Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-18
https://doi.org/10.22049/cco.2026.30643.2562

CCO
Commun. Comb. Optim.

Research Article

Spectra of complement of power graphs on some finite groups

Komal Kumari*, Pratima Panigrahi†

Department of Mathematics, Indian Institute of Technology Kharagpur, India
*`komalkumari1223w@gmail.com`
†`pratima@maths.iitkgp.ac.in`

Received: 22 May 2025; Accepted: 1 May 2026

Abstract: The power graph $\mathscr{P}(G)$ of a group G is an undirected graph with all the elements of G as vertices and where any two vertices are adjacent if and only if one is the integral power of the other. So far, no spectral results had been done for the complement of power graph on any group. In this paper, we compute the adjacency, Laplacian, and signless Laplacian eigenvalues of the complement of power graphs on finite cyclic, dihedral, and quaternion groups. Also we determine all the linearly independent eigenvectors corresponding to these eigenvalues. Moreover, we see that these eigenvectors, except possibly two, are common to all the above three type of matrices.

Keywords: eigenvalue, eigenvector, cyclic group, dihedral group, quaternion group.

AMS Subject classification: 05C75, 05C50, 15A18, 20D99

1. Introduction

Here we deal with simple and undirected graphs only. A *simple undirected graph* H is represented by the pair $H = (V(H), E(H))$, where $V(H)$ represents the vertex set and $E(H)$ represents the edge set made up of distinct, unordered pairs of vertices. The notation $v \sim w$ for $v, w \in V(H)$ indicates that v and w are adjacent in H. A graph is considered to be *r-regular* if each vertex has degree r. The *complete graph* K_n on n vertices is a graph in which every pair of distinct vertices is adjacent. A subgraph S of H is said to be an *induced subgraph* if a pair of vertices in S are adjacent whenever the pair is adjacent in H. The *complement* of a graph H, represented by $\overline{H}$, is the graph with $V(\overline{H}) = V(H)$ and where vertices v and w are adjacent if and only

---

* *Corresponding Author*

if v and w are non-adjacent in H. For more graph theoretical terminologies one may refer [11]. The *disjoint union* $G_1 \cup G_2$ of graphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The *join* of G_1 and G_2, represented by $G_1 \vee G_2$, is the graph $G_1 \cup G_2$ in which each vertex of G_1 is adjacent to every vertex of G_2. For an n-vertex graph H with vertex set $V(H) = \{1, 2, \ldots, n\}$ and a family of vertex disjoint graphs $G_1, G_2, \ldots, G_n$, Cardoso et al. [3] defined the H-join of $G_1, G_2, \ldots, G_n$, denoted as $\tilde{G} = \bigvee_H \{G_1, G_2, \ldots, G_n\}$, as a graph with vertex set $V(\tilde{G}) = \bigcup_{i=1}^{n} V(G_i)$ and edge set

$$E(\tilde{G}) = \left(\bigcup_{i=1}^{n} E(G_i)\right) \bigcup \left(\bigcup_{ij \in E(H)} \{uv : u \in V(G_i), v \in V(G_j)\}\right) \tag{1.1}$$

Let n_i be the number of vertices in G_i and $v_i \in V(G_i)$. Then $\deg_{\tilde{G}(v_i)} = \deg_{G_i}(v_i) + \sum_{ij \in E(H)} n_j$.

In this paper, J, I and O stand for the matrix of all ones, the identity matrix and the zero matrix of suitable order respectively. The *characteristic polynomial* $det(M - \lambda I)$ for any square matrix M, is represented by $\psi(M; \lambda)$, and the *spectrum* of the matrix M is the set of all eigenvalues (with counting multiplicities) of M. An eigenpair of M is a pair (λ, v) where λ is an eigenvalue and v is the corresponding eigenvector of M. Let H be an n-vertex graph. *Adjacency matrix* $A(H)$ of H is an $n \times n$ matrix whose $(i,j)^{th}$ entry is equal to 1 or 0 according to the adjacency or non-adjacency of i^{th} and j^{th} vertices of H. The *degree diagonal matrix* $D(H)$ of H is an $n \times n$ diagonal matrix whose $(i,i)^{th}$ entry is the degree of i^{th} vertex of H. The *Laplacian and signless Laplacian* matrices of H are respectively $L(H) = D(H) - A(H)$ and $SL(H) = D(H) + A(H)$. For an n-vertex graph H, we consider the matrix $C(H) = \alpha J + ((n-1)\beta - \alpha)I - (\alpha A(H) + \beta D(H))$, where $\alpha(\neq 0), \beta \in \mathbb{R}$. This matrix represents adjacency, Laplacian and signless Laplacian matrices of $\overline{H}$ while taking $(\alpha, \beta) = (1, 0), (1, -1)$, and $(1, 1)$, respectively.

Chakarbarty et al. [4] developed the concept of an undirected power graph for groups. The power graph $\mathscr{P}(G)$ of a group G is an undirected graph with all the elements of G as vertices and two vertices u and v are adjacent if and only if $u = v^m$ or $v = u^m$, $m \in \mathbb{Z}$. Chattopadhyay and Panigrahi [5] explored the Laplacian spectrum of $\mathscr{P}(\mathbb{Z}_n)$ and $\mathscr{P}(D_n)$. Banerjee and Adhikari [2] investigated the signless Laplacian spectrum of the power graph for the group $\mathbb{Z}_n$. Panda [8] investigated the Laplacian spectrum of power graphs for several families of finite groups. Romdhini et al. [10] investigated the adjacency and Laplacian spectra of the power graphs associated with dihedral groups D_{2n}. Romdhini et al. [9] further explored the adjacency and Laplacian spectra of the power graphs associated with generalized quaternion groups Q_n. For more results on power graphs one may refer the recent survey [6].

It is to be noted that no spectral results had been done for the complement of power graphs. Here we obtain spectra of the adjacency, Laplacian, and signless Laplacian matrices of the complement of power graphs on the cyclic groups $\mathbb{Z}_{p^r}$ and $\mathbb{Z}_{pq}$, dihedral groups D_{2p^r}, and generalized quaternion groups Q_n. Also we determine all the linearly independent eigenvectors corresponding to these eigenvalues. Moreover, we see that these eigenvectors, except possibly two, are independent of any kind of the above spectra.

2. Spectra of the complement of power graphs on cyclic groups

Notation 1. For integers n and r, $e_{n,r}^T$ denotes the column vector of dimension n, where the first entry is equal to 1, r^{th} entry is equal to -1 and the remaining entries are equal to 0. The all one column vector of dimension n is denoted by $\mathbf{j}_{n,1}$. The all zero column vector of dimension n is denoted by $\mathbf{0}_{n,1}$. For any positive integer n, $\phi(n)$ denotes Euler's totient function, that is $\phi(n)$ is the number of positive integers less than or equal to n which are relatively prime to n.

We consider the group $\mathbb{Z}_n = \{0, 1, \ldots, (n-1)\}$, where addition modulo n is the binary operation. Now we consider a simple graph Ω_n, where the vertices are the positive divisors $d_1, d_2, \ldots, d_t$ (for some integer t) of n, that is, $V(\Omega_n) = \{d_1, d_2, \cdots, d_t\}$, and where two distinct vertices are adjacent if and only if one divides the other.

Theorem 1. *([7], Theorem 2.2) For* $n \in \mathbb{N}$, $\mathscr{P}(\mathbb{Z}_n) = K_{\phi(n)+1} \vee \Omega_n[K_{\phi(d_1)}, K_{\phi(d_2)}, \ldots, K_{\phi(d_t)}]$.

Theorem 2. *([4],Theorem 2.12) Let G be a finite group. Then $\mathscr{P}(G)$ is complete if and only if G is a cyclic group of order 1 or p^r, where p is any prime and $r \in \mathbb{N}$.*

Theorem 3. *Let $n = p^r$, p is a prime number. Then, the eigenvalue of $C(\mathscr{P}(\mathbb{Z}_n))$ is 0 with multiplicity n and the eigenvectors are $X_1 = \boldsymbol{j}_{n,1}^T$, $X_i = \boldsymbol{e}_{n,i}^T$ for $2 \le i \le n$.*

Proof. By Theorem 2, $\overline{\mathscr{P}(\mathbb{Z}_n)}$ is a null graph and hence the result follows. □

Theorem 4. *Let $n = pq$, where p and q are distinct primes. Then, the eigenvalues of $C(\mathscr{P}(\mathbb{Z}_n))$ and their corresponding eigenvectors are as given below:*

(i) The eigenvalue 0 with multiplicity $\phi(pq) + 1$, and corresponding eigenvectors $X_1 = (\boldsymbol{j}_{\phi(pq)+1,1},\ \boldsymbol{0}_{\phi(q),1},\ \boldsymbol{0}_{\phi(p),1})^T$, $X_i = (\boldsymbol{e}_{\phi(pq)+1,i}^T, \boldsymbol{0}_{\phi(q),1},\ \boldsymbol{0}_{\phi(p),1})^T$, $2 \le i \le \phi(pq) + 1$.

(ii) The eigenvalue $(p-1)\beta$ with multiplicity $(q-2)$, and corresponding eigenvectors $Y_{i-1} = (\boldsymbol{0}_{\phi(pq)+1},\ \boldsymbol{e}_{\phi(q),i}^T,\ \boldsymbol{0}_{\phi(p),1})^T$, $2 \le i \le \phi(q)$.

(iii) The eigenvalue $(q-1)\beta$ with multiplicity $(p-2)$, and corresponding eigenvectors $Z_{i-1} = (\boldsymbol{0}_{\phi(pq)+1},\ \boldsymbol{0}_{\phi(p),1},\ \boldsymbol{e}_{\phi(p),i}^T)^T$, $2 \le i \le \phi(q)$.

(iv) The remaining two eigenvalues are $\frac{(p+q-2)\beta-D}{2}$ *and* $\frac{(p+q-2)\beta+D}{2}$ *with corresponding eigenvectors* $W_1 = (\boldsymbol{0}_{\phi(pq)+1,1}, (\beta(p-q)-D)\boldsymbol{j}_{\phi(q),1}, 2\alpha(q-1)\boldsymbol{j}_{\phi(p),1})^T$ *and* $W_2 = (\boldsymbol{0}_{\phi(pq)+1,1}, (\beta(p-q)+D)\boldsymbol{j}_{\phi(q),1}, 2\alpha(q-1)\boldsymbol{j}_{\phi(p),1})^T$ *respectively, where* $D = \sqrt{(p+q-2)^2\beta^2 - 4(p-1)(q-1)(\beta^2-\alpha^2)}$.

