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Research Article

Clean graphs and idempotent graphs over finite rings: an approach based on $\mathbb{Z}_n$

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Abstract: Let R be a finite ring with identity. The idempotent graph $I(R)$ is the graph whose vertex set consists of the non-trivial idempotent elements of R, where two distinct vertices x and y are adjacent if and only if $xy = yx = 0$. The clean graph $Cl(R)$ is a graph whose vertices are of the form (e, u), where e is an idempotent element and u is a unit of R. Two distinct vertices (e, u) and (f, v) are adjacent if and only if $ef = fe = 0$ or $uv = vu = 1$. The graph $Cl_2(R)$ is the subgraph of $Cl(R)$ induced by the set $\{(e, u) : e \text{ is a nonzero idempotent element of } R\}$. In this study, we examine the structure of clean graphs over $\mathbb{Z}_n$ derived from their Cl_2 graphs and investigate their relationship with the structure of their idempotent graphs. Furthermore, we obtain an equivalence between the isomorphism of two clean graphs and the isomorphism of their corresponding idempotent graphs over an Artinian ring.

Keywords: clean graph, idempotent graph, isomorphism graph, local ring, Artinian ring.

AMS Subject classification: 05C25, 05C75, 16U40, 16U60, 13M05

1. Introduction

The study of zero-divisor graph and algebraic graphs over ring was first introduced by Beck in [3], who considered all elements of a commutative ring R as vertices and explored the structure of its zero-divisor graph, primarily focusing on graph colorings. In [2], Anderson and Livingston formally defined the zero-divisor graph of commutative ring R, denoted by $\Gamma(R)$. In their definition, the vertices of $\Gamma(R)$ consist

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of all nonzero zero-divisors of R, with two distinct vertices x and y connected by an edge if and only if $xy = 0$.

In addition to the zero-divisor graph, there exists the unit graph over the ring $\mathbb{Z}_n$, introduced by Grimaldi in [8], and the unit regular graph, introduced by Susanti et al. in [17], and the idempotent graph introduced by Akbari et al. in [1]. The vertex set of the unit graph over $\mathbb{Z}_n$ is $\mathbb{Z}_n$, and two vertices x and y are adjacent if $x+y$ is a unit in $\mathbb{Z}_n$. The vertex set of the unit regular graph over a ring R is R, where two distinct vertices x and y are adjacent if and only if $x + y$ is a unit regular element of R. The vertex set of the idempotent graph over a ring R consists of the nontrivial idempotent elements of R, and two vertices x and y are adjacent if $xy = yx = 0$. Further research on idempotent graphs over matrix rings was conducted by [14], who determined the structure of the idempotent graph over the ring $M_2(\mathbb{F})$, where $\mathbb{F}$ is a field. Then, the study of graphs associated with rings has become an active area of research (see, for example, [8, 15, 16]). The concepts of idempotent graphs and clean rings motivated Habibi et al. in [9] to define a clean graph over a ring R, denoted by $Cl(R)$, where the vertices consist of all pairs of idempotent elements and unit elements of R. Two vertices (e, u) and (f, v) are adjacent if and only if $ef = fe = 0$ or $uv = vu = 1$. This naturally raises the question of how the clean graph over a ring is related to its idempotent graph, which served as the primary motivation for defining the clean graph.

Research on algebraic graphs has potential applications in coding theory. [7] constructed linear codes derived from the incidence matrix of the line graph of the Hamming graph. This motivated [11] to construct linear codes based on the incidence matrix of the unit graph over $\mathbb{Z}_n$, by categorizing cases based on the number of prime factors of n.

In this context, the theory of clean graphs can be utilized for further studies in coding theory. However, additional research is needed, particularly focusing on the construction of linear codes. This requires a thorough investigation of the structure of clean graphs over finite rings, especially $\mathbb{Z}_n$.

Before proceeding, we recall some basic terminology that will be used in this paper. Let R be a ring with identity. An element $e \in R$ is called *idempotent* if $e^2 = e$, and an element $u \in R$ is called a unit if there exists $v \in R$ such that $uv = vu = 1$. The sets of all idempotent elements and all unit elements of R are denoted by $Id(R)$ and $U(R)$, respectively. Additionally, the set $U(R)$ can be partitioned as follows:

$$U'(R) = \{u \in U(R) : u^2 = 1\} \text{ and } U''(R) = U(R) \setminus U'(R).$$

If an element $a \in R$ can be expressed as $a = e + u$, where e is an idempotent and u is a unit in R, then a is called clean. A ring R is said to be clean if every element of R is clean. In [10], Immormino proved that every finite ring is clean. From [6], a commutative ring R is said to be Artinian if whenever $I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$ is a decreasing chain of ideals of R, then there is a positive integer m such that $I_k = I_m$ for all $k \geq m$. Furthermore, there are only finitely many maximal ideals in R. For any undefined notation or terminology in ring theory and further studies related to clean rings, we refer the reader to [12] and [13].

Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ represent the set of vertices and edges, respectively. A graph G is said to be *connected* if there exists a path between every pair of distinct vertices, and *complete* if every pair of distinct vertices is adjacent. The complete graph with n vertices is denoted by K_n. The degree of a vertex v in graph G, denoted by $\deg_G(v)$, refers to the number of edges in graph G that incident to v. Two graphs G_1 and G_2 are said to be isomorphic, denoted $G_1 \cong G_2$, if there exists a bijection $f : V(G_1) \to V(G_2)$ between their vertex sets such that for every pair of vertices $u, v \in V(G_1)$, u and v are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2. For additional background on graph theory and relevant terminology, we refer the reader to [18].

In [9], Habibi et al. defined the subgraphs $Cl_1(R)$ and $Cl_2(R)$ as the induced subgraphs of $Cl(R)$, where the vertex sets are given by

$$Cl_1(R) = \{(0, u) : u \in U(R)\} \text{ and } Cl_2(R) = \{(e, u) : e \in Id(R) \setminus \{0\}, u \in U(R)\}.$$

Moreover, Remark 2.6 in [9] provides a formula for the degree for any vertex $x = (e, u)$ in the graph $Cl_2(R)$ as follows:

$$deg_{Cl_2(R)}(x) = \begin{cases} |Id(R)| + O_e|U(R)| - 2, & \text{if } u \in U'(R), \\ |Id(R)| + O_e|U(R)| - 1, & \text{if } u \in U''(R), \end{cases}$$

where $O_e := |\{f \in Id(R) \setminus \{0\} : ef = fe = 0\}|$. However, a counterexample can be found in the clean graph $Cl_2(\mathbb{Z}_{10})$. Specifically, the vertex $(6, 3)$ has a degree of 6, whereas applying the given formula in the remark results in a degree of 7. To address this discrepancy, this paper provides a correction to the degree formula for vertices in $Cl_2(R)$. Additionally, we investigate the structure of the graphs $Cl_2(\mathbb{Z}_n)$ on clean graphs $Cl_2(\mathbb{Z}_{p^n})$ and find a relation about the structure of clean graph $Cl_2(R)$ with their idempotent graph $I(R)$, for arbitrary ring R. Furthermore, we get corollary about the structure of clean graphs $Cl_2(\mathbb{Z}_{p_1^{n_1} p_2^{n_2}})$, $Cl_2(\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}})$, and $Cl_2(\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}})$, from which we derive a generalization for the structure of $Cl_2(\mathbb{Z}_n)$. Moreover, we show that for Artinian rings, the problem of determining whether two clean graphs are isomorphic reduces to checking the isomorphism of their corresponding idempotent graphs. This observation highlights an important advantage of our approach: the structure of a clean graph can be understood and extended in a more efficient manner by analyzing only its idempotent graph.

