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Research Article

Spectral properties of a new distance-based matrix

Razie Alidehi-Ravandi[†], **Modjtaba Ghorbani**[*]

Department of Mathematics, Faculty of Science, Shahid Rajaee Teacher Training University, Tehran, Iran
[†]razieravandi@sru.ac.ir
[*]mghorbani@sru.ac.ir

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Abstract: In this paper, we introduce and analyze the spectral properties of the Graovac-Ghorbani matrix $\tilde{\mathcal{A}}$. We first calculate the Graovac-Ghorbani index for specific classes of structures, including nicely distance-balanced, vertex-transitive, and edge-transitive graphs. By defining the Laplacian matrix associated with $\tilde{\mathcal{A}}$, we prove that for any connected graph of order $n \geq 3$, the rank of this Laplacian matrix is exactly $n-1$. We establish sharp upper and lower bounds for the spectral radius of $\tilde{\mathcal{A}}$ in terms of graph parameters for various families, including trees, unicycle bipartite graphs, and general bipartite graphs, characterizing the extremal cases such as complete bipartite graphs. Furthermore, the spectral properties of strongly distance-balanced (SDB) graphs are investigated. A key finding of this study is the proof that the $\tilde{\mathcal{A}}$-eigenvalues of any bipartite graph are symmetric with respect to zero. However, we demonstrate that the converse does not hold, implying that a symmetric $\tilde{\mathcal{A}}$-spectrum does not necessarily guarantee the bipartiteness of the graph.

Keywords: Graovac-Ghorbani matrix, spectral radius, Rayleigh quotient theorem.

AMS Subject classification: 05C07, 05C35

1. Introduction

Graph theory has become a foundational domain within mathematics due to its wide-ranging applications in numerous disciplines. It plays a vital role in computer science, physics, and social sciences, where the underlying structures of networks and systems are naturally modeled by graphs. Over the past few years, there has been a growing interest in studying various indices that arise from graph analysis [1, 2, 13–15, 23, 25].

* *Corresponding Author*

These indices offer valuable quantitative measures that help in understanding different aspects of graph structures.

In addition to these indices, the investigation of spectral properties has emerged as a powerful approach to uncover the deeper structural traits and dynamic behaviors of graphs. Spectral graph theory, which studies the eigenvalues and eigenvectors associated with matrix representations of graphs, provides profound insights that are not always evident from purely combinatorial methods. Among the many matrix-based tools employed in spectral graph analysis, one particularly notable representation is the Graovac-Ghorbani matrix, denoted by $\tilde{\mathcal{A}}$.

In this work, our focus is directed toward examining the spectral features of the $\tilde{\mathcal{A}}$-matrix across a broad spectrum of graph families. These include trees, bipartite graphs, unicycle bipartite graphs, complete bipartite graphs, and triangle-free graphs. By deriving both upper and lower bounds for the spectral radius of $\tilde{\mathcal{A}}$, we aim to investigate how variations in graph topology influence the associated spectral quantities.

To support our analysis, we also consider the Laplacian matrix, traditionally defined as $\mathcal{L} = \mathcal{D} - A$, where $\mathcal{D}$ denotes the degree matrix and A the adjacency matrix. This matrix is instrumental in understanding graph connectivity and structural decomposition. Building upon this, we define a new Laplacian matrix based on the $\tilde{\mathcal{A}}$-matrix, referred to as $\tilde{\mathcal{L}}$. Our study includes an exploration of its properties, such as the rank of $\tilde{\mathcal{L}}$, the number of connected components in the graph, and the multiplicity of the zero eigenvalue. These aspects are crucial for characterizing the internal organization and coherence of graph structures.

An interesting finding of our work is that the $\tilde{\mathcal{A}}$-eigenvalues for bipartite graphs are symmetric with respect to zero. This symmetric pattern reflects a distinctive spectral signature of bipartite graphs. Nonetheless, we observe that the converse does not generally hold; in other words, not all graphs exhibiting symmetric $\tilde{\mathcal{A}}$-spectra are bipartite. This asymmetry in implication highlights the nuanced and sometimes unexpected relationships between graph structure and spectral behavior.