Proof. Let $V_1 = \{x \in \mathbb{Z}_n : gcd(x,n) = 1\} \cup \{0\}$, $V_2 = \{x \in \mathbb{Z}_n : gcd(x,n) = p\}$ and $V_3 = \{x \in \mathbb{Z}_n : gcd(x,n) = q\}$. Then,

$$C(\mathscr{P}(\mathbb{Z}_n)) = \begin{pmatrix} C(\mathscr{P}(V_1))_{\phi(pq)+1,\phi(pq)+1} & O_{\phi(pq)+1,\phi(q)} & O_{\phi(pq)+1,\phi(p)} \\ O_{\phi(q),\phi(pq)+1} & (C(\mathscr{P}(V_2)) + (p-1)\beta I)_{\phi(q),\phi(q)} & \alpha J_{\phi(q),\phi(p)} \\ O_{\phi(p),\phi(pq)+1} & \alpha J_{\phi(p),\phi(q)} & (C(\mathscr{P}(V_3)) + (q-1)\beta I)_{\phi(p),\phi(p)} \end{pmatrix},$$

where $C(\mathscr{P}(V_1)) = O_{\phi(pq)+1,\phi(pq)+1}$, $C(\mathscr{P}(V_2)) = O_{\phi(q),\phi(q)}$ and $C(\mathscr{P}(V_2)) = O_{\phi(p),\phi(p)}$. Let $X_1 = (\mathbf{j}_{\phi(pq)+1,1}, \mathbf{0}_{\phi(q),1}, \mathbf{0}_{\phi(p),1})^T$ be a vector. Now,

$$C(\mathscr{P}(\mathbb{Z}_n))X_1 = \begin{pmatrix} 0.\mathbf{j}_{\phi(pq)+1,1} \\ \mathbf{0}_{\phi(q),1} \\ \mathbf{0}_{\phi(p),1} \end{pmatrix} = 0 \begin{pmatrix} \mathbf{j}_{\phi(pq)+1,1} \\ \mathbf{0}_{\phi(q),1} \\ \mathbf{0}_{\phi(p),1} \end{pmatrix}.$$

Thus, 0 is an eigenvalue of $C(\mathscr{P}(\mathbb{Z}_n))$ with corresponding eigenvector X_1. Let $X_i = (\mathbf{e}^T_{\phi(pq)+1,i}, \mathbf{0}_{\phi(q),1}, \mathbf{0}_{\phi(p),1})^T$, $2 \leq i \leq \phi(pq)+1$. Then,

$$C(\mathscr{P}(\mathbb{Z}_n))X_i = \begin{pmatrix} 0.\mathbf{e}^T_{\phi(pq)+1,i} \\ \mathbf{0}_{\phi(q),1} \\ \mathbf{0}_{\phi(p),1} \end{pmatrix} = 0 \begin{pmatrix} \mathbf{e}^T_{\phi(pq)+1,i} \\ \mathbf{0}_{\phi(q),1} \\ \mathbf{0}_{\phi(p),1} \end{pmatrix} = 0X_i.$$

So, 0 is an eigenvalue of $C(\mathscr{P}(\mathbb{Z}_n))$ with multiplicity $\phi(pq)+1$, and corresponding eigenvectors are X_i, for $1 \leq i \leq \phi(pq)+1$. Let $Y_{i-1} = (\mathbf{0}_{\phi(pq)+1,1}, \mathbf{e}^T_{\phi(q),i}, \mathbf{0}_{\phi(p),1})^T$, $2 \leq i \leq \phi(q)$ be a vector. Then,

$$C(\mathscr{P}(\mathbb{Z}_n))Y_{i-1} = \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (p-1)\beta\mathbf{e}^T_{\phi(q),i} \\ \mathbf{0}_{\phi(p),1} \end{pmatrix} = (p-1)\beta \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ \mathbf{e}^T_{\phi(q),i} \\ \mathbf{0}_{\phi(p),1} \end{pmatrix} = (p-1)\beta Y_{i-1}.$$

Therefore, $(p-1)\beta$ is an eigenvalue of $C(\mathscr{P}(\mathbb{Z}_n))$ with multiplicity $(q-2)$, and corresponding eigenvectors are Y_{i-1}, $2 \leq i \leq \phi(pq)$. Let $Z_{i-1} = (\mathbf{0}_{\phi(pq)+1,1}, \mathbf{0}_{\phi(p),1}, \mathbf{e}^T_{\phi(p),i})^T$, $2 \leq i \leq \phi(q)$ be a vector. Then,

$$C(\mathscr{P}(\mathbb{Z}_n))Z_{i-1} = \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ \mathbf{0}_{\phi(p),1} \\ (q-1)\beta\mathbf{e}^T_{\phi(p),i} \end{pmatrix} = (q-1)\beta \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ \mathbf{0}_{\phi(p),1} \\ \mathbf{e}^T_{\phi(p),i} \end{pmatrix} = (q-1)\beta Z_{i-1}.$$

Therefore, $(q-1)\beta$ is an eigenvalue of $C(\mathscr{P}(\mathbb{Z}_n))$ with multiplicity $(p-2)$, and corresponding eigenvectors are Z_{i-1}, for $2 \leq i \leq (p-2)$. Let $D = \sqrt{(\beta(p+q-2))^2 + 4(p-1)(q-1)(\alpha^2-\beta^2)}$, and $W_1 = (\mathbf{0}_{\phi(pq)+1,1}, (\beta(p-q)-D)\mathbf{j}_{\phi(q),1}, 2\alpha(q-1)\mathbf{j}_{\phi(p),1})^T$ be a vector. Now,

$$
\begin{aligned}
C(\mathscr{P}(\mathbb{Z}_n))W_1 &= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (C(\mathscr{P}(V_2)) + (p-1)\beta I)_{\phi(q),\phi(q)}(\beta(p-q)-D)\mathbf{j}_{\phi(q),1} + \alpha J_{\phi(q),\phi(p)} 2\alpha(q-1)\mathbf{j}_{\phi(p),1} \\ \alpha J_{\phi(p),\phi(q)}(\beta(p-q)-D)\mathbf{j}_{\phi(q),1} + (C(\mathscr{P}(V_3)) + (q-1)\beta I)_{\phi(p),\phi(p)} 2\alpha(q-1)\mathbf{j}_{\phi(p),1} \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (p-1)\beta((p-q)\beta - D)\mathbf{j}_{\phi(q),1} + 2\alpha^2(p-1)(q-1)\mathbf{j}_{\phi(q),1} \\ \alpha(q-1)((p-q)\beta - D)\mathbf{j}_{\phi(p),1} + 2\alpha\beta(q-1)^2\mathbf{j}_{\phi(p),1} \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (p-1)\beta((p-1)\beta - (q-1)\beta - D)\mathbf{j}_{\phi(q),1} + 2\alpha^2(p-1)(q-1)\mathbf{j}_{\phi(q),1} \\ \alpha(q-1)((p-1)\beta - (q-1)\beta - D)\mathbf{j}_{\phi(p),1} + 2\alpha\beta(q-1)^2\mathbf{j}_{\phi(p),1} \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (\beta^2(p-1) - \beta^2(p-1)(q-1) - D(p-1)\beta + 2\alpha^2(p-1)(q-1))\mathbf{j}_{\phi(q),1} \\ (\alpha\beta(p-1)(q-1) - D\alpha(q-1) + \alpha\beta(q-1)^2)\mathbf{j}_{\phi(p),1} \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (\frac{2\beta^2(p-1)^2 - 2\beta D(p-1) - 2(p-1)(q-1)\beta^2 + 4\alpha^2(p-1)(q-1)}{2})\mathbf{j}_{\phi(q),1} \\ (\alpha(q-1)((p-1)\beta + (q-1)\beta - D))\mathbf{j}_{\phi(p),1} \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (\frac{(p-1+q-1)\beta - D}{2}((p-1)\beta - (q-1)\beta - D))\mathbf{j}_{\phi(q),1} \\ (2\alpha(q-1)\frac{(p+q-2)\beta - D}{2})\mathbf{j}_{\phi(p),1} \end{pmatrix}, \\
&= \frac{(p+q-2)\beta - D}{2} \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (\beta(p-q)-D)\mathbf{j}_{\phi(q),1} \\ 2\alpha(q-1)\mathbf{j}_{\phi(p),1} \end{pmatrix} = \frac{(p+q-2)\beta - D}{2} W_1.
\end{aligned}
$$

Thus, $\frac{(p+q-2)\beta-D}{2}$ is an eigenvalue of $C(\mathscr{P}(\mathbb{Z}_n))$ corresponding to the eigenvector W_1. Let $W_2 = (\mathbf{0}_{\phi(pq)+1,1}, (\beta(p-q)+D)\mathbf{j}_{\phi(q),1}, 2\alpha(q-1)\mathbf{j}_{\phi(p),1})^T$ be a vector. Now,

$$
\begin{aligned}
C(\mathscr{P}(\mathbb{Z}_n))W_2 &= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (C(\mathscr{P}(V_2)) + (p-1)\beta I)_{\phi(q),\phi(q)}(\beta(p-q)+D)\mathbf{j}_{\phi(q),1} + \alpha J_{\phi(q),\phi(p)} 2\alpha(q-1)\mathbf{j}_{\phi(p),1} \\ \alpha J_{\phi(p),\phi(q)}(\beta(p-q)+D)\mathbf{j}_{\phi(q),1} + (C(\mathscr{P}(V_3)) + (q-1)\beta I)_{\phi(p),\phi(p)} 2\alpha(q-1)\mathbf{j}_{\phi(p),1} \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (p-1)\beta((p-q)\beta + D)\mathbf{j}_{\phi(q),1} + 2\alpha^2(p-1)(q-1)\mathbf{j}_{\phi(q),1} \\ \alpha(q-1)((p-q)\beta + D)\mathbf{j}_{\phi(p),1} + 2\alpha\beta(q-1)^2\mathbf{j}_{\phi(p),1} \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (p-1)\beta((p-1)\beta - (q-1)\beta + D)\mathbf{j}_{\phi(q),1} + 2\alpha^2(p-1)(q-1)\mathbf{j}_{\phi(q),1} \\ \alpha(q-1)((p-1)\beta - (q-1)\beta + D)\mathbf{j}_{\phi(p),1} + 2\alpha\beta(q-1)^2\mathbf{j}_{\phi(p),1} \end{pmatrix}, \\
&= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (\beta^2(p-1) - \beta^2(p-1)(q-1) + D(p-1)\beta + 2\alpha^2(p-1)(q-1))\mathbf{j}_{\phi(q),1} \\ (\alpha\beta(p-1)(q-1) + D\alpha(q-1) + \alpha\beta(q-1)^2)\mathbf{j}_{\phi(p),1} \end{pmatrix},
\end{aligned}
$$