2. Result

The following theorem provides a correction to the vertex degree formula in the graph $Cl_2(R)$.

Theorem 1. *Let R be a ring with identity. For every $x = (e,u) \in V(Cl_2(R))$ we have*

$$deg_{Cl_2(R)}(x) = \begin{cases} |Id(R)| + O_e\,(|U(R)| - 1) - 2, & \text{if } u \in U'(R), \\ |Id(R)| + O_e\,(|U(R)| - 1) - 1, & \text{if } u \in U''(R). \end{cases}$$

Proof. For every $x = (e,u) \in V(Cl_2(R))$ we have two cases as follows.

(i) If $u \in U'(R)$, then $u^2 = 1$. For every vertex $(f,v) \in V(Cl_2(R))$,

$$(f,v)(e,u) \in E(Cl_2(R)) \iff fe = ef = 0 \text{ or } uv = 1.$$

(i.a) Suppose $uv = 1$, it means $v = u$, thus $(f,u)(e,u) \in E(Cl_2(R)$ for every $f \in Id(R) \setminus \{0, e\}$. There are $|Id(R)| - 2$ possibilities.

(i.b) Suppose $fe = ef = 0$, then $(f,v)(e,u) \in E(Cl_2(R))$ for every $v \in U(R)$. However, the vertex (f,u) has been included in the previous case, and there are $O_e\,(|U(R)| - 1)$ remaining possibilities.

(ii) If $u \in U''(R)$, then $u^2 \neq 1$. For every vertex $(f,v) \in V(Cl_2(R))$,

$$(f,v)(e,u) \in E(Cl_2(R)) \iff fe = ef = 0 \text{ or } uv = 1.$$

(ii.a) Suppose $uv = 1$, it means $v \neq u$, thus $(f,v)(e,u) \in E(Cl_2(R)$ for every $f \in Id(R) \setminus \{0\}$. There are $|Id(R)| - 1$ possibilities.

(ii.b) Suppose $fe = ef = 0$, then $(f,v)(e,u) \in E(Cl_2(R))$ for every $v \in U(R)$. However, the vertex (f,u^{-1}) has been included in the previous case, and there are $O_e\,(|U(R)| - 1)$ remaining possibilities.

□

2.1. Structure of clean graphs over $\mathbb{Z}_{p^n}$

This section is concerned with the structural description of the clean graph associated with $\mathbb{Z}_{p^n}$. We examine how the properties of clean elements determine the adjacency relation and shape the overall structure of the graph. The following key lemma is instrumental in determining the structure and characteristics of the clean graph over $\mathbb{Z}_{p^n}$.

Lemma 1. *Given ring $\mathbb{Z}_{p^n}$ with a prime number p and a natural number n. Let $a \in \mathbb{Z}_{p^n}$.*

(i) If $p \neq 2$, then $a^2 \equiv 1 \pmod{p^n}$ if and only if $a \in \{1, p^n - 1\}$.

(ii) If $p = 2$ and $n \geq 3$, then $a^2 \equiv 1 \pmod{p^n}$ if and only if $a \in \{1, 2^{n-1} - 1, 2^{n-1} + 1, 2^n - 1\}$.

Proof. Let $a \in \mathbb{Z}_{p^n}$.

(i) Let $a \in \{1, p^n - 1\}$. If $a = 1$, then $a^2 \equiv 1 \pmod{p^n}$. On the other hand, if $a = p^n - 1$, then $a \equiv (-1) \pmod{p^n}$. Hence, $a^2 \equiv 1 \pmod{p^n}$.

Conversely, let $a^2 \equiv 1 \pmod{p^n}$. It follows that $a^2 - 1 \equiv 0 \pmod{p^n}$ and so $(a-1)(a+1) \equiv 0 \pmod{p^n}$. Thus, we get $p^n \mid (a-1)$ or $p^n \mid (a+1)$. It means $a - 1 \equiv 0 \pmod{p^n}$ or $a + 1 \equiv 0 \pmod{p^n}$. Hence $a \equiv 1 \pmod{p^n}$ or $a \equiv (-1) \pmod{p^n} \equiv p^n - 1 \pmod{p^n}$. As a result $a \in \{1, p^n - 1\}$.

(ii) Assume that $a \in \{1, 2^{n-1} - 1, 2^{n-1} + 1, 2^n - 1\}$. If $a = 1$, then $a^2 \equiv 1 \pmod{2^n}$. Suppose that $a = 2^{n-1} \pm 1$. It follows that $a^2 \equiv (2^{n-1} \pm 1)^2 \pmod{p^n}$ and so $a^2 = \left((2^n)(2^{n-2}) \pm 2^n + 1\right) \pmod{p^n}$. Thus $a^2 \equiv 1 \pmod{2^n}$. Suppose now that $a = 2^n - 1$. Then $a \equiv (-1) \pmod{2^n}$ and hence $a^2 \equiv 1 \pmod{2^n}$.
Conversely, let $a^2 \equiv 1 \pmod{p^n}$. Analogously to the previous point, we obtain

$$(a-1)(a+1) \equiv 0 \pmod{2^n}.$$

Consequently, it follows that $2^c \mid a+1$ and $2^{n-c} \mid a-1$ for $0 \leq c \leq n$. Assuming $2 \leq c \leq n-2$, we can write $a + 1 = 2^c t$ and $a - 1 = 2^{n-c} k$, where $t, k \in \mathbb{Z}^+$. Hence, $2^c t - 2^{n-c} k = 2$ if and only if $2\left(2^{c-2} t - 2^{n-c-2} k\right) = 1$. This is impossible since t and k are integers.

Thus, the possible values of c that satisfy the condition are $\{0, 1, n-1, n\}$. Consequently, there are several possible cases: $2^0 \mid a+1$ and $2^n \mid a-1$, or $2^1 \mid a+1$ and $2^{n-1} \mid a-1$, or $2^{n-1} \mid a+1$ and $2^1 \mid a-1$, or $2^n \mid a+1$ and $2^0 \mid a-1$. Therefore, the possible values of a are $\{1, 2^{n-1} - 1, 2^{n-1} + 1, 2^n - 1\}$. □

Lemma 1 characterizes the elements of $\mathbb{Z}_{p^n}$ that are self-inverse under multiplication. This characterization plays a central role in determining the adjacency relations in the clean graph, and consequently in describing the structure of $Cl_2(\mathbb{Z}_{p^n})$ as presented in Theorem 2.

