. $v_0N_1N_2N_3$ in $\check{\Delta}_{S_3}$) and its faces of positive dimensions can be replaced by the simplex N_1N_2 (resp. $N_1N_2N_3$) without altering its homotopy type (we are actually considering a sequence of elementary collapses [18, p. 94]). Clearly, $|\Delta_{S_2}| \simeq |\check{\Delta}_{S_2}| \simeq \mathbb{S}^1$ and $|\Delta_{S_3}| \simeq |\check{\Delta}_{S_3}| \simeq \Sigma|\check{\Delta}_{S_2}| \simeq \Sigma\,\mathbb{S}^1 \simeq \mathbb{S}^2$.

Let now S be any other subset of V_T containing $n - s_0$ vertices. Since $S \neq S_r$, then $N(v_0) \cap S$ contains at least one pendant vertex. We set $P = \{v \in N(v_0) \cap S \mid \deg_T(v) = 1\}$ and distinguish two cases.

Case 1. $S \cap \bigcup_{i=1}^r (N[v_i] - \{v_0\}) = \varnothing$. Then, possibly apart from v_0, the set S only consists of pendant vertices in $N(v_0)$. Thus, Δ_S is a either a full simplex Δ_{n-s_0-1} (if $v_0 \notin S$) or $\Delta_{n-s_0-2} \cup \Delta_0$ (if $v_0 \in \Delta_S$). In both cases, the geometric realization of Δ_S has trivial homology in positive dimensions.

Case 2. $S \cap \bigcup_{i=1}^r (N[v_i] - \{v_0\}) \neq \varnothing$. We first note that the geometric realizations of Δ_S and $\Delta_{S-\{v_0\}}$ have the same homotopy type. Moreover, after identifying all vertices in P with a single point N_0, we realize that $|\Delta_{S-\{v_0\}}| \simeq C|\Delta_{S-(\{v_0\}\cup P)}|$ (the vertex of the cone is N_0), and the latter is contractible. Hence, $\widetilde{H}_{r-1}(\Delta_S;\mathbb{K}) = 0$.

So far, we have proved that $\beta_{n-(s_0+r),n-s_0}(T) = 1$. If $(s_0+r) \leqslant \frac{n}{2}$, by (3.9) we obtain $p := \mathrm{pd}(T) = n - (s_0+r)$. According to (3.6), $\rho := \mathrm{reg}(R/I(T)) = r$. Thus, in our case, $\beta_{\tau,\tau+\rho}(T) = \beta_{n-(s_0+r),n-s_0}(T) = 1$. The result now follows from Lemma 1. □

Remark 1. It is known that if an ideal $I \subset R$ is Cohen-Macaulay, then I has a unique extremal Betti number (see [3, Lemma 3]). Moreover, by [28, Theorem 6.3.4] the only trees with diameter 4 whose edge ideal is Cohen-Macaulay are those of type $T(1;1,\dots,1)$. Thus, trees of type $T(s_0;s_1,\dots,s_r)$ with n vertices, $s_0 + r \leqslant n/2$ and at least one s_i not equal to 1 represent examples of graphs whose edge ideal I has a unique extremal Betti number, although I is not Cohen-Macaulay. □

4. Betti numbers of the linear strand

Theorem 4, i.e. Kimura's purely combinatorial description of Betti numbers for the edge ideal of forests leads to very explicit formulæ for all the Betti numbers of $I(T)$, for T being a tree with regularity 3 and diameter up to 4. By Theorem 6, trees of this kind are necessarily caterpillars of type $T(s_0;s_1,s_2)$ (see Fig. 3 and Fig. 7). Graphs of type $T(0;s_1,s_2)$ are also a special type of double brooms.