$$= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (\frac{2\beta^2(p-1)^2+2\beta D(p-1)-2(p-1)(q-1)\beta^2+4\alpha^2(p-1)(q-1)}{2})\mathbf{j}_{\phi(q),1} \\ (\alpha(q-1)((p-1)\beta+(q-1)\beta+D))\mathbf{j}_{\phi(p),1} \end{pmatrix},$$

$$= \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (\frac{(p-1+q-1)\beta+D}{2}((p-1)\beta-(q-1)\beta+D))\mathbf{j}_{\phi(q),1} \\ (2\alpha(q-1)\frac{(p+q-2)\beta+D}{2})\mathbf{j}_{\phi(p),1} \end{pmatrix},$$

$$= \frac{(p+q-2)\beta+D}{2} \begin{pmatrix} \mathbf{0}_{\phi(pq)+1,1} \\ (\beta(p-q)+D)\mathbf{j}_{\phi(q),1} \\ 2\alpha(q-1)\mathbf{j}_{\phi(p),1} \end{pmatrix} = \frac{(p+q-2)+D}{2}W_2.$$

Thus, $\frac{(p+q-2)+D}{2}$ is an eigenvalue of $C(\overline{\mathscr{P}(\mathbb{Z}_n)})$ corresponding to the eigenvector W_2.

□

Remark 1. We note that the eigenvectors of $C(\overline{\mathscr{P}(\mathbb{Z}_n)})$ obtained in Theorem 4 $(i)-(iii)$ are independent of α and β. So $A(\overline{\mathscr{P}(\mathbb{Z}_n)})$, $L(\overline{\mathscr{P}(\mathbb{Z}_n)})$, and $SL(\overline{\mathscr{P}(\mathbb{Z}_n)})$ have the same eigenvectors, may be except possibly two.

Corollary 1. *Let $n = pq$, where p and q are distinct primes. Then, the eigenvalues of $A(\overline{\mathscr{P}(\mathbb{Z}_n)})$ and their corresponding eigenvectors are as given below:*

(i) The eigenvalue 0 with multiplicity $pq-2$, and corresponding eigenvectors are $X_1 = (\boldsymbol{j}_{\phi(pq)+1,1},\ \boldsymbol{0}_{\phi(q),1},\ \boldsymbol{0}_{\phi(p),1})^T$, $X_i = (\boldsymbol{e}^T_{\phi(pq)+1,i}, \boldsymbol{0}_{\phi(q),1},\ \boldsymbol{0}_{\phi(p),1})^T$, $2 \le i \le \phi(pq)+1$, $Y_{i-1} = (\boldsymbol{0}_{\phi(pq)+1}, \boldsymbol{e}^T_{\phi(q),i},\ \boldsymbol{0}_{\phi(p),1})^T$, $2 \le i \le \phi(q)$, $Z_{i-1} = (\boldsymbol{0}_{\phi(pq)+1}, \boldsymbol{0}_{\phi(p),1},\ \boldsymbol{e}^T_{\phi(p),i})^T$, $2 \le i \le \phi(q)$.

(ii) The remaining two eigenvalues are $-\sqrt{(p-1)(q-1)}$ and $\sqrt{(p-1)(q-1)}$ with corresponding eigenvectors $W_1 = (\boldsymbol{0}_{\phi(pq)+1,1},\ -2\sqrt{(p-1)(q-1)}\boldsymbol{j}_{\phi(q),1}, 2(q-1)\boldsymbol{j}_{\phi(p),1})^T$ and $W_2 = (\boldsymbol{0}_{\phi(pq)+1,1},\ 2\sqrt{(p-1)(q-1)}\boldsymbol{j}_{\phi(q),1}, 2(q-1)\boldsymbol{j}_{\phi(p),1})^T$, respectively.

Corollary 2. *Let $n = pq$, where p and q are distinct primes. Then, the eigenvalues of $L(\overline{\mathscr{P}(\mathbb{Z}_n)})$ and their corresponding eigenvectors are as given below:*

(i) The eigenvalue 0 with multiplicity $\phi(pq)+2$, and corresponding eigenvectors are $X_1 = (\boldsymbol{j}_{\phi(pq)+1,1},\ \boldsymbol{0}_{\phi(q),1},\ \boldsymbol{0}_{\phi(p),1})^T$, $X_i = (\boldsymbol{e}^T_{\phi(pq)+1,i}, \boldsymbol{0}_{\phi(q),1},\ \boldsymbol{0}_{\phi(p),1})^T$, $2 \le i \le \phi(pq)+1$ and $W_1 = (\boldsymbol{0}_{\phi(pq)+1,1}, 2(1-q)\boldsymbol{j}_{\phi(q),1}, -2(q-1)\boldsymbol{j}_{\phi(p),1})^T$.

(ii) The eigenvalue $(p-1)$ with multiplicity $(q-2)$, and corresponding eigenvectors are $Y_{i-1} = (\boldsymbol{0}_{\phi(pq)+1},\ \boldsymbol{e}^T_{\phi(q),i},\ \boldsymbol{0}_{\phi(p),1})^T$, $2 \le i \le \phi(q)$.

(iii) The eigenvalue $(q-1)$ with multiplicity $(p-2)$, and corresponding eigenvectors are $Z_{i-1} = (\boldsymbol{0}_{\phi(pq)+1},\ \boldsymbol{0}_{\phi(p),1},\ \boldsymbol{e}^T_{\phi(p),i})^T$, $2 \le i \le \phi(q)$.

(iv) The eigenvalue $(p+q-2)$ with corresponding eigenvector $W_2 = (\boldsymbol{0}_{\phi(pq)+1,1}, 2(p-1)\boldsymbol{j}_{\phi(q),1}, -2(q-1)\boldsymbol{j}_{\phi(p),1})^T$.

Example 1. Let $n = 6$ and $(\alpha, \beta) = (-1, 1)$. Then $C(\mathscr{P}(\mathbb{Z}_6)) = 6I - J - (-A(\mathscr{P}(\mathbb{Z}_6)) + D(\mathscr{P}(\mathbb{Z}_6)))$ is the complement of the Laplacian matrix of $\mathscr{P}(\mathbb{Z}_6)$. Therefore,

$$C(\mathscr{P}(\mathbb{Z}_6)) = L(\overline{\mathscr{P}(\mathbb{Z}_6)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}.$$

By Theorem 4, the eigenvalues of $\overline{L(\mathscr{P}(\mathbb{Z}_6))}$ are $0, 0, 0, 0, 1$, and 3. And their corresponding eignevctors are respectively $(1, 1, 1, 0, 0, 0)^T$, $(1, -1, 0, 0, 0, 0)^T$, $(1, 0, -1, 0, 0, 0)^T$, $((0, 0, 0, -4, -4, -4))$, $(0, 0, 0, 1, -1, 0)^T$ and $(0, 0, 0, 2, 2, -4)$.

Corollary 3. *Let $n = pq$, where p and q are distinct primes. Then, the eigenvalues of $SL(\overline{\mathscr{P}(\mathbb{Z}_n)})$ and their corresponding eigenvectors are as given below:*

- *(i) The eigenvalue 0 with multiplicity $\phi(pq) + 2$, and the corresponding eigenvectors are $X_1 = (\boldsymbol{j}_{\phi(pq)+1,1}, \boldsymbol{0}_{\phi(q),1}, \boldsymbol{0}_{\phi(p),1})^T$, $X_i = (\boldsymbol{e}^T_{\phi(pq)+1,i}, \boldsymbol{0}_{\phi(q),1}, \boldsymbol{0}_{\phi(p),1})^T$, $2 \le i \le \phi(pq) + 1$, and $W_1 = (\boldsymbol{0}_{\phi(pq)+1,1}, 2(1-q)\boldsymbol{j}_{\phi(q),1}, 2(q-1)\boldsymbol{j}_{\phi(p),1})^T$.*
- *(ii) The eigenvalue $(p-1)$ with multiplicity $(q-2)$, and the corresponding eigenvectors are $Y_{i-1} = (\boldsymbol{0}_{\phi(pq)+1}, \boldsymbol{e}^T_{\phi(q),i}, \boldsymbol{0}_{\phi(p),1})^T$, $2 \le i \le \phi(q)$.*
- *(iii) The eigenvalue $(q-1)$ with multiplicity $(p-2)$, and the corresponding eigenvectors are $Z_{i-1} = (\boldsymbol{0}_{\phi(pq)+1}, \boldsymbol{0}_{\phi(p),1}, \boldsymbol{e}^T_{\phi(p),i})^T$, $2 \le i \le \phi(q)$.*
- *(iv) The eigenvalue $(p+q-2)$ with corresponding eigenvector $W_2 = (\boldsymbol{0}_{\phi(pq)+1,1}, 2(p-1)\boldsymbol{j}_{\phi(q),1}, 2(q-1)\boldsymbol{j}_{\phi(p),1})^T$.*

3. Spectra of the complement of power graphs on dihedral groups

The dihedral group (D_{2n}) with $n \ge 3$ is defined by $\langle a, b | a^n = e = b^2, bab = a^{-1}\rangle$, where $o(a) = n$ and $o(b) = 2$. Now, we partition $V(D_{2n})$ into three sets. Let $V_1 = \{e\}$, $V_2 = \langle a\rangle/\{e\}$ and $V_3 = \{a^i b : 0 \le i \le n-1\}$. So, $|V_1| = 1$, $|V_2| = n-1$ and $|V_3| = n$.