Theorem 2. *Given ring $\mathbb{Z}_{p^n}$ with a prime number p and a natural number n. It holds that*

$$Cl_2(\mathbb{Z}_{p^n}) = \begin{cases} K_1, & \text{if } p = 2, n = 1, \\ 2K_1, & \text{if } p = 2, n = 2, \\ 4K_1 \cup \left(2^{n-1} - 2^{n-2} - 2\right) K_2, & \text{if } p = 2, n \geq 3, \\ 2K_1 \cup \left(\frac{p^n - p^{n-1}}{2} - 1\right) K_2, & \text{if } p \neq 2, n \geq 1. \end{cases}$$

Proof. Suppose $p = 2$ and $n = 1$, we have $\mathbb{Z}_{p^n} = \mathbb{Z}_2$, then $Cl_2(\mathbb{Z}_2) = K_1$. Suppose $p = 2$ and $n = 2$, we have $Cl_2(\mathbb{Z}_{p^n}) = Cl_2(\mathbb{Z}_4) = 2K_1$. For any prime number and any natural number not included in the cases above, the following holds:

$$\begin{aligned} V(Cl_2(\mathbb{Z}_{p^n})) &= \{(1, u) : u \in U(\mathbb{Z}_{p^n})\} \\ &= \{(1, u) : u \in \mathbb{Z}_{p^n} \setminus \langle p \rangle\}. \end{aligned}$$

Using Lemma 1, we get

$$|U'(\mathbb{Z}_{p^n})| = \begin{cases} 2, & \text{if } p \neq 2, \\ 4, & \text{if } p = 2 \text{ and } n \geq 3. \end{cases}$$

Since $|U''(\mathbb{Z}_{p^n})| = |U(\mathbb{Z}_{p^n})| - |U'(\mathbb{Z}_{p^n})|$, we derive

$$|U''(\mathbb{Z}_{p^n})| = \begin{cases} p^n - p^{n-1} - 2, & \text{if } p \neq 2, \\ 2^n - 2^{n-1} - 4, & \text{if } p = 2 \text{ and } n \geq 3. \end{cases}$$

Consequently, we obtain

$$Cl_2(\mathbb{Z}_{p^n}) = \begin{cases} 2K_1 \cup \frac{p^n - p^{n-1} - 2}{2} K_2, & \text{if } p \neq 2, \\ 4K_1 \cup \frac{2^n - 2^{n-1} - 4}{2} K_2, & \text{if } p = 2 \text{ and } n \geq 3 \end{cases}$$

$$Cl_2(\mathbb{Z}_{p^n}) = \begin{cases} 2K_1 \cup \left(\frac{p^n - p^{n-1}}{2} - 1 \right) K_2, & \text{if } p \neq 2, \\ 4K_1 \cup \left(2^{n-1} - 2^{n-2} - 2 \right) K_2, & \text{jika } p = 2 \text{ and } n \geq 3. \end{cases}$$

□

2.2. Structure of clean graphs over $\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}}$, $\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3} p_4^{n_4}}$, and generalized structure of clean graph over $\mathbb{Z}_n$

Before discussing the structure of graphs over $\mathbb{Z}_n$ for any natural number n, we introduce a generalization of the shuriken graph, referred to as the shuriken operation on a graph, as defined below.

Definition 1. Let $G = (V(G), E(G))$ be a graph and let n, t be the positive integers such that $n - t$ is even. The (t, n)-shuriken graph of G, denoted by $Shu_n^t(G)$, is constructed from G by first adding a new vertex z and then creating n copies of the resulting graph. Let G_i' for $1 \leq i \leq n$ denote the i-th copy of G after the addition of the new vertex. The vertex set and edge set of the graph $Shu_n^t(G)$ are given by:

$$\begin{aligned} V(Shu_n^t(G)) =& \bigcup_{i=1}^{n} \{z_i, v_i : v \in V(G)\} \text{ and} \\ E(Shu_n^t(G)) =& \{u_i v_j : uv \in E(G), i, j \in \{1, 2, \ldots, n\}\} \\ & \cup \{u_i v_i : u_i, v_i \in V(G_i'), u_i \neq v_i, i \in \{1, 2, \ldots, t\}\} \\ & \cup \left\{ u_i v_{n+t+1-i} : u_i \in V(G_i'), v_{n+t+1-i} \in V(G_{n+t+1-i}') \right. \\ & \left. \qquad i \in \left\{ t+1, t+2, \ldots, \frac{n+t}{2} \right\} \right\}. \end{aligned}$$

Given graph P_3, with $V(P_3) = \{a, b, c\}$. The $(2,4)$-shuriken graph of P_3, $Shu_4^2(P_3)$ is presented in Figure 2.2.

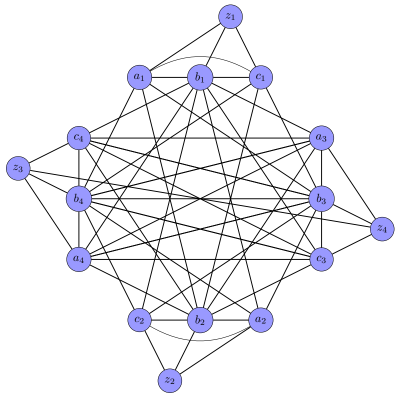

Figure 1. **Graph $Shu_4^2(P_3)$. The visualization of the shuriken graph obtained from the path graph P_3 with parameters $t = 2$ and $n = 4$.**

We now present a discussion on the properties of the (t, n)-shuriken graph for any given graph G, as presented in the following theorems.

Theorem 3. *Given any graph G and positive integers $n \geq t \geq 2$ such that $n - t$ is even. The (t, n)-shuriken graph of G is disconnected if and only if the graph G is a null graph.*

Proof. Let G be null graph. Let $|V(G)| = k$. We have

$$E(Shu_n^t(G)) = \{u_i v_i : u_i, v_i \in V(Shu_n^t(G)), u_i \neq v_i, i \in \{1, 2, \ldots, n\}\} \cup \left\{ u_i v_{n+t+1-i} : u_i, v_{n+t+1-i} \in V(Shu_n^t(G)) \quad i \in \left\{t+1, t+2, \ldots, \frac{n+t}{2}\right\} \right\}.$$

There exists sets of vertices $X_1 = \{z_1, v_1 : v \in V(G)\}$ and $X_2 = \{z_2, v_2 : v \in V(G)\} \subseteq V(Shu_n^t(G))$, such that no two points in X_1 and X_2 are connected to each other.