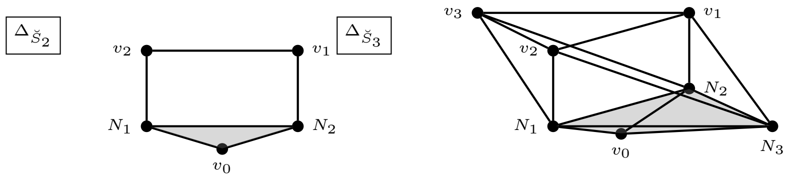


Figure 6. **The complexes $\Delta_{\check{S}_2} \simeq \Delta_{S_2}$ and $\Delta_{\check{S}_3} \simeq \Delta_{S_3}$.**

Theorem 9. *The Betti numbers for $I(T(s_0; s_1, s_2))$ which are possibly nonzero are given by the following formulæ.*

$$\beta_{q,q+1}(T_4(s_0; s_1, s_2)) = \begin{cases} s_0 + s_1 + s_2 + 2 & \text{if } q = 1 \\ \binom{s_0+2}{q} + \binom{s_1+1}{q} + \binom{s_2+1}{q} & \text{if } q \geqslant 2; \end{cases} \tag{4.1}$$

$$\beta_{q,q+2}(T_4(s_0; s_1, s_2)) = c_q(s_0; s_1, s_2) + c'_q(s_0; s_1, s_2) \tag{4.2}$$

where

$$c_q(s_0; s_1, s_2) = \sum_{i=\max\{1,q-s_2\}}^{\min\{s_1,q-1\}} \binom{s_1}{i}\binom{s_2}{q-i},$$

$$c'_q(s_0; s_1, s_2) = \sum_{i=\max\{2,q-s_2\}}^{\min\{s_1+1,q-1\}} \binom{s_1}{i-1}\binom{s_2}{q-i},$$

and $c_q(s_0; s_1, s_2)$ (resp. $c_q(s_0; s_1, s_2)$) is understood to be 0 when

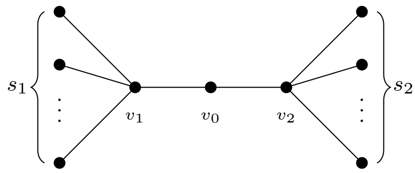
$$\max\{1, q - s_2\} > \min\{s_1, q - 1\} \qquad \textit{(resp. } \max\{2, q - s_2\} > \min\{s_1 + 1, q - 1\}\textit{)}.$$

Proof. By looking at Fig. 3, Equation (4.1) is straightforwardly deduced from (3.4). In order to prove (4.2), it is not restrictive to assume that every strongly disjoint collection of *two* star subgraphs $\{B_1, B_2\}$ satisfies the following properties for $i \in \{1, 2\}$: i) v_i is the non-pendant vertex of B_i, ii) B_i contains a pendant edge attached to v_i. Now we use Theorem 4, and note that, among all the strongly disjoint collections of *two* star subgraphs involving $q + 2$ vertices, $c_q(s_0; s_1, s_2)$ of them does not contain v_0 in their vertex set, and the remaining ones are $c'_q(s_0; s_1, s_2)$. □

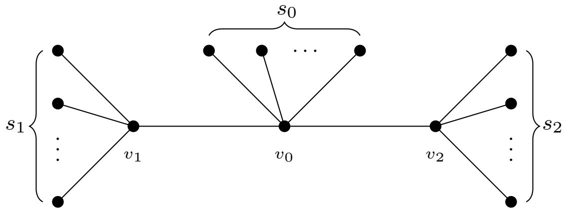
Remark 2. The pendant edges attached to u_0 in a tree of diameter 4 (see Fig 3) cannot be involved in any strongly disjoint collection of *two* star subgraphs. Thus, as expected from Theorem 4, and as confirmed by (4.2),

$$\beta_{q,q+2}(T(s_0; s_1, s_2)) = \beta_{q,q+2}(T(s'_0; s_1, s_2)) \quad \text{for all } q > 0 \text{ and for all } s'_0 > s_0 \geqslant 0.$$

In other words, for every fixed pair of positive integers (s_1, s_2), the Betti tables for the sequence of trees $\{T(j; s_1, s_2)\}_{j\geqslant 0}$ all share their third row. □



Double Broom $T_4(0; s_1, s_2)$



Caterpillar $T_4(s_0; s_1, s_2)$

Figure 7. **Types of trees of diameter** 4.

Apart from (4.2) the Betti numbers off the linear strand for the edge ideal of a tree T of diameter 4 and regularity 3 may be computed in function of the f-vector of the independence complex $\Delta(T)$ introduced in Section 2. A first step toward this goal is the next lemma.