Overall, the findings presented in this paper contribute to a more comprehensive understanding of how the $\tilde{\mathcal{A}}$-matrix encodes structural information. Furthermore, these results open new avenues for future investigations that seek to connect spectral properties with other graph invariants, offering potential applications in both theoretical research and practical domains.

2. Preliminary

For the study of the properties and characteristics of graphs and their associated matrices, see references [3, 4, 6, 8]. All graphs in this article are assumed to be simple and connected. A graph G is defined by its vertices and edges, denoted by $V(G)$ and $E(G)$, respectively.

Definition 1. A non-empty graph is considered nicely distance-balanced (NDB) if there exists a positive integer k such that for any two adjacent vertices u and v in the graph, there are exactly k vertices that are closer to u than to v, and exactly k vertices that are closer to v than to u.

Definition 2. A graph G is said to be strongly distance-balanced (SDB) if for any edge $uv \in E(G)$ and any integer $k \geq 0$, the number of vertices at distance k from u and at distance $k+1$ from v is equal to the number of vertices at distance $k+1$ from u and at distance k from v.

For an edge $e = uv \in E(G)$, let $n_u = n_u(e \mid G)$ and $n_v = n_v(e \mid G)$ represent the number of vertices in G that are closer to vertex u than to vertex v, and the number of vertices in G that are closer to vertex v than to vertex u, see [18]. The edge $e = uv$ is called perfectly distance balanced (pdb) if $n_u = n_v = 1$.

Remark 1. It is worth noting that every strongly distance-balanced graph is distance-balanced, meaning that for every edge $e = uv$, we have $n_u = n_v$.

Given a connected graph G, Graovac and Ghorbani [17] introduced the ABC_{GG}-index ($\mathcal{A}$-index for short) as

$$\mathcal{A}(G) = \sum_{e=uv \in E(G)} \sqrt{\frac{n_u + n_v - 2}{n_u n_v}}.$$

It is a variant of the atom-bond connectivity index [28]. In recent years, there has been a significant increase in interest regarding the mathematical properties of the $\mathcal{A}$-index, resulting in numerous published studies; see e.g. [9–11, 22, 26, 30]. A thorough survey of the $\mathcal{A}$-index was recently compiled in [27], which includes a comprehensive bibliography to facilitate future research. Considering recent advances and the current understanding of the $\mathcal{A}$-index, there are ample opportunities for further exploration of its characteristics.

Additionally, the reverse Szeged index, denoted as $RSz(G)$, is defined in [16] as follows:

$$RSz(G) = 2 \sum_{e=uv \in E} \frac{1}{n_u n_v}.$$

Following Gutman [19], the product version of the Graovac-Ghorbani index can be defined as

$$P\mathcal{A}(G) = \prod_{e=uv \in E(G)} \sqrt{\frac{n_u + n_v - 2}{n_u n_v}}.$$

Definition 3. A matrix $\tilde{\mathcal{A}} = (\tilde{a}_{ij})_{n\times n}$ is defined based on $\mathcal{A}$-index, where entries $\tilde{a}_{ij}$ are given as

$$\tilde{a}_{ij} = \begin{cases} \sqrt{\dfrac{n_i + n_j - 2}{n_i n_j}} & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise} \end{cases}.$$

The eigenvalues of $\tilde{\mathcal{A}}$ are referred to as the $\tilde{\mathcal{A}}$-eigenvalues of G. The largest eigenvalue of this matrix is referred to as $\tilde{\mathcal{A}}$-spectral radius of the graph.

The Laplacian matrix of a graph provides more information on graph structures than its adjacency matrix. Consequently, it is natural to generalize the Laplacian matrix of G to the Laplacian $\tilde{\mathcal{A}}$-matrix as $\tilde{\mathcal{L}} = \tilde{\mathcal{D}} - \tilde{\mathcal{A}}$, where $\tilde{\mathcal{D}} = (\tilde{d}_{ij})_{n\times n}$ is the diagonal matrix of G, with

$$\tilde{d}_{ij} = \begin{cases} \sum_{j=1}^{n} \tilde{a}_{ij} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