Conversely, let G be a graph that is not null. This means that there exists an edge $uv \in E(G)$. Let $x_i, y_j \in V(Shu_n^t(G))$ be arbitrary, with $i, j \in 1, 2, \ldots, n$. This implies that $x_i \in V(G'_i)$ and $y_j \in V(G'_j)$. The following cases are considered:

1. Case $x_i \neq u_i$ and $y_j \neq v_j$. Several paths are available as follows:

$$\begin{aligned}
&x_i - u_i - v_j - y_j, && \text{if } 1 \leq i, j \leq t,\\
&x_i - u_i - v_{n+t+1-j} - y_j, && \text{if } 1 \leq i \leq t \text{ and } t+1 \leq j \leq n,\\
&x_i - u_{n+t+1-i} - v_j - y_j, && \text{if } t+1 \leq i \leq n \text{ and } 1 \leq j \leq t,\\
&x_i - u_{n+t+1-i} - v_{n+t+1-j} - y_j, && \text{if } t+1 \leq i, j \leq n.
\end{aligned}$$

2. Case $x_i = u_i$ and $y_j \neq v_j$. The following paths are obtained:

$$\begin{aligned}
&x_i - v_j - y_j, && \text{if } 1 \leq i, j \leq t,\\
&x_i - v_{n+t+1-j} - y_j, && \text{if } 1 \leq i \leq t \text{ and } t+1 \leq j \leq n,\\
&x_i - v_j - y_j, && \text{if } t+1 \leq i \leq n \text{ and } 1 \leq j \leq t,\\
&x_i - v_{n+t+1-j} - y_j, && \text{if } t+1 \leq i, j \leq n.
\end{aligned}$$

3. Case $x_i \neq u_i$ and $y_j = v_j$. The following paths are obtained:

$$\begin{aligned}
&x_i - u_i - y_j, && \text{if } 1 \leq i, j \leq t,\\
&x_i - u_i - y_j, && \text{if } 1 \leq i \leq t \text{ and } t+1 \leq j \leq n,\\
&x_i - u_{n+t+1-i} - y_j, && \text{if } t+1 \leq i \leq n \text{ and } 1 \leq j \leq t,\\
&x_i - u_{n+t+1-i} - y_j, && \text{if } t+1 \leq i, j \leq n.
\end{aligned}$$

4. If $x_i = u_i$ and $y_j = v_j$, then $x_i y_j \in E(Shu_n^t(G))$.

□

Having established the connectivity condition of the (t, n)-shuriken graph, we now investigate whether the shuriken construction preserves the graph isomorphism, as stated in Proposition 1.

Proposition 1. *Given any connected graphs G_1 and G_2, and positive integers t and n, with $2 \leq t < n$ and $n - t$ even. The following holds:*

$$Shu_n^t(G_1) \cong Shu_n^t(G_2) \iff G_1 \cong G_2.$$

Proof. Since $Shu_n^t(G_1) \cong Shu_n^t(G_2)$, we have

$$|V(Shu_n^t(G))| = |V(Shu_n^t(H))| \text{ and } |E(Shu_n^t(G))| = |E(Shu_n^t(H))|.$$

Thus, $|V(G_1)| = |V(G_2)| = k$. Let $i, j \in \{t+1, t+2, \ldots, n\}$ be arbitrary. Let

$$X_1 = \{u_i : u \in V(G_1)\}$$

and

$$X_2 = \{v_j : v \in V(G_2)\}.$$

The following bijective functions can be constructed.

$$\psi : V(\langle X_1 \rangle_{Shu_n^t(G_1)}) \to V(G_1) \text{ and } \phi : V(\langle X_2 \rangle_{Shu_n^t(G_2)}) \to V(G_2),$$

with $\psi(u_i) = u$ for each $u_i \in V(\langle X_1 \rangle_{Shu_n^t(G_1)})$ and $\phi(v_j) = v$ for any $v_j \in V(\langle X_2 \rangle_{Shu_n^t(G_2)})$. Let $x_i, y_i \in X_1$ and $u_j, v_j \in X_2$, such that $x_i y_i \in E(\langle X_1 \rangle_{Shu_n^t(G_1)})$ and $u_j v_j \in E(\langle X_2 \rangle_{Shu_n^t(G_2)})$. We have $x, y \in V(G_1)$ and $u, v \in V(G_2)$, such that $x_i y_i \in E(Shu_n^t(G_1))$ and $u_j v_j \in E(Shu_n^t(G_2))$. Since $i, j \in \{t+1, t+2, \ldots, n\}$, it must be $\psi(x_i)\psi(y_i) = xy \in E(G_1)$ and $\phi(u_j)\phi(v_j) = uv \in E(G_2)$. Conversely, let $x_i, y_i \in X_1$ and $u_j, v_j \in X_2$, such that $xy \in E(G_1)$ and $uv \in E(G_2)$. Based on the definition of adjacency in the graphs $Shu_n^t(G_1)$ and $Shu_n^t(G_2)$, we obtain $x_i y_i \in E(Shu_n^t(G_1))$ and $u_j v_j \in E(Shu_n^t(G_2))$. In other word, $x_i y_i \in E(\langle X_1 \rangle_{Shu_n^t(G_1)})$ and $u_j v_j \in E(\langle X_2 \rangle_{Shu_n^t(G_2)})$. Hence,

$$\langle X_1 \rangle_{Shu_n^t(G_1)} \cong G_1 \text{ and } \langle X_2 \rangle_{Shu_n^t(G_2)} \cong G_2.$$

Since $Shu_n^t(G_1) \cong Shu_n^t(G_2)$, there exists a graph isomorphism

$$f : Shu_n^t(G_1) \to Shu_n^t(G_2).$$

Observe that:

$$\begin{aligned}
deg_{Shu_n^t(G_1)}(z_i) &= deg_{Shu_n^t(G_2)}(z_i) = k, \text{ for } 1 \le i \le t, \\
deg_{Shu_n^t(G_1)}(z_i) &= deg_{Shu_n^t(G_2)}(z_i) = k+1, \text{ for } t+1 \le i \le n, \\
deg_{Shu_n^t(G_1)}(u_i) &= n(deg_{G_1}(u)) + k, \text{ for } u \in V(G_1) \text{ and } 1 \le i \le t, \\
deg_{Shu_n^t(G_1)}(u_i) &= n(deg_{G_1}(v)) + k + 1, \text{ for } u \in V(G_1) \text{ and } t+1 \le i \le n, \\
deg_{Shu_n^t(G_2)}(v_i) &= n(deg_{G_2}(v)) + k, \text{ for } v \in V(G_2) \text{ and } 1 \le i \le t, \\
deg_{Shu_n^t(G_2)}(v_i) &= n(deg_{G_2}(v)) + k + 1, \text{ for } v \in V(G_2) \text{ and } t+1 \le i \le n,
\end{aligned}$$

therefore

$$f(\{z_i : i = 1, 2, \ldots, t\}) = \{z_i : i = 1, 2, \ldots, t\}$$

and

$$f(\{z_i : i = t+1, t+2, \ldots, n\}) = \{z_i : i = t+1, t+2, \ldots, n\}.$$

Furthermore,

$$\begin{aligned}
N_{Shu_n^t(G_1)}(z_i) &= \{u_i : u \in V(G_1)\}, \text{ for } 1 \le i \le t, \\
N_{Shu_n^t(G_1)}(z_a) &= \{z_{n+t+1-a}, u_{n+t+1-a} : u \in V(G_1)\}, \text{ for } t+1 \le a \le n, \\
N_{Shu_n^t(G_2)}(z_j) &= \{v_j : v \in V(G_2)\}, \text{ for } 1 \le j \le t, \\
N_{Shu_n^t(G_2)}(z_b) &= \{z_{n+t+1-b}, v_{n+t+1-b} : v \in V(G_2)\}, \text{ for } t+1 \le b \le n.
\end{aligned}$$