Lemma 2. *Let T be a tree such that* $\operatorname{reg}(I(T)) = 3$*, and let* $(h'_0, \ldots, h'_d)$ *be the* h'*-vector associated to the complex* $\Delta(T)$ *(see Section 2). Then,*

$$\beta_{q,q+2}(T) = \beta_{q+1,q+2}(T) + (-1)^q h'_{q+2}.$$

Proof. In the case at hand, Equation (2.5) becomes

$$\mathcal{H}(\mathbb{K}[\Delta], t) = \frac{1}{(1-t)^n}\left(1 + \sum_{q>0}(-1)^q(\beta_{q,q+1}t^{q+1} + \beta_{q,q+2}t^{q+2})\right)$$

A comparison with (2.4) leads to the result. □

The remainder of the paper will be devoted to prove the next Theorem 10, which highlights a structural feature of caterpillar trees of diameter 4: their nonlinear strand is entirely determined by the linear strand together with the face numbers of the independence complex. This provides a direct bridge between homological invariants and combinatorial data of the associated simplicial complex.

Theorem 10. *Let* $T := T(s_0; s_1, s_2)$ *be a caterpillar of diameter* 4 *and* n *vertices, and let* f_{i-1} *be the number of* i*-dimensional faces of* $\Delta(T)$*. Then,*

$$\beta_{q,q+2}(T) = \beta_{q,q+1}(T) + \sum_{k=0}^{q+2}(-1)^k\binom{n-k}{q+2-k}f_{k-1}. \tag{4.3}$$

The proof of Theorem 10 essentially consists in proving the equality

$$h'_{q+2} = \sum_{k=0}^{q+2} (-1)^{k+q} \binom{n-k}{q+2-k} f_{k-1} \tag{4.4}$$

More precisely, Theorem 10 immediately follows from (4.4) and Lemma 2. Clearly, (4.3) is useful only after computing the f-vector of $\Delta(T(s_0; s_1, s_2))$. That is why we insert here the next two propositions.

Proposition 2. *Let $T := T_4(s_0; s_1, s_2)$ be a caterpillar tree of order $n = s_0 + s_1 + s_2 + 3$ and $s_0 > 0$ (see Fig. 7). Then, the number of $(i-1)$-dimensional faces for the complex $\Delta(T)$ is*

$$f_{i-1}(T) = \binom{n-3}{i} + \binom{s_1+s_2}{i-1} + \binom{s_0+s_2}{i-1} + \binom{s_0+s_1}{i-1} + \binom{s_0}{i-2}.$$

Proof. For $0 \leqslant j \leqslant 2$ we denote by $u_{j1}, \dots, u_{js_j}$ the vertices in $N_T(v_j)$ that are pendant. The simplicial complex $\Delta(T)$ has the following five facets

$F_1 = \langle s_{01}, \dots, s_{0s_0}, s_{11}, \dots, s_{1s_1}, s_{21}, \dots, s_{2s_2} \rangle$, $F_2 = \langle v_0, s_{11}, \dots, s_{1s_1}, s_{21}, \dots, s_{2s_2} \rangle$,
$F_3 = \langle v_1, s_{01}, \dots, s_{0s_0}, s_{21}, \dots, s_{2s_2} \rangle$, $F_4 = \langle v_2, s_{01}, \dots, s_{0s_0}, s_{11}, \dots, s_{1s_1} \rangle$,
$F_5 = \langle v_1, v_2, s_{01}, \dots, s_{0s_0} \rangle$.

Fig. 8 helps to figure out the shape of $\Delta(T)$.

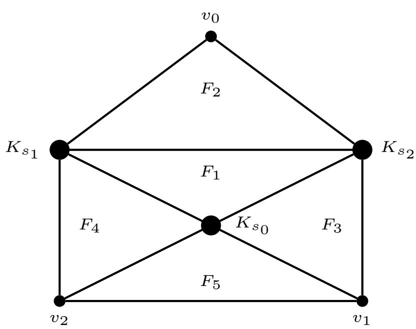


Figure 8. **The complex** $\Delta(T(s_0; s_1, s_2)$

The simplex F_1 has $\binom{s_0+s_1+s_2}{i}$ $(i-1)$-dimensional faces. The $(i-1)$-dimensional faces in '$F_2 - F_1$' (i.e. those belonging to F_2, but not to to F_1) are precisely those containing v_0 among their vertices. The total number of them is $\binom{s_1+s_2}{i-1}$. For a similar reason, the $(i-1)$-dimensional faces of '$F_3 - F_1$' and '$F_4 - F_1$' are $\binom{s_0+s_2}{i-1}$ and $\binom{s_0+s_1}{i-1}$ respectively. Lastly, the $(i-1)$-dimensional faces of F_5 which are not faces of $F_1 \cup F_3 \cup F_4$ contain both v_1 and v_2 in their set of vertices. That is why we find $\binom{s_0}{i-2}$ faces of this type. □