The eigenvalues of $\tilde{\mathcal{L}}$ are represented by $\zeta_1 \geq \zeta_2 \geq \ldots \geq \zeta_n$ known as Laplacian $\tilde{\mathcal{A}}$-eigenvalues of G. The largest eigenvalue of Laplacian $\tilde{\mathcal{A}}$-matrix of G is defined as the Laplacian $\tilde{\mathcal{A}}$-spectral radius of G. We denote the eigenvalues of $\tilde{\mathcal{A}}$ as $\vartheta_1 \geq \vartheta_2 \geq \ldots \geq \vartheta_n$, and the eigenvalues of the adjacency matrix A as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.
The vertex-arc incidence matrix of an oriented graph D is represented by the matrix $R(D) = (r_{ij})_{n\times m}$, in which the entries are defined as

$$r_{ij} = \begin{cases} -(\dfrac{n_i + n_j - 2}{n_i n_j})^{\frac{1}{4}} & \text{if } v_i \text{ is the initial vertex of } e \\ 0 & \text{if } v_i \text{ and } e \text{ are not incident} \\ (\dfrac{n_i + n_j - 2}{n_i n_j})^{\frac{1}{4}} & \text{if } v_i \text{ is the terminal vertex of } e \end{cases}.$$

For any orientation of D, it is not difficult to see that $\tilde{\mathcal{L}} = R(D)R^T(D)$. Therefore, $\tilde{\mathcal{L}}$ is a positive semi-definite matrix. It is evident that zero is an eigenvalue of $\tilde{\mathcal{L}}$ with an eigenvector $\mathbf{j}$, which is the all-one vector.

3. Spectral Properties of Graovac–Ghorbani Matrix

In this section, we investigate the spectral properties of the Graovac-Ghorbani matrix $\tilde{\mathcal{A}}$ for several classes of graphs. In particular, we compute the Graovac–Ghorbani index $\mathcal{A}(G)$ for NDB graphs, vertex-transitive graphs, edge-transitive graphs, and bipartite edge-transitive graphs, and establish upper and lower bounds for $\mathcal{A}(G)$ in the class of trees.

Moreover, we study structural spectral parameters associated with $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{L}}$, including the rank of $\tilde{\mathcal{L}}$, and the multiplicity of the zero eigenvalue. Finally, we derive an explicit formula for the characteristic polynomial of $\tilde{\mathcal{A}}$ and analyze the behavior of $\tilde{\mathcal{A}}$-eigenvalues for specific graph families, such as cycles, complete bipartite graphs, star graphs, and certain classes of bipartite and triangle-free graphs.

Lemma 1. *[12] Let $f(x,y)=\sqrt{\frac{x+y-2}{xy}}$ for $2\leq x\leq n-1$ and $1\leq y\leq n-1$. The following properties can be established:*
(i) The function $f(x,1)$ is increasing with respect to x, and therefore

$$\frac{\sqrt{2}}{2}=f(2,1)\leq f(x,1)\leq f(n-1,1)=\sqrt{\frac{n-2}{n-1}}.$$

(ii) $f(x,2)=\frac{\sqrt{2}}{2}$.
(iii) For $x\geq y\geq 3$, the function $f(x,y)$ is decreasing with respect to both x and y, resulting in the bounds

$$\frac{\sqrt{2n-4}}{n-1}=f(n-1,n-1)\leq f(x,y)\leq f(3,3)=\frac{2}{3}.$$

Note 1. Suppose that G is a NDB graph on n vertices and m edges. Since G is NDB, for all edges $e=uv$, we have $n_u=n_v=k$, for some $k\in\mathbb{N}$ and so

$$\mathcal{A}(G)=m\frac{\sqrt{2k-2}}{k}.$$

Note 2. If a graph G is vertex-transitive with n vertices and m edges, then for each edge $e=uv$, we obtain $n_u=n_v=t$, for some $t\in\mathbb{N}$, and thus

$$\mathcal{A}(G)=m\frac{\sqrt{2t-2}}{t}.$$

Note 3. If G is edge-transitive with n vertices and m edges, for two edges $e=uv$ and $f=xy$, we have $\{n_x,n_y\}=\{n_u,n_v\}$. Choosing $a=n_x$ and $b=n_y$ we yield that