Consider $u_n \in V(Shu_n^t(G_1))$ for any $u \in V(G_1)$. Since $u_n z_{t+1} \in E(Shu_n^t(G_1))$, we have $f(u_n)f(z_{t+1}) \in E(Shu_n^t(G_2))$. Assume that $f(u_n) = v_m$ and $f(z_{t+1}) = z_{n+t+1-m}$ with $t+1 \leq m \leq n$. Consequently,

$$f(\{u_n : u \in V(G_1)\}) = \{v_m : v \in V(G_2)\}.$$

Let $X_1 = \{u_n : u \in V(G_1)\}$ and $X_2 = \{v_m : v \in V(G_2)\}$. As f is a graph isomorphism, it follows that $\langle X_1 \rangle_{Shu_n^t(G_1)} \cong \langle X_2 \rangle_{Shu_n^t(G_2)}$. Hence,

$$G_1 \cong \langle X_1 \rangle_{Shu_n^t(G_1)} \cong \langle X_2 \rangle_{Shu_n^t(G_2)} \cong G_2.$$

For the converse, if $G_1 \cong G_2$, it follows trivially that $Shu_n^t(G_1) \cong Shu_n^t(G_2)$. □

The preservation of structure under shuriken construction naturally leads to its application in algebraic graph theory. The following theorem shows that the clean graph over a ring can be represented as a shuriken graph over its idempotent graph.

Theorem 4. *Let R be an arbitrary ring with an identity element. It holds that*

$$Cl_2(R) \cong Shu_{|U(R)|}^{|U'(R)|}(I(R)).$$

Proof. Let $Id(R) \setminus \{0\} = \{e_1 = 1, e_2, e_3, \ldots, e_k\}$ and

$$U(R) = \{u_1, u_2, \ldots, u_t, u_{t+1}, u_{t+2}, \ldots, u_n\}$$

where

$$U'(R) = \{u_1, u_2, \ldots, u_t\} \text{ and } u_i u_{n+t+1-i} = u_{n+t+1-i} u_i = 1$$
$$\text{for each } i = t+1, t+2, \ldots, \frac{n+t}{2}.$$

We get $V(Cl_2(R)) = \{(e_i, u_j) : i = 1, 2, \ldots, k, j = 1, 2, \ldots, n\}$. On the other hand, $V(I(R)) = \{e_2, e_3, \ldots, e_k\}$, thus

$$\begin{aligned}
V(Shu_{|U(R)|}^{|U'(R)|}(I(R))) &= V(Shu_n^t(I(R))) \\
&= \bigcup_{i=1}^{n} \{e_{1i}, v_i : v \in V(I(R))\} \\
&= \{e_{11}, e_{12}, \ldots, e_{1n}, e_{21}, e_{22}, \ldots, e_{2n}, \\
&\quad e_{31}, e_{32}, \ldots, e_{3n}, \ldots, e_{k1}, e_{k2}, \ldots, e_{kn}\}
\end{aligned}$$

and

$$E(Shu_n^t(I(R))) = \left\{ e_{li}e_{j(n+t+1-i)} : l,j \in \{1,2,\ldots,k\}, i \in \left\{t+1, t+2, \ldots, \frac{n+t}{2}\right\} \right\}$$
$$\cup \{e_{ia}e_{jb} : e_ie_j \in E(I(R)), a,b \in \{1,2,\ldots,n\}\}$$
$$\cup \{e_{li}e_{ji} : l,j \in \{1,2,\ldots,k\}, l \neq j, i \in \{1,2,\ldots,t\}\}.$$

Consequently, we obtain $|V(Cl_2(R))| = nk = |V(Shu_n^t(I(R)))|$. Subsequently, define the function $f : V(Cl_2(R)) \to V(Shu_n^t(I(R)))$, where

$$f(e_j, u_i) = e_{ji}$$

for all $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,k$. Let $(e_{j_1}, u_{i_1}), (e_{j_2}, u_{i_2}) \in V(Cl_2(R))$ such that $f((e_{j_1}, u_{i_1})) = f((e_{j_2}, u_{i_2}))$, we get

$$e_{j_1i_1} = e_{j_2i_2} \iff j_1 = j_2 \text{ and } i_1 = i_2 \iff (e_{j_1}, u_{i_1}) = (e_{j_2}, u_{i_2}).$$

Hence, f is bijective function.
Let $(e_{j_1}, u_{i_1}), (e_{j_2}, u_{i_2}) \in V(Cl_2(R))$ such that $(e_{j_1}, u_{i_1})(e_{j_2}, u_{i_2}) \in E(Cl_2(R))$. We have two possibilites, $e_{j_1}e_{j_2} = e_{j_2}e_{j_1} = 0$ or $u_{i_1}u_{i_2} = u_{i_2}u_{i_1} = 1$.

1. If $e_{j_1}e_{j_2} = e_{j_2}e_{j_1} = 0$, then $e_{j_1}e_{j_2} \in E(I(R))$. Hencce,

$$f((e_{j_1}, u_{i_1}))f((e_{j_2}, u_{i_2})) = e_{j_1i_1}e_{j_2i_2} \in E(Shu_n^t(I(R))).$$

2. If $u_{i_1}u_{i_2} = u_{i_2}u_{i_1} = 1$, then $i_2 = n+t+1-i_1$ with $i_1 \in \{t+1, t+2, \ldots, n\}$ or $i_1 = i_2 \in \{1,2,\ldots,t\}$. Consequently,

$$f((e_{j_1}, u_{i_1}))f((e_{j_2}, u_{i_2})) = e_{j_1i_1}e_{j_2(n+t+1-i_1)} \in E(Shu_n^t(I(R))), \text{ or}$$
$$f((e_{j_1}, u_{i_1}))f((e_{j_2}, u_{i_2})) = e_{j_1i_1}e_{j_2i_1} \in E(Shu_n^t(I(R)))$$

Let $(e_{j_1}, u_{i_1}), (e_{j_2}, u_{i_2}) \in V(Cl_2(R))$ such that

$$f(e_{j_1}, u_{i_1})f(e_{j_2}, u_{i_2}) \in E(Shu_n^t(I(R))).$$

In other words, $e_{j_1i_1}e_{j_2i_2} \in E(Shu_n^t(I(R)))$. Based on the definition of the elements in $E(Shu_n^t(I(R)))$, the following three possibilities are obtained:

1. If $e_{j_1}e_{j_2} \in I(R)$, then $e_{j_1}e_{j_2} = e_{j_2}e_{j_1} = 0$. Hence, $(e_{j_1}, u_{i_1})(e_{j_2}, u_{i_2}) \in E(Cl_2(R))$.

2. If $i_1 = i_2 \in \{1, 2, \ldots, t\}$ and $e_{j_1} \neq e_{j_2}$, we have $u_{i_1}u_{i_2} = u_{i_2}u_{i_1} = 1$. Thus, $(e_{j_1}, u_{i_1})(e_{j_2}, u_{i_2}) \in E(Cl_2(R))$.