Proposition 3. *Let $T_0 := T(0; s_1, s_2)$ be a double broom tree as in Figure 7. Then, the number of $(i-1)$-dimensional faces for the complex $\Delta(T_0)$ is*

$$f_{i-1}(T_0) = \begin{cases} s_1 + s_2 + 3 & \text{if } i = 1, \\ \binom{s_1+s_2+1}{2} + s_1 + s_2 + 1 & \text{if } i = 2, \\ \binom{s_1+s_2+1}{i} + \binom{s_1}{i-1} + \binom{s_2}{i-1} & \text{if } i \geqslant 3. \end{cases}$$

Proof. The complex $\Delta(T_4(0; s_1, s_2))$ can be thought as a subcomplex of $\Delta(T_4(s_0; s_1, s_2))$ considered in Proposition 2. Following the notation introduced along its proof, we need to count the $(i-1)$-dimensional faces not containing s_{0j}'s of F_1, '$F_2 - F_1$', '$F_3 - F_1$', '$F_4 - F_1$' and $F' = F_5 - (F_1 \cup F_3 \cup F_4)$. Recall now that $f_0(T_0) = |V_{T_0}| = s_1 + s_2 + 3$. For $i > 1$ the statement follows easily from the equality

$$\binom{n-3}{i} + \binom{s_1+s_2}{i-1} = \binom{s_1+s_2}{i} + \binom{s_1+s_2}{i-1} = \binom{s_1+s_2+1}{i}$$

(the former only holds for $s_0 = 0$) once you realize that $v_1 v_2$ is the only face in F' having positive dimension. □

The following corollary is an immediate consequence of Propositions 2 and 3.

Corollary 2. *Let $d-1$ be the maximal dimension for a simplex belonging to $\Delta(T(s_0; s_1, s_2))$. Then*

$$d - 1 = \begin{cases} n-3 & \text{if } s_0 = 0, \\ n-4 & \text{if } s_0 \geqslant 0. \end{cases}$$

In the remainder of the paper we denote by $f_i(G)$, $h_i(G)$ and $h'_i(G)$ the i-th components of the f-vector, the h-vector and the h'-vector associated to the independent complex $\Delta(G)$.

Proposition 4. *For $T_0 := T(0; s_1, s_2)$ and $T := T(s_0; s_1, s_2)$ with $s_0 \geqslant 0$, the following equalities involving the h- and the h'-vectors hold.*

$$h_q(T_0) = \sum_{i=0}^{q} (-1)^{q-i} \binom{s_1+s_2+1-i}{q-i} f_{i-1}(T_0), \quad \text{with } 0 \leqslant q \leqslant s_1 + s_2 + 1,$$

and

$$h_q(T) = \sum_{i=0}^{q} (-1)^{q-i} \binom{s_0+s_1+s_2-i}{q-i} f_{i-1}(T), \quad \text{with } 0 \leqslant q \leqslant s_0 + s_1 + s_2.$$

Moreover,

$$h'_q(T_0) = h_q(T_0) - 2h_{q-1}(T_0) + h_{q-2}(T_0) \quad \textit{with } 0 \leqslant q \leqslant s_1 + s_2 + 3,$$

and

$$h'_q(T) = h_q(T) - 3h_{q-1}(T) + 3h_{q-2}(T) - h_{q-3}(T) \quad \textit{with } 0 \leqslant q \leqslant s_0 + s_1 + s_2 + 3,$$

where the h_{-b}'s and the h'_{-b}'s for $b > 0$ are understood to be 0.