Theorem 5. *[1] Let $n = p^r$, p is a prime number and $r \in \mathbb{N}$. Let $\mathscr{P}(D_{2n})$ be the power graph of D_{2n} and $deg(v)$ be the degree of vertex v, then*

1. *$deg(v) = 2n - 1$, $\forall v \in V_1$,*
2. *$deg(v) = n - 1$, $\forall v \in V_2$,*
3. *$deg(v) = 1$, $\forall v \in V_3$.*

Theorem 6. *Let $n = p^r$, where p is a prime number. Then, the eigenvalues of $C(\mathscr{P}(D_{2n}))$ and their corresponding eigenvectors are as follow:*

- *(i) The eigenvalue 0 with the associated eigenvector $X_1^T = (1, \boldsymbol{0}_{(n-1),1}, \boldsymbol{0}_{n,1})$.*

(ii) *The eigenvalue* $n\beta$ *with multiplicity* $(n-2)$, *and the associated eigenvectors are* $X_j = (0, \boldsymbol{e}_{(n-1),j},\ \boldsymbol{0}_{n,1})^T$ *for* $2 \leq i \leq n-1$.

(iii) *The eigenvalue* $-\alpha + (2n-2)\beta$ *with multiplicity* $(n-1)$, *and the corresponding eigenvectors are* $Y_j = (\boldsymbol{0}, \boldsymbol{0}_{(n-1),1}, \boldsymbol{e}_{(n,j)})^T$, $2 \leq j \leq n$.

(iv) *The remaining two eigenvalues are* $\frac{n\beta+(2\beta+\alpha)(n-1)+D}{2}$ *and* $\frac{n\beta+(2\beta+\alpha)(n-1)-D}{2}$ *with corresponding eigenvectors* $Z_1 = (0, (\alpha + 2\beta - (\alpha+\beta)n + D)\boldsymbol{j}_{(n-1),1}, 2\alpha(n-1)\boldsymbol{j}_{n,1})^T$ *and* $Z_2 = (0, (\alpha + 2\beta - (\alpha+\beta)n - D)\boldsymbol{j}_{(n-1),1}, 2\alpha(n-1)\boldsymbol{j}_{n,1})^T$ *respectively, where* $D = \sqrt{(n\beta + (2\beta+\alpha)(n-1))^2 - 4n(n-1)(\beta(2\beta+\alpha) - \alpha^2)}$.

Proof. We have

$$C(\mathscr{P}(D_{2n})) = \begin{pmatrix} C(\mathscr{P}(V_1)) & O_{1,(n-1)} & O_{1,n} \\ O_{(n-1),1} & (C(\mathscr{P}(V_2)) + n\beta I)_{(n-1),(n-1)} & \alpha J_{(n-1),n} \\ O_{n,1} & \alpha J_{n,(n-1)} & (C(\mathscr{P}(V_3)) + (n-1)\beta I)_{n,n} \end{pmatrix},$$

where $C(\mathscr{P}(V_1)) = 0$, $C(\mathscr{P}(V_2)) = O_{n-1,n-1}$, and $C(\mathscr{P}(V_3)) = \alpha(J-I)_{n,n} + (n-1)\beta I_{n,n}$. Let $X_1 = (1, \mathbf{0}_{(n-1),1}, \mathbf{0}_{n,1})^T$ be a vector. Now,

$$C(\mathscr{P}(D_{2n}))X_1 = \begin{pmatrix} 0.1 \\ \mathbf{0}_{(n-1),1} \\ \mathbf{0}_{n,1} \end{pmatrix} = 0 \begin{pmatrix} 1 \\ \mathbf{0}_{(n-1),1} \\ \mathbf{0}_{n,1} \end{pmatrix}.$$

Thus, 0 is an eigenvalue of $C(\mathscr{P}(D_{2n}))$ corresponding to the eigenvector X_1. Let $X_j = (\mathbf{0}, \mathbf{e}_{(n-1),j}, \mathbf{0}_{n,1})^T$ for $2 \leq i \leq n-1$ be a vector. Then,

$$C(\mathscr{P}(D_{2n}))X_j = \begin{pmatrix} 0 \\ (C(\mathscr{P}(V_2)) + n\beta I)\mathbf{e}_{(n-1),j} \\ \mathbf{0}_{n,1} \end{pmatrix} = \begin{pmatrix} 0 \\ (0 + n\beta)\mathbf{e}_{(n-1),j} \\ \mathbf{0}_{n,1} \end{pmatrix} = n\beta \begin{pmatrix} 0 \\ \mathbf{e}_{(n-1),j} \\ \mathbf{0}_{n,1} \end{pmatrix}.$$

Therefore, $n\beta$ is an eigenvalue of $C(\mathscr{P}(D_{2n}))$ with multiplicity $(n-2)$, and corresponding eigenvectors are X_j, $2 \leq j \leq n-1$. Let $Y_{j-1} = (0, \mathbf{0}_{(n-1),1}, \mathbf{e}_{(n,j)})^T$, $2 \leq j \leq n$. Now,

$$C(\mathscr{P}(D_{2n}))Y_{j-1} = \begin{pmatrix} 0 \\ \mathbf{0}_{(n-1),1} \\ (C(\mathscr{P}(V_3)) + (n-1)\beta I)\mathbf{e}_{(n,j)} \end{pmatrix},$$

$$C(\mathscr{P}(D_{2n}))Y_{j-1} = \begin{pmatrix} 0 \\ \mathbf{0}_{(n-1),1} \\ (-\alpha + (n-1)\beta)\mathbf{e}_{(n,j)} + (n-1)\beta\mathbf{e}_{(n,j)} \end{pmatrix} = (-\alpha + 2(n-1)\beta)Y_{j-1}.$$

Thus, $(-\alpha + 2(n-1)\beta)$ is an eigenvalue of $C(\mathscr{P}(D_{2n}))$ with multiplicity $(n-1)$, and corresponding eigenvectors are Y_{j-1}, $2 \leq j \leq n$.

Let $D = \sqrt{(n\beta + (2\beta+\alpha)(n-1))^2 - 4n(n-1)(\beta(2\beta+\alpha)-\alpha^2)}$, and $Z_1 = (0, (\alpha + 2\beta - (\alpha+\beta)n + D)\mathbf{j}_{(n-1),1}, 2\alpha(n-1)\mathbf{j}_{n,1})^T$ be a vector. Now,

$$\begin{aligned}
C(\mathscr{P}(D_{2n}))Z_1 &= \begin{pmatrix} 0 \\ (C(\mathscr{P}(V_2)) + n\beta I)_{(n-1),(n-1)}(\alpha+2\beta-(\alpha+\beta)n+D)\mathbf{j}_{(n-1),1} + \alpha J_{(n-1),n}2\alpha(n-1)\mathbf{j}_{n,1} \\ \alpha J_{n,(n-1)}(\alpha+2\beta-(\alpha+\beta)n+D)\mathbf{j}_{(n-1),1} + (C(\mathscr{P}(V_3)) + (n-1)\beta I)_{n,n}2\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ n\beta(\alpha+2\beta-(\alpha+\beta)n+D)\mathbf{j}_{(n-1),1} + 2\alpha^2 n(n-1)\mathbf{j}_{(n-1),1} \\ \alpha(n-1)(\alpha+2\beta-(\alpha+\beta)n+D)\mathbf{j}_{n,1} + ((n-1)\alpha + 2(n-1)\beta)2\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \frac{2n\beta(\alpha+2\beta-(\alpha+\beta)n+D)+4\alpha^2 n(n-1)}{2}\mathbf{j}_{(n-1),1} \\ (\alpha+2\beta-(\alpha+\beta)n+D+2(n-1)\alpha+4(n-1)\beta)\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ (\frac{n\beta+(2\beta+\alpha)(n-1)+D}{2})(\alpha+2\beta-(\alpha+\beta)n+D)\mathbf{j}_{(n-1),1} \\ (n\beta+(2\beta+\alpha)(n-1)+D)\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \frac{n\beta+(2\beta+\alpha)(n-1)+D}{2}\begin{pmatrix} 0 \\ (\alpha+2\beta-(\alpha+\beta)n+D)\mathbf{j}_{(n-1),1} \\ 2\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \frac{n\beta+(2\beta+\alpha)(n-1)+D}{2}Z_1.
\end{aligned}$$

Thus, Z_1 is an eigenvector of $C(\mathscr{P}(D_{2n}))$ corresponding to the eigenvalue $\frac{n\beta+(2\beta+\alpha)(n-1)+D}{2}$. Let $Z_2 = (0, (\alpha + 2\beta - (\alpha+\beta)n - D)\mathbf{j}_{(n-1),1}, 2\alpha(n-1)\mathbf{j}_{n,1})^T$ be vector. Now,

$$\begin{aligned}
C(\mathscr{P}(D_{2n}))Z_2 &= \begin{pmatrix} 0 \\ (C(\mathscr{P}(V_2)) + n\beta I)_{(n-1),(n-1)}(\alpha+2\beta-(\alpha+\beta)n-D)\mathbf{j}_{(n-1),1} + \alpha J_{(n-1),n}2\alpha(n-1)\mathbf{j}_{n,1} \\ \alpha J_{n,(n-1)}(\alpha+2\beta-(\alpha+\beta)n-D)\mathbf{j}_{(n-1),1} + (C(\mathscr{P}(V_3)) + (n-1)\beta I)_{n,n}2\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ n\beta(\alpha+2\beta-(\alpha+\beta)n-D)\mathbf{j}_{(n-1),1} + 2\alpha^2 n(n-1)\mathbf{j}_{(n-1),1} \\ \alpha(n-1)(\alpha+2\beta-(\alpha+\beta)n-D)\mathbf{j}_{n,1} + ((n-1)\alpha + 2(n-1)\beta)2\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \frac{2n\beta(\alpha+2\beta-(\alpha+\beta)n-D)+4\alpha^2 n(n-1)}{2}\mathbf{j}_{(n-1),1} \\ (\alpha+2\beta-(\alpha+\beta)n-D+2(n-1)\alpha+4(n-1)\beta)\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ (\frac{n\beta+(2\beta+\alpha)(n-1)-D}{2})(\alpha+2\beta-(\alpha+\beta)n-D)\mathbf{j}_{(n-1),1} \\ (n\beta+(2\beta+\alpha)(n-1)-D)\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \frac{n\beta+(2\beta+\alpha)(n-1)-D}{2}\begin{pmatrix} 0 \\ (\alpha+2\beta-(\alpha+\beta)n-D)\mathbf{j}_{(n-1),1} \\ 2\alpha(n-1)\mathbf{j}_{n,1} \end{pmatrix} \\
&= \frac{n\beta+(2\beta+\alpha)(n-1)-D}{2}Z_2.
\end{aligned}$$

Thus, Z_2 is an eigenvector of $C(\mathscr{P}(D_{2n}))$ corresponding to the eigenvalue $\frac{n\beta+(2\beta+\alpha)(n-1)-D}{2}$. □

Remark 2. We note that the eigenvectors of $C(\overline{\mathscr{P}(D_{2p^r})})$ obtained in Theorem 6 $(i)-(iii)$ are independent of α and β. So $A(\overline{\mathscr{P}(D_{2p^r})})$, $L(\overline{\mathscr{P}(D_{2p^r})})$, and $SL(\overline{\mathscr{P}(D_{2p^r})})$ have the same eigenvectors, may be except possibly two.