3. If $i_2 = n + t + 1 - i_1 \in \{t + 1, t + 2, \ldots, n\}$, then

$$u_{i_1}u_{i_2} = u_{i_1}u_{n+t+1-i_1} = 1 = u_{n+t+1-i_1}u_{i_1} = u_{i_2}u_{i_1}.$$

Hence, $(e_{j_1}, u_{i_1})(e_{j_2}, u_{i_2}) \in E(Cl_2(R))$.

□

After establishing the general structure theorem for clean graphs by relating them to the idempotent graph of a ring, we can now apply this result to determine the clean graph structures over rings such as $\mathbb{Z}_{p_1^{n_1}p_2^{n_2}}$, $\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}}$, $\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}}$ and more broadly over $\mathbb{Z}_n$ for any positive integer n.

Corollary 1. *For any distinct prime numbers p_1, p_2 and natural numbers n_1, n_2, the following holds*

$$Cl_2(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}}) \cong Shu^{|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})|}_{(p_1^{n_1}-p_1^{n_1-1})(p_2^{n_2}-p_2^{n_2-1})}(P_2).$$

Building upon Corollary 1, we now specify the explicit form of the clean graph over $\mathbb{Z}_{p_1^{n_1}p_2^{n_2}}$ according to the values of $p_1^{n_1}$ and $p_2^{n_2}$. The following proposition provides a detailed classification of its structure.

Proposition 2. *For any distinct prime numbers p_1, p_2 and natural numbers n_1, n_2, the following holds*

$$Cl_2(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}}) \cong \begin{cases} Shu^2_{p_2^{n_2}-p_2^{n_2-1}}(P_2), & \text{if } p_1^{n_1} = 2, \\ Shu^4_{2(p_2^{n_2}-p_2^{n_2-1})}(P_2), & \text{if } p_1^{n_1} = 4, \\ Shu^8_{(p_1^{n_1}-p_1^{n_1-1})(p_2^{n_2}-p_2^{n_2-1})}(P_2), & \text{if } p_1 = 2, n_1 \geq 3, \\ Shu^4_{(p_1^{n_1}-p_1^{n_1-1})(p_2^{n_2}-p_2^{n_2-1})}(P_2), & \text{if } p_1, p_2 \neq 2. \end{cases}$$

Proof. From Corollary 1, we know that

$$Cl_2(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}}) \cong Shu^{|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})|}_{(p_1^{n_1}-p_1^{n_1-1})(p_2^{n_2}-p_2^{n_2-1})}(P_2).$$

Based on Lemma 1, the following four cases are considered.

1. Case $p_1^{n_1} = 2$. Therefore, $p_2 \neq 2$. We have $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})| = p_2^{n_2} - p_2^{n_2-1}$ and

$$\left(|U'(\mathbb{Z}_{p_1^{n_1}})| = 1,\ |U'(\mathbb{Z}_{p_2^{n_2}})| = 2\right) \Rightarrow |U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})| = 2.$$

2. Case $p_1^{n_1} = 4$. It must be that $p_2 \neq 2$. We have $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})| = 2(p_2^{n_2} - p_2^{n_2-1})$ and $\Big(|U'(\mathbb{Z}_{p_1^{n_1}})| = 2,\, |U'(\mathbb{Z}_{p_2^{n_2}})| = 2\Big) \Rightarrow |U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})| = 4.$

3. Case $p_1 = 2$ and $n_1 \geq 3$. This implies $p_2 \neq 2$. We have

$$|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})| = (p_1^{n_1} - p_1^{n_1-1})(p_2^{n_2} - p_2^{n_2-1})$$

and

$$(|U'(\mathbb{Z}_{p_1^{n_1}})| = 4,\, |U'(\mathbb{Z}_{p_2^{n_2}})| = 2) \Rightarrow |U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})| = 8.$$

4. Case $p_1, p_2 \neq 2$. We have $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})| = (p_1^{n_1} - p_1^{n_1-1})(p_2^{n_2} - p_2^{n_2-1})$ and

$$\Big(|U'(\mathbb{Z}_{p_1^{n_1}})| = 2,\, |U'(\mathbb{Z}_{p_2^{n_2}})| = 2\Big) \Rightarrow |U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}})| = 4.$$

□

For example, the clean graph $Cl_2(\mathbb{Z}_{40}) \cong Shu^8_{(2^3-2^2)(5^1-5^0)}(P_2) = Shu^8_{16}(P_2)$, which is illustrated in the Figure 2.2.

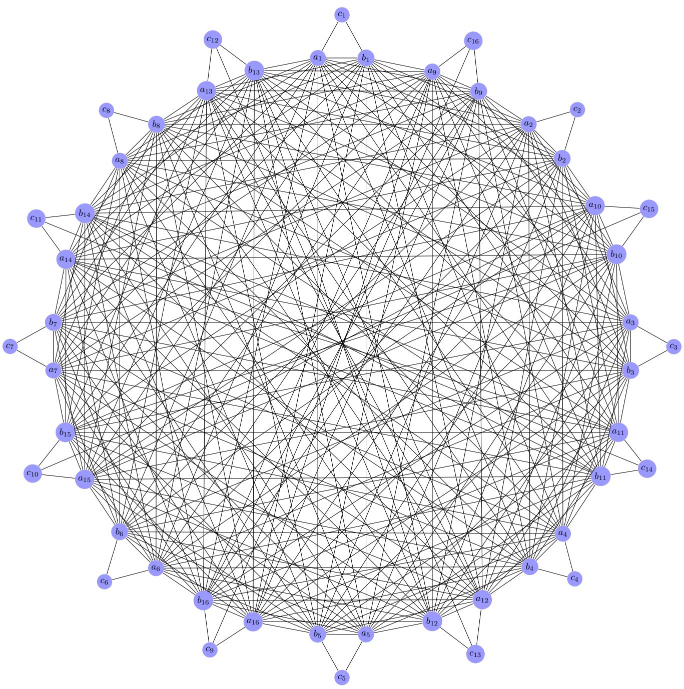

Figure 2. **Graph $Cl_2(\mathbb{Z}_{40})$. The visualization of the clean graph $Cl_2(\mathbb{Z}_{40})$, which is isomorphic to the shuriken graph constructed from K_2 with parameters $t = 8$ and $n = 16$.**

Before presenting the next result, we explicitly describe the idempotent graph associated with the ring $\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}}$. The graph G_1 illustrated in Figure 2 represents the adjacency relations among its idempotent elements. Using this structure, the following corollary characterizes the clean graph over the ring as a shuriken graph over G_1.

Corollary 2. *For any distinct prime numbers p_1, p_2, p_3 and natural numbers n_1, n_2, n_3, the following holds*

$$Cl_2(\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}}) \cong Shu_{|U(\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}})|}^{|U'(\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}})|}(G_1),$$

where the structure of the graph G_1 is shown in the Figure 2:

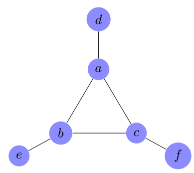

Figure 3. **Graph G_1. The visualization of the idempotent graph $I(\mathbb{Z}_{p_1^{n_1}} \mathbb{Z}_{p_2^{n_2}} \mathbb{Z}_{p_3^{n_3}})$, where p_1, p_2, p_3 are distinct prime numbers and n_1, n_2, n_3 are positive integers.**

Based on Corollary 2, we can derive a more explicit form of the clean graph depending on the values of $p_1^{n_1}$, $p_2^{n_2}$, and $p_3^{n_3}$, as shown in the following proposition.