$$\mathcal{A}(G)=m\sqrt{\frac{a+b-2}{ab}},$$

which conclude that for the adjacency matrix A

$$\tilde{\mathcal{A}}=\sqrt{\frac{a+b-2}{ab}}A.$$

Example 1. Since the star graph S_n is edge-transitive, Note 3 implies that for a pendant edge $e=uv$, we have $n_u=1$ and $n_v=n-1$. Hence, for the star graph S_n, $\tilde{\mathcal{A}}=\sqrt{\frac{n-2}{n-1}}A$. This implies that

$$Spec_{\tilde{\mathcal{A}}}(S_n)=\sqrt{\frac{n-2}{n-1}}Spec(S_n),$$

or equivalently

$$Spec_{\tilde{\mathcal{A}}}(S_n)=\begin{pmatrix}-\sqrt{n-2} & 0 & \sqrt{n-2}\\ 1 & n-2 & 1\end{pmatrix}.$$

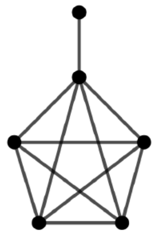

Figure 1. **The graph $K_5 + e$.**

Example 2. *i*) For the graph $K_{n-1} + e$, (a complete graph on $n-1$ vertices with an additional pendant edge as depicted in Figure 1), we obtain

$$\mathcal{A}(K_{n-1}+e) = \sqrt{\frac{n-2}{n-1}} + (n-2)\sqrt{\frac{2+1-2}{2}} = \sqrt{\frac{n-2}{n-1}} + \frac{\sqrt{2}(n-2)}{2}.$$

ii) The $\mathcal{A}$-index of a cycle graph is

$$\mathcal{A}(C_n) = \begin{cases} n\sqrt{\frac{n-2}{\frac{n^2}{4}}} = 2\sqrt{n-2} & \text{if } \ 2 \mid n \\ n\sqrt{\frac{n-3}{\frac{(n-1)^2}{4}}} = \frac{2n}{n-1}\sqrt{n-3} & \text{if } \ 2 \nmid n \end{cases}.$$

iii) For the star graph S_n, the $\mathcal{A}$-index is given by

$$\mathcal{A}(S_n) = (n-1)\sqrt{\frac{n-2}{n-1}} = \sqrt{(n-1)(n-2)}.$$

iv) It's clear that the $\mathcal{A}$-index of a complete graph K_n is zero-valued,

$$\mathcal{A}(K_n) = 0.$$

v) For the graph $K_n \backslash e$, we have

$$\mathcal{A}(K_n \backslash e) = (2n-4)\sqrt{\frac{1+2-2}{2}} = \sqrt{2}(n-2).$$

Theorem 1. *[29] If T is a tree on n vertices, then*

$$\mathcal{A}(P_n) \leq \mathcal{A}(T) \leq \mathcal{A}(S_n).$$

Note 4. If $G \ncong K_n$ and P is a shortest path between two vertices in G with length at least two, then no edge of P is *pdb*.

Lemma 2. *Let $G \ncong K_n$ and define the set $F = \{e = uv \in E; n_u = n_v = 1\}$. Then the subgraph $G \backslash F$ is connected.*

Proof. By Note 4, no edge on the shortest path between any two vertices belongs to F . Therefore, every such path remains intact in $G - F$, proving $G - F$ is connected. □

Let $R(G)$ represent the vertex-arc incidence matrix of an arbitrary orientation of G.

Lemma 3. *Let $G \ncong K_n$ be a graph on $n \geq 3$ vertices. Then rank$(R(G)) = n - 1$.*