Corollary 4. *Let* $n = p^r$*. The eigenvalues of* $A(\overline{\mathscr{P}(D_{2n})})$ *and their corresponding eigenvectors are as follows:*

(i) The eigenvalue 0 *with multiplicity* $n-1$*, and the corresponding eigenvectors are* $X_1 = (1, \boldsymbol{0}_{(n-1),1},\ \boldsymbol{0}_{n,1})^T$*,* $X_j = (\boldsymbol{0}, \boldsymbol{e}_{(n-1),j},\ \boldsymbol{0}_{n,1})^T$ *for* $2 \leq i \leq n-1$*.*

(ii) The eigenvalue -1 *with multiplicity* $(n-1)$*, and the corresponding eigenvectors are* $Y_j = (0,\ \boldsymbol{0}_{(n-1),1},\ \boldsymbol{e}_{(n,j)})^T$*,* $2 \leq j \leq n$*.*

(iii) And the remaining two eigenvalues are $\frac{(n-1)+\sqrt{(n-1)^2+4n(n-1)}}{2}$ *and* $\frac{(n-1)-\sqrt{(n-1)^2+4n(n-1)}}{2}$ *with corresponding eigenvectors* $Z_1 = (0, (1-n+\sqrt{(n-1)^2+4n(n-1)})\boldsymbol{j}_{(n-1),1}, 2(n-1)\boldsymbol{j}_{n,1})^T$ *and* $Z_2 = (0, (1-n-\sqrt{(n-1)^2+4n(n-1)})\boldsymbol{j}_{(n-1),1}, 2(n-1)\boldsymbol{j}_{n,1})^T$*, respectively.*

Corollary 5. *Let* $n = p^r$*. The eigenvalues of* $L(\overline{\mathscr{P}(D_{2n})})$ *and their corresponding eigenvectors are as follows:*

(i) The eigenvalue 0 *with multiplicity* 2*, and the associated eigenvectors are* $X_1 = (1, \boldsymbol{0}_{(n-1),1}, \boldsymbol{0}_{n,1})^T$*,* $Z_1 = (0, 2(1-n)\boldsymbol{j}_{(n-1),1}, -2(n-1)\boldsymbol{j}_{n,1})^T$*.*

(ii) The eigenvalue n *with multiplicity* $(n-2)$*, and the corresponding eigenvectors are* $X_j = (0,\ \boldsymbol{e}_{(n-1),j},\ \boldsymbol{0}_{n,1})^T$ *for* $2 \leq i \leq n-1$*.*

(iii) The eigenvalue $(2n-1)$ *with multiplicity* $(n-1)$*, and the corresponding eigenvectors are* $Y_j = (0,\ \boldsymbol{0}_{(n-1),1}, \boldsymbol{e}_{(n,j)})^T$*,* $2 \leq j \leq n$*.*

(iv) The eigenvalue $2n-1$ *with corresponding eigenvector* $Z_2 = (0, 2n\boldsymbol{j}_{(n-1),1}, -2(n-1)\boldsymbol{j}_{n,1})^T$*.*

Corollary 6. *Let* $n = p^r$*. The eigenvalues of* $SL(\overline{\mathscr{P}(D_{2n})})$ *and their corresponding eigenvectors are as follows:*

(i) The eigenvalue 0 *with the associated eigenvector* $X_1 = (1, \boldsymbol{0}_{(n-1),1}, \boldsymbol{0}_{n,1})^T$*.*

(ii) The eigenvalue n *with multiplicity* $(n-2)$*, and the corresponding eigenvectors are* $X_j = (0,\ \boldsymbol{e}_{(n-1),j},\ \boldsymbol{0}_{n,1})^T$ *for* $2 \leq i \leq n-1$*.*

(iii) The eigenvalue $(2n-3)$ *with multiplicity* $(n-1)$*, and the corresponding eigenvectors are* $Y_j = (0,\ \boldsymbol{0}_{(n-1),1},\ \boldsymbol{e}_{(n,j)})^T$*,* $2 \leq j \leq n$*.*

(iv) And the remaining two eigenvalues are $\frac{4n-3+\sqrt{8n^2-16n+9}}{2}$ *and* $\frac{4n-3-\sqrt{8n^2-16n+9}}{2}$ *with corresponding eigenvectors* $Z_1 = (0, (3-2n+\sqrt{8n^2-16n+9})\boldsymbol{j}_{(n-1),1}, 2(n-1)\boldsymbol{j}_{n,1})^T$ *and* $Z_2 = (0, (3-2n-\sqrt{8n^2-16n+9})\boldsymbol{j}_{(n-1),1}, 2(n-1)\boldsymbol{j}_{n,1})^T$*, respectively.*

Example 2. Let $n = 4$ and $(\alpha, \beta) = (1, 1)$. Then $C(\mathscr{P}(D_8)) = J + 6I - (A(\mathscr{P}(D_8)) + D(\mathscr{P}(D_8))$ is the signless Laplacian matrix of $\overline{\mathscr{P}(D_8)}$.Therefore,

$$C(\mathscr{P}(D_8)) = SL(\overline{\mathscr{P}(D_8)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 4 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 6 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 6 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 6 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 6 \end{pmatrix}.$$

By Theorem 6, eigenvalues of $SL(\overline{\mathscr{P}(D_8)})$ are $0, 4, 4, 5, 5, 5, 2.2280$, and 10.7720. Also by the same theorem, eigenvectors corresponding to eigenvalues $0, 4, 4, 5, 5$, and 5 are respectively $(1, 0, 0, 0, 0, 0, 0, 0)^T$, $(0, 1, -1, 0, 0, 0, 0, 0)^T$, $(0, 1, 0, -1, 0, 0, 0, 0)$, $(0, 0, 0, 0, 1, -1, 0, 0)^T$, $(0, 0, 0, 0, 1, 0, -1, 0)^T$, $(0, 0,\ 0,\ 0,\ 1, 0, 0, -1)^T$, $(0, -5 - \sqrt{73}, -5 - \sqrt{73}, -5 - \sqrt{73}, 6, 6, 6, 6)^T$ and $(0, -5 + \sqrt{73}, -5 + \sqrt{73}, -5 + \sqrt{73}, 6, 6, 6, 6)^T$.

4. Spectra of the complement of power graphs on generalized quaternion groups

For $n \geq 2$, the dicyclic group $Q_n = < a, b | a^{2n} = e, b^2 = a^n, ab = ba^{-1} >$ is of order $4n$. If n is a power of 2, then Q_n is known as the generalized quaternion group. Let $T_1 {=} \{e, a^n\}$, $T_2 {=} \{a, a^2, \ldots, a^{n-1}, a^{n+1}, \ldots, a^{2n-1}\}$, and for $3 \leq i \leq n + 2$, $T_i {=} \{a^{i-3}b, a^{n+i-3}b\}$. Therefore, $T_1 \cup T_2 \cup T_3 \cup T_4 \ldots \cup T_{n+2}$ is a partition of $V(\mathscr{P}(Q_n))$. We see that T_2 induces the complete graph K_{2n-2} and each T_i induces K_2, for $1 \leq i \leq n + 2$ $(i \neq 2)$. Let T be the star graph $K_{1,n+1}$ with $V(K_{1,n+1}) = \{1, 2, \ldots, n + 2\}$ and where 1 is a non-pendent vertex. Then

$$\mathscr{P}(Q_n) = \bigvee_T \{\mathscr{P}(T_1), \mathscr{P}(T_2), \ldots, \mathscr{P}(T_{n+2})\}$$

Theorem 7. *[1] Let $\mathscr{P}(Q_n)$ be the power graph of Q_n and $deg(v)$ be the degree of vertex v, then*

1. *$deg(v) = 4n - 1$, $\forall\ v \in T_1$,*
2. *$deg(v) = 2n - 1$, $\forall\ v \in T_2$,*
3. *$deg(v) = 3$, $\forall\ v \in \bigcup_{i=3}^{n+2} T_i$.*

Theorem 8. *The eigenvalues of $C(\mathscr{P}(Q_n))$ and their corresponding eigenvectors are as given below:*

(i) The eigenvalue 0 with multiplicity 2, and the associated eigenvectors are $X_1 = (\boldsymbol{e}^T_{2,2}, \boldsymbol{0}_{(2n-2),1}, \underbrace{\boldsymbol{0}_{2,1}, \ldots, \boldsymbol{0}_{2,1}}_{n\ \ times})^T$, $X_2 = (\boldsymbol{j}_{2,1}, \boldsymbol{0}_{(2n-2),1}, \underbrace{\boldsymbol{0}_{2,1}, \ldots, \boldsymbol{0}_{2,1}}_{n\ \ times})^T$.

(ii) The eigenvalue $2n\beta$ *with multiplicity* $(2n-3)$*, and the associated eigenvectors are* $Y_{j-1} = (\boldsymbol{O}_{2,1},\ \boldsymbol{e}_{(2n-2),j}, \underbrace{\boldsymbol{O}_{2,1}, \dots, \boldsymbol{O}_{2,1}}_{n \ \ times})^T$ *for* $2 \leq j \leq 2n-2$.