Proposition 3. *For any distinct prime numbers p_1, p_2, p_3 and natural numbers n_1, n_2, n_3, the following holds*

$$Cl_2(\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}}) \cong \begin{cases} Shu^4_{\left(p_2^{n_2} - p_2^{n_2-1}\right)\left(p_3^{n_3} - p_3^{n_3-1}\right)}(G_1), & \text{if } p_1^{n_1} = 2, \\ Shu^8_{2\left(p_2^{n_2} - p_2^{n_2-1}\right)\left(p_3^{n_3} - p_3^{n_3-1}\right)}(G_1), & \text{if } p_1{}^{n_1} = 4, \\ Shu^{16}_m(G_1), & \text{if } p_1 = 2, n_1 \geq 3, \\ Shu^8_m(G_1), & \text{if } p_1, p_2, p_3 \neq 2, \end{cases}$$

where $m = \left(p_1^{n_1} - p_1^{n_1-1}\right)\left(p_2^{n_2} - p_2^{n_2-1}\right)\left(p_3^{n_3} - p_3^{n_3-1}\right)$.

Proof. Based on Corollary 2, we have

$$Cl_2(\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}}) \cong Shu_{|U(\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}})|}^{|U'(\mathbb{Z}_{p_1^{n_1} p_2^{n_2} p_3^{n_3}})|}(G_1).$$

From Lemma 1, we obtain four possibilities.

1. Assume that $p_1^{n_1} = 2$.
Therefore, $p_2, p_3 \neq 2$. We get $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}})| = \left(p_2^{n_2} - p_2^{n_2-1}\right)\left(p_3^{n_3} - p_3^{n_3-1}\right)$ and $|U'(\mathbb{Z}_{p_1^{n_1}})| = 1, |U'(\mathbb{Z}_{p_2^{n_2}})| = 2, |U'(\mathbb{Z}_{p_3^{n_3}})| = 2$ leading that $|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}})| = 4$.

2. Assume that $p_1^{n_1} = 4$.
Hence, $p_2, p_3 \neq 2$. We get $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}})| = 2(p_2^{n_2} - p_2^{n_2-1})(p_3^{n_3} - p_3^{n_3-1})$ and $|U'(\mathbb{Z}_{p_1^{n_1}})|=2$, $|U'(\mathbb{Z}_{p_2^{n_2}})| = 2, |U'(\mathbb{Z}_{p_3^{n_3}})| = 2$ implying that $|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}})| = 8$.

3. Assume that $p_1 = 2$ and $n_1 \geq 3$.
It must be $p_2, p_3 \neq 2$. Thus, $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}})|=\left(p_1^{n_1} - p_1^{n_1-1}\right)\left(p_2^{n_2} - p_2^{n_2-1}\right)\left(p_3^{n_3} - p_3^{n_3-1}\right)$ and $|U'(\mathbb{Z}_{p_1^{n_1}})| = 4$, $|U'(\mathbb{Z}_{p_2^{n_2}})| = 2$, $|U'(\mathbb{Z}_{p_3^{n_3}})| = 2$ leading to $|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}})| = 16$.

4. Assume that $p_1, p_2, p_3 \neq 2$.
We have $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}})| = \left(p_1^{n_1} - p_1^{n_1-1}\right)\left(p_2^{n_2} - p_2^{n_2-1}\right)\left(p_3^{n_3} - p_3^{n_3-1}\right)$ and $|U'(\mathbb{Z}_{p_1^{n_1}})|=2$, $|U'(\mathbb{Z}_{p_2^{n_2}})| = 2$, $|U'(\mathbb{Z}_{p_3^{n_3}})| = 2$ leading that $|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}})| = 8$.

□

Following the previous case with three prime factors, we now analyze the case of four distinct primes. The idempotent graph of $\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}}$, denoted by G_2 and shown in Figure 4, is used to describe the clean graph $Cl_2(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})$, which is isomorphic to a shuriken graph over G_2, as established in the following proposition.

Proposition 4. *For any distinct prime numbers p_1, p_2, p_3, p_4 and natural numbers n_1, n_2, n_3, n_4, the following holds*

$$Cl_2(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}}) \cong \begin{cases} Shu_m^8(G_2), & \text{if } p_1^{n_1} = 2, \\ Shu_{2m}^{16}(G_2), & \text{if } {p_1}^{n_1} = 4, \\ Shu^{32}_{\left(p_1^{n_1} - p_1^{n_1-1}\right)m}(G_2), & \text{if } p_1 = 2, n_1 \geq 3, \\ Shu^{16}_{\left(p_1^{n_1} - p_1^{n_1-1}\right)m}(G_2), & \text{if } p_1, p_2, p_3, p_4 \neq 2, \end{cases}$$

where $m = \left(p_2^{n_2} - p_2^{n_2-1}\right)\left(p_3^{n_3} - p_3^{n_3-1}\right)\left(p_4^{n_4} - p_4^{n_4-1}\right)$ and the structure of the graph G_2 is shown in the following figure:

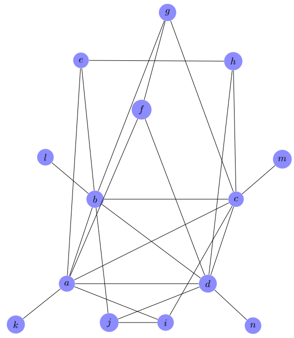

Figure 4. **Graph G_2. The visualization of the idempotent graph $I(\mathbb{Z}_{p_1^{n_1}}\mathbb{Z}_{p_2^{n_2}}\mathbb{Z}_{p_3^{n_3}}\mathbb{Z}_{p_4^{n_4}})$, where p_1, p_2, p_3, p_4 are distinct prime numbers and n_1, n_2, n_3, n_4 are positive integers**

Proof. Based on Theorem 4, we have

$$Cl_2(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}}) \cong Shu_{|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})|}^{|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})|}(I(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})).$$

In this case, $G_2 = I(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})$, with $a = (1,0,0,0)$, $b = (0,1,0,0)$, $c = (0,0,1,0)$, $d = (0,0,0,1)$, $e = (0,0,1,1)$, $f = (0,1,1,0)$, $g = (1,0,0,1)$, $h = (1,1,0,0)$, $i = (0,1,0,1)$, $j = (1,0,1,0)$, $k = (0,1,1,1)$, $l = (1,0,1,1)$, $m = (1,1,0,1)$, and $n = (1,1,1,0)$. From Lemma 1, we have four possibilities below.