Proof. Assume that $0 \neq y$ is a vector of the left zero vector space for $R(G)$. That is $y^T R(G) = 0$, where 0 is the zero vector. Suppose $v_i v_j \in E(G)$ whose corresponding direction is from v_i to v_j in the digraph D obtained from G. By the above equality, we have

$$(y_i - y_j)(\frac{n_i + n_j - 2}{n_i n_j})^{\frac{1}{4}} = 0.$$

If $\dfrac{n_i + n_j - 2}{n_i n_j} = 0$, then $e = u_i u_j \in F$. According to Lemma 2, the removal of this edge from graph G does not affect its connectivity. Hence, by removing all these edges, we can suppose in the resulting graph $\dfrac{n_i + n_j - 2}{n_i n_j} \neq 0$ and thus $y_i = y_j$. So all components of y equal 1, indicating that the dimension of the left zero vector space of $R(G)$ is at most one. Then we have rank$(R(G)) \geq n - 1$. On the other hand, we find that the sum of elements in each column of matrix $R(G)$ is zero, which means the rows of $R(G)$ are linearly dependent. So, rank$(R(G)) \leq n - 1$. □

Suppose the nullity (multiplicity of eigenvalue zero) of a matrix M is denoted by $\eta(M)$.

Theorem 2. *Let $G \ncong K_n$ be a graph of order $n \geq 3$. Then $\eta(\tilde{\mathcal{L}}) = 1$.*

Proof. It is evident that

$$rank(\tilde{\mathcal{L}}) = rank(R(G)R^T(G)) = rank(R(G)).$$

By Lemma 3, we obtain

$$rank(\tilde{\mathcal{L}}) = n - 1.$$

Since $\tilde{\mathcal{L}}$ is a real symmetric matrix of order n, the multiplicity of the eigenvalue zero is $n - rank(\tilde{\mathcal{L}}) = n - (n - 1) = 1$. □

If G is a graph with a diameter of d, then G will have at least $d+1$ distinct adjacency eigenvalues, at least $d+1$ distinct Laplacian eigenvalues, and at least $d+1$ distinct signless Laplacian eigenvalues [5, 24].

The proof provided in [5] demonstrates that the above result holds true for any non-negative symmetric matrix $M = (M_{ij})_n$ indexed by the vertices of a graph G, where $M_{ij} > 0$ if and only if vertices v_i and v_j are adjacent. As a result, the following corollary can be immediately deduced.

Proposition 1. *If G is a graph with no pdb-edge and has diameter d and k distinct $\tilde{\mathcal{A}}$-eigenvalues, then $k \geq d+1$.*

Another immediate consequence is the next result.

Theorem 3. *The graph G has exactly one $\tilde{\mathcal{A}}$-eigenvalue if and only if $G \cong K_n$.*

Proof. If G has $n = 1$ or 2 vertices, the proof is complete. Suppose G has $n \geq 3$ vertices. Since G is connected with exactly one eigenvalue ϑ, necessarily $\vartheta = 0$ and thus all entries of the $\tilde{\mathcal{A}}$ are zero, which implies that for each edge $e = uv$, we have $n_u = n_v = 1$. Thus, all triples of vertices are in a common triangle, and so G is a complete graph. Conversely, if $G \cong K_n$, the proof is clear. □

Theorem 4. *There is no graph G with exactly two distinct $\tilde{\mathcal{A}}$-eigenvalues.*

Proof. Suppose G has exactly two distinct $\tilde{\mathcal{A}}$-eigenvalues and has no *pdb*-edge, then by Corollary 1 its diameter is 1. Therefore, G is necessarily K_n, which contradicts the assumption that G has no *pdb*-edge. Hence, there is at least a *pdb*-edge $e = uv$ such that $n_u = n_v = 1$. Since the rows of the $\tilde{\mathcal{A}}$ associated to u and v are the same, the nullity of the $\tilde{\mathcal{A}}$ is greater than or equal to one. Thus G has exactly one non-zero $\tilde{\mathcal{A}}$-eigenvalue ϑ. Since the sum of all eigenvalues equals the trace of the $\tilde{\mathcal{A}}$, we obtain $\vartheta = 0$, a contradiction. This completes the proof. □

The Perron-Frobenius theorem provides fundamental insights into the spectral properties of positive matrices. Specifically, for a positive matrix A with spectral radius $\rho(A) = r$, the theorem states the following:

1) The number r is a positive real number such that for any other eigenvalue λ, $|\lambda| < r$.

2) The eigenvalue r is simple, meaning that it has algebraic multiplicity one. Both the right and left eigenspaces associated with r are one-dimensional [20].