(iii) The eigenvalue $(4n-4)\beta$ *with multiplicity* n*, and the associated eigenvectors are* $\boldsymbol{W}_i = (\boldsymbol{O}_{2,1},\ \boldsymbol{O}_{2n-2,1},\ \boldsymbol{w}_1, \dots,\ \boldsymbol{w}_{(n-1)},\ \boldsymbol{w}_n)^T$*,* $i = 1, 2..., n$*, where for* $l = 1, 2, \dots, n$*,*

$$\boldsymbol{w}_l = \begin{cases} \boldsymbol{e}^T_{2,2}, & \textit{if} \ \ l = i, \\ \boldsymbol{O}_{2,1}, & \textit{otherwise}. \end{cases}$$

(iv) The eigenvalue $-2\alpha + (4n-4)\beta$ *with multiplicity* $n-1$*, and the associated eigenvectors are* $\boldsymbol{Z}_i = (\boldsymbol{O}_{2,1},\ \boldsymbol{O}_{2n-2,1},\ \boldsymbol{j}_{2,1}, \boldsymbol{z}_1, \dots,\ \boldsymbol{z}_{(n-2)},\ \boldsymbol{z}_{(n-1)})^T$*,* $i = 1, 2..., (n-1)$*, where for* $l = 1, 2, \dots, (n-1)$*,*

$$\boldsymbol{z}_l = \begin{cases} \text{-}\boldsymbol{j}_{2,1}, & \textit{if} \ \ l = i, \\ \boldsymbol{O}_{2,1}, & \textit{otherwise}. \end{cases}$$

(v) And the remaining two eigenvalues are $\frac{2(n-1)\alpha+(6n-4)\beta-D}{2}$*,* $\frac{2(n-1)\alpha+(6n-4)\beta+D}{2}$ *with corresponding eigenvectors* $U_1 = (\boldsymbol{O}_{2,1}, \frac{((2-2n)\alpha+(4-2n)\beta-D)}{2}\boldsymbol{j}_{(2n-2),1}, 2\alpha(n-1)j_{2n,1})^T$*,* $U_2 = (\boldsymbol{O}_{2,1}, \frac{((2-2n)\alpha+(4-2n)\beta+D)}{2}\ \boldsymbol{j}_{(2n-2),1}, 2\alpha(n-1)j_{2n,1})^T$ *respectively, where* $D = 2\sqrt{((n-1)\alpha + (3n-2)\beta)^2 - 4n(n-1)(\alpha\beta + 2\beta^2 - \alpha^2)}$*.*

Proof. We have

$$C(\mathscr{P}(Q_n)) = \begin{pmatrix} C(\mathscr{P}(T_1)) & O_{2,(2n-2)} & O_{2,2} & O_{2,2} & \dots & O_{2,2} \\ O_{(2n-2),2} & C(\mathscr{P}(T_2)) + 2n\beta I & \alpha J_{(2n-2),2} & \alpha J_{(2n-2),2} & \dots & \alpha J_{(2n-2),2} \\ O_{2,2} & \alpha J_{2,(2n-2)} & C(\mathscr{P}(T_3)) + (4n-4)\beta I & \alpha J_{2,2} & \dots & \alpha J_{2,2} \\ O_{2,2} & \alpha J_{2,(2n-2)} & \alpha J_{2,2} & C(\mathscr{P}(T_4)) + (4n-4)\beta I & \dots & \alpha J_{2,2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ O_{2,2} & \alpha J_{2,(2n-2)} & \alpha J_{2,2} & \alpha J_{2,2} & \dots & C(\mathscr{P}(T_{n+2})) + (4n-4)\beta I \end{pmatrix},$$

where $C(\mathscr{P}(T_i)) = \alpha J_{2,2} - \alpha I_{2,2} - \alpha A(\mathscr{P}(T_i)) = O_{2,2}$ for $1 \leq i(\neq 2) \leq (n+2)$ and $C(\mathscr{P}(T_2)) = \alpha J_{(2n-2),(2n-2)} - \alpha I_{(2n-2),(2n-2)} - \alpha A(\mathscr{P}(T_2)) = O_{(2n-2),(2n-2)}$. Let $X_1 = (\mathbf{e}^T_{2,2}, \mathbf{0}_{(2n-2),1}, \underbrace{\mathbf{0}_{2,1}, \dots, \mathbf{0}_{2,1}}_{n \ \ times})^T$ be a vector. Now,

$$C(\mathscr{P}(Q_n))X_1 = \begin{pmatrix} (-\alpha + \alpha)e^T_{2,2} \\ 0_{(2n-2),1} \\ 0_{2,1} \\ \vdots \\ 0_{2,1} \end{pmatrix} = 0 \begin{pmatrix} e^T_{2,2} \\ 0_{(2n-2),1} \\ 0_{2,1} \\ \vdots \\ 0_{2,1} \end{pmatrix}.$$

So, 0 is an eigenvalue of $C(\mathscr{P}(Q_n))$ corresponding to the eigenvector $X_1 = (\mathbf{e}^T_{2,2}, \mathbf{0}_{(2n-2),1}, \underbrace{\mathbf{0}_{2,1}, \dots, \mathbf{0}_{2,1}}_{n \ \ times})^T$. Let $X_2 = (\mathbf{j}^T_{2,1}, \mathbf{0}_{(2n-2),1}, \underbrace{\mathbf{0}_{2,1}, \dots, \mathbf{0}_{2,1}}_{n \ \ times})$ be a vector. Now,

$$C(\mathscr{P}(Q_n))X_2 = \begin{pmatrix} 0.\mathbf{j}_{2,1}^T \\ \mathbf{0}_{(2n-2),1} \\ \mathbf{0}_{2,1} \\ \vdots \\ \mathbf{0}_{2,1} \end{pmatrix} = 0 \begin{pmatrix} \mathbf{j}_{2,1}^T \\ \mathbf{0}_{(2n-2),1} \\ \mathbf{0}_{2,1} \\ \vdots \\ \mathbf{0}_{2,1} \end{pmatrix}.$$

Thus, X_2 is an eigenvector of $C(\mathscr{P}(Q_n))$ corresponding to the eigenvalue 0. Therefore, 0 is an eigenvalue of $C(\mathscr{P}(Q_n))$ with multiplicity 2, and corresponding eigenvectors are X_1, X_2. Let $Y_{j-1} = (\mathbf{0}_{2,1}, \mathbf{e}_{(2n-2),j}, \underbrace{\mathbf{0}_{2,1}, \ldots, \mathbf{0}_{2,1}}_{n \quad times})^T$ for $2 \leq j \leq 2n-2$ be vector.

Now, $$C(\mathscr{P}(Q_n))Y_{j-1} = \begin{pmatrix} \mathbf{0}_{2,1} \\ (C(\mathscr{P}(T_2)) + 2n\beta I)\mathbf{e}_{(2n-2),j} \\ \mathbf{0}_{2,1} \\ \vdots \\ \mathbf{0}_{2,1} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{2,1} \\ (2n)\beta\mathbf{e}_{(2n-2),j} \\ \mathbf{0}_{2,1} \\ \vdots \\ 0_{2,1} \end{pmatrix} = 2n\beta \begin{pmatrix} \mathbf{0}_{2,1} \\ \mathbf{e}_{(2n-2),j} \\ \mathbf{0}_{2,1} \\ \vdots \\ \mathbf{0}_{2,1} \end{pmatrix}.$$

Thus, $2n\beta$ is an eigenvalue of $C(\mathscr{P}(Q_n))$ with multiplicity $(2n-3)$, and corresponding eigenvectors are Y_{j-1} for $2 \leq j \leq (2n-2)$.
Let $\mathbf{W}_i = (\mathbf{0}_{2,1}, \mathbf{0}_{2n-2,1}, \mathbf{w}_1, \ldots, \mathbf{w}_{(n-1)}, \mathbf{w}_n)^T$, $i = 1, 2..., n$, where for $l = 1, 2, \ldots, n$,

$$\mathbf{w}_l = \begin{cases} \mathbf{e}_{2,2}^T, & \text{if } \ l = i, \\ \mathbf{0}_{2,1}, & \text{otherwise.} \end{cases}$$

Now, $$C(\mathscr{P}(Q_n))W_i = \begin{pmatrix} \mathbf{0}_{2,1} \\ \mathbf{0}_{(2n-2),1} \\ \mathbf{0}_{2,1} \\ \vdots \\ (C(\mathscr{P}(T_i)) + (4n-4)\beta I)\mathbf{e}_{2,2} \\ \vdots \\ \mathbf{0}_{2,1} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{2,1} \\ \mathbf{0}_{(2n-2),1} \\ \mathbf{0}_{2,1} \\ \vdots \\ (4n-4)\beta\mathbf{e}_{2,2} \\ \vdots \\ \mathbf{0}_{2,1} \end{pmatrix} = (4n-4)\beta \begin{pmatrix} \mathbf{0}_{2,1} \\ \mathbf{0}_{(2n-2),1} \\ \mathbf{0}_{2,1} \\ \vdots \\ \mathbf{e}_{2,2} \\ \vdots \\ \mathbf{0}_{2,1} \end{pmatrix}.$$

Thus, $(4n-4)\beta$ is an eigenvalue of $C(\mathscr{P}(Q_n))$ with multiplicity n, and corresponding eigenvectors are W_i, for $1 \leq i \leq n$. Let $\mathbf{Z}_i = (\mathbf{0}_{2,1}, \mathbf{0}_{2n-2,1}, \mathbf{j}_{2,1}, \mathbf{z}_1, \ldots, \mathbf{z}_{(n-2)}, \mathbf{z}_{(n-1)})^T$, $i = 1, 2, \ldots, n-1$, where for $l = 1, 2, \ldots, (n-1)$,

$$\mathbf{z}_l = \begin{cases} \text{-}\mathbf{j}_{2,1}, & \text{if } \ l = i, \\ \mathbf{0}_{2,1}, & \text{otherwise.} \end{cases}$$

be a vector. Now,

$$C(\mathscr{P}(Q_n))Z_i = \begin{pmatrix} \mathbf{0}_{2,1} \\ \mathbf{0}_{(2n-2),1} \\ (C(\mathscr{P}(T_3)) + (4n-4)\beta I)\mathbf{j}_{2,1} - \alpha J_{2,2}\mathbf{j}_{2,1} \\ J_{2,2}\mathbf{0}_{2,1} \\ \vdots \\ \alpha J_{2,2}\mathbf{j}_{2,1} - (C(\mathscr{P}(T_i)) + (4n-4)\beta I)\mathbf{j}_{2,1} \\ \vdots \\ J_{2,2}\mathbf{0}_{2,1} \end{pmatrix} = \begin{pmatrix} \mathbf{0}_{2,1} \\ \mathbf{0}_{(2n-2),1} \\ (-2\alpha + (4n-4)\beta)\mathbf{j}_{2,1} \\ \mathbf{0}_{2,1} \\ \vdots \\ -(-2\alpha + (4n-4)\beta)\mathbf{j}_{2,1} \\ \vdots \\ \mathbf{0}_{2,1} \end{pmatrix}.$$