Proof. In order to specialize Equations (2.2) and (2.3) to the cases T_0 and T, the number d must be replaced, according to Corollary 2, with $n-2 = s_1 + s_2 + 1$ for the tree T_0, and with $n - 3 = s_0 + s_1 + s_2$ for the tree T. ☐

We are finally able to prove (4.4) (and, consequently, Theorem 10). Let us deal with T_0 and T separately.
In the first formula of Proposition 4, we find the number $f_{i-1}(T_0)$ multiplied by $(-1)^{q-i}\binom{a-i}{q-i}$, where $a := s_1 + s_2 + 1$. Therefore,

$$\begin{aligned} h'_{q+2}(T_0) &= h_{q+2}(T_0) - 2h_{q-1}(T_0) + h_{q-2}(T_0) \\ &= (-1)^{q+2-i}\left(\binom{a-i}{q+2-i} + 2\binom{a-i}{q+1-i} + \binom{a-i}{q-i}\right) f_{i-1}(T_0) \\ &= (-1)^{q+i}\left(\binom{a-i}{q+2-i} + \binom{a-i}{q+1-i} + \binom{a-i}{q+1-i} + \binom{a-i}{q-i}\right) f_{i-1}(T_0) \qquad (4.5) \\ &= (-1)^{q+i}\left(\binom{a-i+1}{q+2-i} + \binom{a-i+1}{q+1-i}\right) f_{i-1}(T_0) \\ &= (-1)^{q+i}\binom{a-i+2}{q+2-i} f_{i-1}(T_0). \end{aligned}$$

Thus, Equation (4.4) is proved for T_0, since $a + 2 = n$. Now, (4.4) also holds for T: if $b = s_0 + s_1 + s_2 = n - 3$,

$$\begin{aligned} h'_{q+2}(T) &= h_{q+2}(T) - 3h_{q+1}(T) + 3h_q(T) - h_{q-1}(T) \\ &= (-1)^{q+2-i}\left(\binom{b-i}{q+2-i} + 2\binom{b-i}{q+1-i} + \binom{b-i}{q-i}\right. \\ &\qquad\qquad \left. + \binom{b-i}{q+1-i} + 2\binom{b-i}{q-i} + \binom{b-i}{q-1-i}\right) f_{i-1}(T) \\ &= (-1)^{q+i}\left(\binom{b-i+2}{q+2-i} + \binom{b-i+2}{q+1-i}\right) f_{i-1}(T) \\ &= (-1)^{q+i}\binom{b-i+3}{q+2-i} f_{i-1}(T) \\ &= (-1)^{q+i}\binom{n-i}{q+2-i} f_{i-1}(T). \end{aligned}$$

Just note that, along the sequence of equalities considere here, we used twice the identity

$$\binom{m}{s} + 2\binom{m}{s-1} + \binom{m}{s-2} = \binom{m+2}{s}$$

extrapolated from (4.5).

Example 2. The graph $T(0;3,2)$ is a caterpillar tree with 8 vertices and diameter 4. Following (4.1), the nonzero Betti numbers of $R/I(T(0;3,2))$ in the linear strand are

$$\beta_{1,2}=7,\quad \beta_{2,3}=\binom{4}{2}+\binom{3}{2}+1=10,\quad \beta_{3,4}=\binom{4}{3}+\binom{3}{3}=5,\quad \beta_{4,5}=\binom{4}{4}=1.$$

The nonzero Betti numbers off the linear strand, computed according to (4.2), are:

$$\begin{aligned}
&\beta_{1,3}=0, && \beta_{4,6}=\tbinom{3}{2}\tbinom{2}{2}+\tbinom{3}{3}\tbinom{2}{1}+\tbinom{3}{1}\tbinom{2}{2}+\tbinom{3}{2}\tbinom{2}{1}=14\\
&\beta_{2,4}=\tbinom{3}{1}\tbinom{2}{1}=6 && \beta_{5,7}=\tbinom{3}{3}\tbinom{2}{2}+\tbinom{3}{2}\tbinom{2}{2}+\tbinom{3}{3}\tbinom{2}{1}=6\\
&\beta_{3,5}=\tbinom{3}{1}\tbinom{2}{2}+\tbinom{3}{2}\tbinom{2}{1}+\tbinom{3}{1}\tbinom{2}{1}=15 && \beta_{6,8}=\tbinom{3}{3}\tbinom{2}{2}=1.
\end{aligned}$$