1. $p_1^{n_1} = 2$.
   It must be that $p_2, p_3, p_4 \neq 2$. We get $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})| = (p_2^{n_2} - p_2^{n_2-1})(p_3^{n_3} - p_3^{n_3-1})(p_4^{n_4} - p_4^{n_4-1})$ and $|U'(\mathbb{Z}_{p_1^{n_1}})| = 1$, $|U'(\mathbb{Z}_{p_2^{n_2}})| = 2$, $|U'(\mathbb{Z}_{p_3^{n_3}})| = 2$, $|U'(\mathbb{Z}_{p_4^{n_4}})| = 2$ implying that $|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})| = 8$.

2. $p_1^{n_1} = 4$.
   It follows that $p_2, p_3, p_4 \neq 2$. Therefore, $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})| = 2\,(p_2^{n_2} - p_2^{n_2-1})(p_3^{n_3} - p_3^{n_3-1})(p_4^{n_4} - p_4^{n_4-1})$ and $|U'(\mathbb{Z}_{p_1^{n_1}})| = 2$, $|U'(\mathbb{Z}_{p_2^{n_2}})| = 2$, $|U'(\mathbb{Z}_{p_3^{n_3}})| = 2$, $|U'(\mathbb{Z}_{p_4^{n_4}})| = 2$ leading that $|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})| = 16$.

3. $p_1 = 2$ and $n_1 \geq 3$.
   It must be $p_2, p_3, p_4 \neq 2$. Thus, $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})| = (p_1^{n_1} - p_1^{n_1-1})(p_2^{n_2} - p_2^{n_2-1})(p_3^{n_3} - p_3^{n_3-1})(p_4^{n_4} - p_4^{n_4-1})$, and by ($|U'(\mathbb{Z}_{p_1^{n_1}})| = 4$, $|U'(\mathbb{Z}_{p_2^{n_2}})| = 2$, $|U'(\mathbb{Z}_{p_3^{n_3}})| = 2$, $|U'(\mathbb{Z}_{p_4^{n_4}})| = 2$ it follows that $|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})| = 32$.

4. $p_1, p_2, p_3, p_4 \neq 2$.
   We have $|U(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})| = (p_1^{n_1} - p_1^{n_1-1})(p_2^{n_2} - p_2^{n_2-1})(p_3^{n_3} - p_3^{n_3-1})(p_4^{n_4} - p_4^{n_4-1})$, and $|U'(\mathbb{Z}_{p_1^{n_1}})| = 2$, $|U'(\mathbb{Z}_{p_2^{n_2}})| = 2$, $|U'(\mathbb{Z}_{p_3^{n_3}})| = 2$, $|U'(\mathbb{Z}_{p_4^{n_4}})| = 2$ leading that $|U'(\mathbb{Z}_{p_1^{n_1}p_2^{n_2}p_3^{n_3}p_4^{n_4}})| = 16$.

□

The results obtained for specific cases naturally extend to a general framework with any finite number of distinct prime factors.

Corollary 3. *For any natural number n, there exist distinct prime numbers $p_1, p_2, \dots, p_k$ and positive integers $n_1, n_2, \dots, n_k$ such that $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$. The following holds*

$$Cl_2(\mathbb{Z}_n) \cong \begin{cases} Shu_m^{2^{k-1}}(I(\mathbb{Z}_n)), & \text{if } 2 \mid n \text{ and } 4 \nmid n \\ Shu_m^{2^{k+1}}(I(\mathbb{Z}_n)), & \text{if } 8 \mid n \\ Shu_m^{2^{k}}(I(\mathbb{Z}_n)), & \text{otherwise} \end{cases}$$

where $m = (p_1^{n_1} - p_1^{n_1-1})(p_2^{n_2} - p_2^{n_2-1}) \dots (p_k^{n_k} - p_k^{n_k-1})$.

2.3. Relationship between isomorphism two clean graph and isomorphism two idempotent graph over ring Artinian

We note that if R is a local ring, then R does not have a non-trivial idempotent element. Consequently, we can not find the idempotent graph over R. Furthermore, in [6], it is proved that every Artinian ring is isomorphic to the direct product of a finite number of Artinian local rings. Let R be an Artinian ring with $n > 1$ distinct maximal ideals. Assume $R = R_1 \times R_2 \times \cdots \times R_n$, where R_i is a local ring for $i = 1, 2, \dots, n$. We obtain

$$Id(R) = \{(e_1, e_2, \dots, e_n) : e_i \in \{0, 1\} \text{ for all } i = 1, 2, \dots, n\}.$$

As a result, for any $a = (a_1, a_2, \dots, a_n), b = (b_1, b_2, \dots, b_n) \in Id(R) \setminus \{\mathbf{0}, \mathbf{1}\}$, it means there is $s, t \in \{1, 2, \dots .n\}$ such that $a_s = b_t = 0$. We can get $e = (e_1, e_2, \dots, e_n)$ and $f = (f_1, f_2, \dots, f_n)$ in $Id(R) \setminus \{\mathbf{0}, \mathbf{1}\}$, where

$$e_i = \begin{cases} 1, & \text{if } i = s \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_i = \begin{cases} 1, & \text{if } i = t \\ 0, & \text{otherwise} \end{cases}$$

such that $a - e - f - b$ is a walk in the graph $I(R)$ if $s \neq t$. On the other hand, if $s = t$, then $a - e - b$ is a walk in the graph $I(R)$. Hence, the graph $I(R)$ is connected.

Theorem 5. *Let R and S be Artinian rings with identity, each having more than one distinct maximal ideal. The following three statements are equivalent:*

(1) $Cl(R) \cong Cl(S)$

(2) $Cl_2(R) \cong Cl_2(S)$

(3) $I(R) \cong I(S)$, $|U(R)| = |U(S)|$, and $|U'(R)| = |U'(S)|$.

Proof. From Theorem 1 in [5], we get (1)⇔(2). Therefore, it is sufficient to prove (2)⇔(3). From Corollary 1 in [5] and Lemma 2 in [5], we know that $|Id(R)| = |Id(S)|$, $|U(R)| = |U(S)| = k$, and $|U'(R)| = |U'(S)| = t$. Since $Cl_2(R) \cong Cl_2(S)$, we have $Shu_k^t(I(R)) \cong Shu_k^t(I(S))$. Since graphs $I(R)$ and $I(S)$ are connected, based on Proposition 1, we get $I(R) \cong I(S)$. The converse also involves using Proposition 1. □

Conclusion

This study investigates the structure of clean graphs over the ring $\mathbb{Z}_n$, with particular focus on the subgraph $Cl_2(\mathbb{Z}_n)$ formed by pairs of nonzero idempotents and units. We examine its relationship with the idempotent graph $I(\mathbb{Z}_n)$ and establish a strong structural correspondence between a clean graph and its associated idempotent graph. Formally, the structure of the clean graph $Cl_2(R)$, where R is a ring with identity, is the (t, n)-shuriken of its idempotent graph $I(R)$, where t and n denote the number of self-inverse units in R and the total number of units in R, respectively. These results show how the algebraic properties of idempotents and units in a ring influence the topology of their associated graphs. Moreover, they enrich the theoretical framework of algebraic graph theory and support further studies on structural and combinatorial properties. Moreover, for two Artinian rings with identity, each having more than one distinct maximal ideal, we demonstrate that the isomorphism of their clean graphs can be determined by checking the isomorphism of their corresponding idempotent graphs.

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