This implies that

$$C(\mathscr{P}(Q_n))Z_i = (-2\alpha + (4n-4)\beta) \begin{pmatrix} \mathbf{0}_{2,1} \\ \mathbf{0}_{(2n-2),1} \\ \mathbf{j}_{2,1} \\ \mathbf{0}_{2,1} \\ \vdots \\ -\mathbf{j}_{2,1} \\ \vdots \\ \mathbf{0}_{2,1} \end{pmatrix}.$$

Thus, $(-2\alpha + (4n-4)\beta)$ is an eigenvalue of $C(\mathscr{P}(Q_n))$ with multiplicity $n-1$, and corresponding eigenvectors are Z_i, for $1 \leq i \leq (n-1)$. Let $D = 2\sqrt{((n-1)\alpha + (3n-2)\beta)^2 - 4n(n-1)(\alpha\beta + 2\beta^2 - \alpha^2)}$, and $U_1 = (\mathbf{0}_{2,1}, \frac{((2-2n)\alpha+(4-2n)\beta-D)}{2}\mathbf{j}_{(2n-2),1}, 2\alpha(n-1)j_{2n,1})^T$ be a vector. Now,

$$\begin{aligned} C(\mathscr{P}(Q_n))U_1 &= \begin{pmatrix} \mathbf{0}_{2,1} \\ 2n\beta\frac{((2-2n)\alpha+(4-2n)\beta-D)}{2}\mathbf{j}_{(2n-2),1} + 4\alpha^2 n(n-1)\mathbf{j}_{(2n-2),1} \\ \alpha(2n-2)\frac{((2-2n)\alpha+(4-2n)\beta-D)}{2}\mathbf{j}_{2n,1} + (2\alpha\beta(n-1)(4n-4) + 4\alpha^2(n-1)^2)\mathbf{j}_{2n,1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0}_{2,1} \\ (n\beta((2-2n)\alpha + (4-2n)\beta - D) + 4\alpha^2 n(n-1))\mathbf{j}_{(2n-2),1} \\ (\alpha(n-1)((2-2n)\alpha + (4-2n)\beta - D) + 8\alpha\beta(n-1)^2 + 4\alpha^2(n-1)^2)\mathbf{j}_{2n,1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0}_{2,1} \\ (-2\alpha\beta(n-1) - 2\beta^2 n(n-2) - Dn\beta) + 4\alpha^2 n(n-1))\mathbf{j}_{(2n-2),1} \\ \alpha(n-1)((2-2n)\alpha + (4-2n)\beta - D + 8\beta(n-1) + 4\alpha(n-1))\mathbf{j}_{2n,1} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0}_{2,1} \\ \frac{(2(n-1)\alpha+(6n-4)\beta-D)((2-2n)\alpha+(4-2n)\beta-D)}{4}\mathbf{j}_{(2n-2),1} \\ \alpha(n-1)(2(n-1)\alpha + (6n-4)\beta - D)\mathbf{j}_{2n,1} \end{pmatrix} \\ &= \frac{2(n-1)\alpha + (6n-4)\beta - D}{2} \begin{pmatrix} \mathbf{0}_{2,1} \\ \frac{((2-2n)\alpha+(4-2n)\beta-D)}{2}\mathbf{j}_{(2n-2),1} \\ 2\alpha(n-1)j_{2n,1} \end{pmatrix} \\ &= \frac{2(n-1)\alpha + (6n-4)\beta - D}{2}U_1. \end{aligned}$$

Thus, $\frac{2(n-1)\alpha+(6n-4)\beta-D}{2}$ is an eigenvalue of $C(\mathscr{P}(Q_n))$, corresponding to the eigenvector U_1. Let $U_2 = (\mathbf{0}_{2,1}, \frac{((2-2n)\alpha+(4-2n)\beta+D)}{2}\mathbf{j}_{(2n-2),1}, 2\alpha(n-1)j_{2n,1})^T$ be a vector. Now,

$$\begin{aligned}
C(\mathscr{P}(Q_n))U_2 &= \begin{pmatrix} \mathbf{0}_{2,1} \\ 2n\beta\frac{((2-2n)\alpha+(4-2n)\beta+D)}{2}\mathbf{j}_{(2n-2),1} + 4\alpha^2 n(n-1)\mathbf{j}_{(2n-2),1} \\ \alpha(2n-2)\frac{((2-2n)\alpha+(4-2n)\beta+D)}{2}\mathbf{j}_{2n,1} + (2\alpha\beta(n-1)(4n-4)+4\alpha^2(n-1)^2)\mathbf{j}_{2n,1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0}_{2,1} \\ (n\beta((2-2n)\alpha+(4-2n)\beta+D)+4\alpha^2 n(n-1))\mathbf{j}_{(2n-2),1} \\ (\alpha(n-1)((2-2n)\alpha+(4-2n)\beta+D)+8\alpha\beta(n-1)^2+4\alpha^2(n-1)^2)\mathbf{j}_{2n,1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0}_{2,1} \\ (-2\alpha\beta(n-1)-2\beta^2 n(n-2)+Dn\beta)+4\alpha^2 n(n-1))\mathbf{j}_{(2n-2),1} \\ \alpha(n-1)((2-2n)\alpha+(4-2n)\beta+D+8\beta(n-1)+4\alpha(n-1))\mathbf{j}_{2n,1} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{0}_{2,1} \\ \frac{(2(n-1)\alpha+(6n-4)\beta+D)((2-2n)\alpha+(4-2n)\beta+D)}{4}\mathbf{j}_{(2n-2),1} \\ \alpha(n-1)(2(n-1)\alpha+(6n-4)\beta+D)\mathbf{j}_{2n,1} \end{pmatrix} \\
&= \frac{2(n-1)\alpha+(6n-4)\beta+D}{2}\begin{pmatrix} \mathbf{0}_{2,1} \\ \frac{((2-2n)\alpha+(4-2n)\beta+D)}{2}\mathbf{j}_{(2n-2),1} \\ 2\alpha(n-1)j_{2n,1} \end{pmatrix} \\
&= \frac{2(n-1)\alpha+(6n-4)\beta+D}{2}U_2.
\end{aligned}$$

Thus, $\frac{2(n-1)\alpha+(6n-4)\beta+D}{2}$ is an eigenvalue of $C(\mathscr{P}(Q_n))$, corresponding to the eigenvector U_2. ☐

Remark 3. We note that the eigenvector of $C(\mathscr{P}(Q_n))$ obtained in Theorem 8 $(i)-(iv)$ are independent of α and β. So $A(\overline{\mathscr{P}(Q_n)})$, $L(\overline{\mathscr{P}(Q_n)})$, and $SL(\overline{\mathscr{P}(Q_n)})$ have the same eigenvectors, may be except possibly two.

Corollary 7. *The eigenvalues of $A(\overline{\mathscr{P}(Q_n)})$ and their corresponding eigenvectors are as given below:*

(i) The eigenvalue 0 with multiplicity $3n-1$, and associated eigenvectors are $X_1 = (\boldsymbol{e}^T_{2,2},\ \boldsymbol{0}_{(2n-2),1},\ \underbrace{\boldsymbol{0}_{2,1},\dots,\boldsymbol{0}_{2,1}}_{n\ \ times})^T$, $X_2 = (\boldsymbol{j}_{2,1}, 0_{(2n-2),1}, \underbrace{\boldsymbol{0}_{2,1},\dots,\boldsymbol{0}_{2,1}}_{n\ \ times})^T$, $Y_{j-1} = (\boldsymbol{0}_{2,1},$ $\boldsymbol{e}_{(2n-2),j}, \underbrace{\boldsymbol{0}_{2,1},\dots,\boldsymbol{0}_{2,1}}_{n\ \ times})^T$ for $2 \le j \le 2n-2$, $\boldsymbol{W}_i = (\boldsymbol{0}_{2,1},\ \boldsymbol{0}_{2n-2,1},\ \boldsymbol{w}_1,\dots,\ \boldsymbol{w}_{(n-1)},$ $\boldsymbol{w}_{(n)})^T$, $i = 1, 2..., n$, where for $l = 1, 2, \dots, n$,

$$\boldsymbol{w}_l = \begin{cases} \boldsymbol{e}^T_{2,2}, & if\ \ l = i, \\ \boldsymbol{0}_{2,1}, & otherwise. \end{cases}$$

(ii) The eigenvalue -2 with multiplicity $n-1$, and associated eigenvectors are $\boldsymbol{Z}_i = (\boldsymbol{0}_{2,1},$ $\boldsymbol{0}_{2n-2,1}, \boldsymbol{j}_{2,1}, \boldsymbol{z}_1, \dots,\ \boldsymbol{z}_{(n-2)},\ \boldsymbol{z}_{(n-1)})^T$, $i = 1, 2, \dots, (n-1)$, where for $l = 1, 2, \dots, (n-1)$,

$$\boldsymbol{z}_l = \begin{cases} \text{-}\boldsymbol{j}_{2,1}, & if\ \ l = i, \\ \boldsymbol{0}_{2,1}, & otherwise. \end{cases}$$

(iii) The remaining two eigenvalues are $\frac{2(n-1)-\sqrt{20n^2-24n+4}}{2}$ *and* $\frac{2(n-1)+\sqrt{20n^2-24n+4}}{2}$ *with corresponding eigenvectors* $U_1 = (\boldsymbol{0}_{2,1}, \frac{(2-2n-\sqrt{20n^2-24n+4})}{2}\boldsymbol{j}_{(2n-2),1}, 2(n-1)j_{2n,1})^T$ *and* $U_2 = (\boldsymbol{0}_{2,1}, \frac{(2-2n+\sqrt{20n^2-24n+4})}{2}\ \boldsymbol{j}_{(2n-2),1}, 2(n-1)j_{2n,1})^T$*, respectively.*

Example 3. Let $n = 2,$ and $(\alpha, \beta) = (1, 0)$. Then, $C(\mathscr{P}(Q_8) = J - I - A(\mathscr{P}(Q_8)$ is the adjacency matrix of $\overline{\mathscr{P}(Q_8)}$. Therefore,

$$C(\mathscr{P}(Q_8) = A(\overline{\mathscr{P}(Q_8)}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

By Theorem 8, we get the eigenvalues of $A(\overline{\mathscr{P}(Q_8)})$ are $0, 0, 0, 0, 0. - 2, -2,$ and 4. Also corresponding eigenvectors are $(1, -1, 0, 0, 0, 0, 0, 0)^T$, $(1, 1, 0, 0, 0, 0, 0, 0)$, $(0, 0, 1, -1, 0, 0, 0, 0)^T$, $(0, 0, 0, 0, 0, 0, 1, -1)^T$, $(0, 0, 0, 0, 1, -1, 0, 0)^T$, $(0, 0, 0, 0, 1, 1, -1, -1)^T$, $(0, 0, -4, -4, 2, 2, 2, 2)^T$ and $(0, 0, 2, 2, 2, 2, 2, 2)^T$.