From Proposition 3, it turns out that the f-vector is $(1,8,21,24,16,6,1)$. Thus, we can retrieve the nonzero Betti numbers off the linear strand by using Theorem 10. We just compute in this alternative way $\beta_{1,3}$ and $\beta_{2,4}$:

$$\beta_{1,3}=\beta_{2,3}+\binom{8}{3}f_{-1}-\binom{7}{2}f_0+\binom{6}{1}f_1-\binom{5}{0}f_2=10+56\cdot 1-21\cdot 8+6\cdot 21-1\cdot 24=0,$$

$$\begin{aligned}
\beta_{2,4}=\beta_{3,4}+\binom{8}{4}f_{-1}-\binom{7}{3}f_0+\binom{6}{2}f_1-\binom{5}{1}f_2+\binom{4}{0}f_3&\\
=5+70\cdot 1-35\cdot 8+15\cdot 21-5\cdot 24+1\cdot 16=6.&
\end{aligned}$$

Our calculations are consistent with the output of Macaulay2, which provides the following Betti table for $T(0;2,3)$:

```
       0 1  2  3  4 5 6
total: 1 7 16 20 15 6 1
    0: 1 .  .  .  . . .
    1: . 7 10  5  1 . .
    2: . .  6 15 14 6 1
```

Figure 9. **Betti table of the minimal free resolution of $R/I(T(0;2,3))$.**

Example 3. The caterpillar $T\cong T(2;2,2)$ has 9 vertices. According to (4.1), the nonzero Betti numbers of $R/(I(T(2;2,2)))$ in the linear strand are

$$\beta_{1,2}=8,\quad \beta_{2,3}=\binom{4}{2}+\binom{3}{2}+\binom{3}{2}=12,\quad \beta_{3,4}=\binom{4}{3}+\binom{3}{3}+\binom{3}{3}=6,\quad \beta_{4,5}=\binom{4}{4}=1.$$

The nonzero Betti numbers off the linear strand, computed through (4.2), are:

$$\beta_{1,3} = 0, \qquad \beta_{4,6} = \binom{2}{2}\binom{2}{2} + \binom{2}{1}\binom{2}{2} + \binom{2}{2}\binom{2}{1} = 5$$

$$\beta_{2,4} = \binom{2}{1}\binom{2}{1} = 4 \qquad \beta_{5,7} = \binom{2}{2} = 1$$

$$\beta_{3,5} = \binom{2}{1}\binom{2}{2} + \binom{2}{2}\binom{2}{1} + \binom{2}{1}\binom{2}{1} = 8.$$

From Proposition 3, the f-vector is $(1, 9, 28, 40, 28, 9, 1)$. As done in Example 2, we may compute the several $\beta_{q,q+2}$ by means of Theorem 10. For instance:

$$\beta_{1,3} = \beta_{2,3} + \binom{9}{3} f_{-1} - \binom{8}{2} f_0 + \binom{7}{1} f_1 - \binom{6}{0} f_2 = 12 + 84 \cdot 1 - 28 \cdot 9 + 7 \cdot 28 - 1 \cdot 40 = 0$$

$$\begin{aligned}\beta_{2,4} = \beta_{3,4} + \binom{9}{4} f_{-1} - \binom{8}{3} f_0 + \binom{7}{2} f_1 - \binom{6}{1} f_2 + \binom{5}{0} f_3 \\ = 6 + 126 \cdot 1 - 56 \cdot 9 + 21 \cdot 28 - 6 \cdot 40 + 1 \cdot 28 = 4\end{aligned}$$

Once again, our computational results are confirmed by the software Macaulay2, which provides the following Betti table for the graph $T(2; 2, 2)$:

	0	1	2	3	4	5
total:	1	8	16	14	6	1
0:	1	.	.	.	.	.
1:	.	8	12	6	1	.
2:	.	.	4	8	5	1

Figure 10. **Betti table of the minimal free resolution of $R/I(T(2; 2, 2))$.**

The explicit relation between the graded Betti numbers and the f-vector established in Theorem 10 shows that trees of diameter 4 exhibit a particularly transparent interaction between homological invariants and the combinatorics of their independence complexes. For chordal graphs, more general connections between Betti numbers and combinatorial data are known; for instance, it has been shown that the total Betti sequence of the edge ideal of a chordal graph coincides with the f-vector of a suitably associated simplicial complex (see [13]). Moreover, recursive descriptions via shellability and linear quotients are available for related classes of ideals. It would be natural to investigate whether explicit closed formulas, in the spirit of Theorem 10 and expressed directly in terms of the independence complex, can be obtained for broader families of graphs beyond the small-diameter case considered here.

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