Corollary 8. *The eigenvalues of* $L(\overline{\mathscr{P}(Q_n)})$ *and their corresponding eigenvectors are as given below:*

(i) The eigenvalue 0 *with multiplicity* 3*, and corresponding eigenvectors are* $X_1 = (\boldsymbol{e}^T_{2,2}, \boldsymbol{0}_{(2n-2),1}, \underbrace{\boldsymbol{0}_{2,1}, \ldots, \boldsymbol{0}_{2,1}}_{n \quad times})^T$, $X_2 = (\boldsymbol{j}_{2,1}, \boldsymbol{0}_{(2n-2),1}, \underbrace{\boldsymbol{0}_{2,1}, \ldots, \boldsymbol{0}_{2,1}}_{n \quad times})^T$ *and* $U_1 = (\boldsymbol{0}_{2,1}, -2(n-1)\boldsymbol{j}_{(2n-2),1}, -2(n-1)j_{2n,1})^T$.

(ii) The eigenvalue $2n$ *with multiplicity* $(2n-3)$*, and corresponding eigenvectors are* $Y_{j-1} = (\boldsymbol{0}_{2,1},\ \boldsymbol{e}_{(2n-2),j}, \underbrace{\boldsymbol{0}_{2,1}, \ldots, \boldsymbol{0}_{2,1}}_{n \quad times})^T$ *for* $2 \le j \le 2n-2$.

(iii) The eigenvalue $4n-4$ *with multiplicity* n*, and corresponding eigenvectors are* $\boldsymbol{W}_i = (\boldsymbol{0}_{2,1},\ \boldsymbol{0}_{2n-2,1},\ \boldsymbol{w}_1, \ldots,\ \boldsymbol{w}_{(n-1)},\ \boldsymbol{w}_n)^T$, $i = 1, 2, \ldots, n$*, where for* $l = 1, 2, \ldots, n$,

$$\boldsymbol{w}_l = \begin{cases} \boldsymbol{e}^T_{2,2}, & if \ \ l = i, \\ \boldsymbol{0}_{2,1}, & otherwise. \end{cases}$$

(iv) The eigenvalue $4n-2$ *with multiplicity* n*, and corresponding eigenvectors are* $U_2 = (\boldsymbol{0}_{2,1}, 2n\boldsymbol{j}_{(2n-2),1},\ -2(n-1)j_{2n,1})^T$ *and* $\boldsymbol{Z}_i = (\boldsymbol{0}_{2,1},\ \boldsymbol{0}_{2n-2,1},\ \boldsymbol{j}_{2,1}, \boldsymbol{z}_1, \ldots,\ \boldsymbol{z}_{(n-2)}, \boldsymbol{z}_{(n-1)})^T$, $i = 1, 2, \ldots, (n-1)$*, where for* $l = 1, 2, \ldots, (n-1)$,

$$\boldsymbol{z}_l = \begin{cases} \text{-}\boldsymbol{j}_{2,1}, & if \ \ l = i, \\ \boldsymbol{0}_{2,1}, & otherwise. \end{cases}$$

Corollary 9. *The eigenvalues of $SL(\overline{\mathscr{P}(Q_n)})$ and their corresponding eigenvectors are as given below:*

(i) The eigenvalue 0 *with multiplicity* 2*, and associated eigenvectors are* $X_1 = (\boldsymbol{e}^T_{2,2}, \boldsymbol{0}_{(2n-2),1}, \underbrace{\boldsymbol{0}_{2,1}, \ldots, \boldsymbol{0}_{2,1}}_{n \quad times})^T$, $X_2 = (\boldsymbol{j}_{2,1}, \boldsymbol{0}_{(2n-2),1}, \underbrace{\boldsymbol{0}_{2,1}, \ldots, \boldsymbol{0}_{2,1}}_{n \quad times})^T$.

(ii) The eigenvalue $2n$ *with multiplicity* $(2n-3)$*, and associated eigenvectors are* $Y_{j-1} = (\boldsymbol{0}_{2,1}, \ \boldsymbol{e}_{(2n-2),j}, \underbrace{\boldsymbol{0}_{2,1}, \ldots, \boldsymbol{0}_{2,1}}_{n \quad times})^T$ *for* $2 \leq j \leq 2n-2$.

(iii) The eigenvalue $4n-4$ *with multiplicity* n*, and associated eigenvectors are* $\boldsymbol{W}_i = (\boldsymbol{0}_{2,1}, \boldsymbol{0}_{2n-2,1}, \ \boldsymbol{w}_1, \ldots, \ \boldsymbol{w}_{(n-1)}, \ \boldsymbol{w}_n)^T$, $i = 1, 2, \ldots, n$*, where for* $l = 1, 2, \ldots, n$,

$$\boldsymbol{w}_l = \begin{cases} \boldsymbol{e}^T_{2,2}, & \text{if} \quad l = i, \\ \boldsymbol{0}_{2,1}, & \text{otherwise.} \end{cases}$$

(iv) The eigenvalue $4n-6$ *with multiplicity* $n-1$*, and associated eigenvectors are* $\boldsymbol{Z}_i = (\boldsymbol{0}_{2,1}, \boldsymbol{0}_{2n-2,1}, \boldsymbol{j}_{2,1}, \boldsymbol{z}_1, \ldots, \ \boldsymbol{z}_{(n-2)}, \ \boldsymbol{z}_{(n-1)})^T$, $i = 1, 2, \ldots, (n-1)$*, where for* $l = 1, 2, \ldots, (n-1)$,

$$\boldsymbol{z}_l = \begin{cases} \text{-}\boldsymbol{j}_{2,1}, & \text{if} \quad l = i, \\ \boldsymbol{0}_{2,1}, & \text{otherwise.} \end{cases}$$

(v) The rest of two eigenvalues are $\frac{8n-6-\sqrt{(8n-6)^2-32n(n-1)}}{2}$ *and* $\frac{8n-6+\sqrt{(8n-6)^2-32n(n-1)}}{2}$ *with corresponding eigenvectors* $U_1 = (\boldsymbol{0}_{2,1}, (3 - 2n - \sqrt{(4n-3)^2 - 8n(n-1)})\boldsymbol{j}_{(2n-2),1}, 2(n-1)j_{2n,1})^T$ *and* $U_2 = (\boldsymbol{0}_{2,1}, (3 - 2n + \sqrt{(4n-3)^2 - 8n(n-1)})\boldsymbol{j}_{(2n-2),1}, 2(n-1)j_{2n,1})^T$*, respectively.*

Concluding Remark: Using the matrix $C(H)$ of a graph H, one can compute adjacency, Laplacian and signless Laplacian spectra of complement of any other graph not necessarily a power graph. For example, we consider the cycle C_4 on 4 vertices. One sees that C_4 is not a power graph because in a power graph there is a vertex (identity element) which is adjacent to every other vertex. Now, $C(C_4) = \alpha J + (3\beta - \alpha)I - (\alpha A(C_4) + \beta D(C_4)) = \begin{bmatrix} \beta & 0 & \alpha & 0 \\ 0 & \beta & 0 & \alpha \\ \alpha & 0 & \beta & 0 \\ 0 & \alpha & 0 & \beta \end{bmatrix}$. Taking $\alpha = 1, \beta = 0$, the spectrum of the matrix $C(C_4)$ is the adjacency spectrum of $\overline{C_4}$.

Acknowledgments: The authors are thankful to the anonymous referees for their valuable and constructive comments which improved the presentation of the paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] F. Ali, S. Fatima, and W. Wang, *On the power graphs of certain finite groups*, Linear Multilinear Algebra **70** (2022), no. 19, 3803–3817.
https://doi.org/10.1080/03081087.2020.1856028.

[2] S. Banerjee and A. Adhikari, *Signless Laplacian spectrum of power graphs of finite cyclic groups*, AKCE Int. J. Graphs Comb. **17** (2020), no. 1, 356–366.
https://doi.org/10.1016/j.akcej.2019.03.009.

[3] D.M. Cardoso, M.A.A. de Freitas, E.A. Martins, and M. Robbiano, *Spectra of graphs obtained by a generalization of the join graph operation*, Discrete Math. **313** (2013), no. 5, 733–741.
https://doi.org/10.1016/j.disc.2012.10.016.

[4] I. Chakrabarty, S. Ghosh, and M.K. Sen, *Undirected power graphs of semigroups*, Semigroup forum **78** (2009), no. 3, 410–426.
https://doi.org/10.1007/s00233-008-9132-y.

[5] S. Chattopadhyay and P. Panigrahi, *On Laplacian spectrum of power graphs of finite cyclic and dihedral groups*, Linear Multilinear Algebra **63** (2015), no. 7, 1345–1355.
https://doi.org/10.1080/03081087.2014.936435.

[6] A. Kumar, L. Selvaganesh, P.J. Cameron, and T.T. Chelvam, *Recent developments on the power graph of finite groups–A survey*, AKCE Int. J. Graphs Comb. **18** (2021), no. 2, 65–94.
https://doi.org/10.1080/09728600.2021.1953359.

[7] Z. Mehranian, A. Gholami, and A.R. Ashrafi, *A note on the power graph of a finite group*, Int. J. Group Theory **5** (2016), no. 1, 1–10.

[8] R.P. Panda, *Laplacian spectra of power graphs of certain finite groups*, Graphs Combin. **35** (2019), no. 5, 1209–1223.
https://doi.org/10.1007/s00373-019-02070-x.

[9] M.U. Romdhini, A. Nawawi, F. Al-Sharqi, and A. Al-Quran, *Spectral power graph of generalized quaternion groups*, Asia Pacific Journal of Mathematics **11** (2024), 36.
https://doi.org/10.28924/APJM/11-36.

[10] ———, *Spectral properties of power graph of dihedral groups*, Eur. J. Pure Appl. Math. **17** (2024), no. 2, 591–603.
https://doi.org/10.29020/nybg.ejpam.v17i2.5036.

[11] D.B. West, *Introduction to Graph Theory*, Upper Saddle River: Prentice hall, 1996.