Proposition 2. *Every connected strongly distance–balanced graph is regular.*

Proof. This follows immediately from Proposition 1.1 in [21]. □

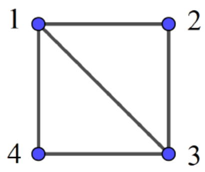

Figure 2. **The graph** K_4/e.

Theorem 5. *If G is a SDB graph, then*
i) $\vartheta = \frac{r\sqrt{2d-2}}{d}$ *is an eigenvalue of G, where G is r-regular and the contribution of each edge is d.*
ii) The multiplicity of $\frac{r\sqrt{2d-2}}{d}$ *is one.*
iii) For other eigenvalues ϑ of G we have $|\vartheta| \leq \frac{r\sqrt{2d-2}}{d}$.

Proof. i, ii) Consider the vector $\mathbf{j} = [1, \ldots, 1]^t$. Clearly

$$\tilde{\mathcal{A}}\mathbf{j} = \frac{r\sqrt{2d-2}}{d}\mathbf{j},$$

since for an arbitrary edge $e = uv$ we have $n_u = n_v = d$.
iii) It is clear by the Perron-Frobenius theorem. □

Theorem 6. *i) All $\tilde{\mathcal{A}}$-eigenvalues of bipartite graph G are symmetric around zero.*
ii) If G is SDB, then $\vartheta_i = \frac{\sqrt{2d-2}}{d}\lambda_i$, *for some d.*

Proof. i) If G is bipartite, then by labeling its vertices properly, its $\tilde{\mathcal{A}}$-matrix can be written as

$$\tilde{\mathcal{A}}(G) = \begin{bmatrix} 0 & W \\ W^T & 0 \end{bmatrix}.$$

The eigenvalues of $\tilde{\mathcal{A}}(G)$ satisfy $\tilde{\mathcal{A}}\mathbf{x} = \vartheta\mathbf{x}$ and $\tilde{\mathcal{A}}\mathbf{x}^* = -\vartheta\mathbf{x}^*$, where $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{x}^* = (x_1, -x_2)^T$. This means that the $\tilde{\mathcal{A}}$-eigenvalues of a bipartite graph are symmetric.
ii) By Theorem 5, since $\tilde{\mathcal{A}} = \frac{\sqrt{2d-2}}{d}A$, the proof is straightforward. □

It is a well-known result that a graph is bipartite if and only if its adjacency spectrum is symmetric around zero. However, the converse of Theorem 6(i) does not hold for the $\tilde{\mathcal{A}}$-matrix. In other words, a symmetric $\tilde{\mathcal{A}}$-spectrum does not imply that the graph is bipartite. For example, consider the graph $\mathcal{K} = K_4/e$ shown in Figure 2. The $\tilde{\mathcal{A}}$-spectrum of $\mathcal{K}$ is $\{[-2]^1, [0]^2, [2]^1\}$, which is symmetric around zero but the graph $\mathcal{K}$ is not bipartite.

4. Spectral Radius of $\tilde{\mathcal{A}}$-matrix

In this section, we investigate the spectral radius of the $\tilde{\mathcal{A}}$-matrix associated with a graph G. Our aim is to derive sharp bounds for the largest $\tilde{\mathcal{A}}$-eigenvalue in terms of fundamental graph parameters such as the order, size, and bipartiteness of the graph. To this end, we first recall a classical inequality for the largest eigenvalue of a nonnegative irreducible symmetric matrix, which allows us to estimate the spectral radius via suitable test vectors. Using this approach together with the Rayleigh quotient technique, we establish several upper and lower bounds for the spectral radius of $\tilde{\mathcal{A}}$. Special attention is paid to bipartite graphs, trees and unicycle graphs, for which the obtained bounds are shown to be tight. Concrete examples are provided to illustrate the effectiveness of the results.

Lemma 4. *[7] Let M be an irreducible symmetric $n \times n$ matrix with non-negative entries m_{ij}, and let α be the largest eigenvalue of M. For any positive vector $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T$, we have*

$$\min_{1\leq i\leq n} \frac{1}{y_i}\sum_{j=1}^{n} m_{ij}y_j \leq \alpha \leq \max_{1\leq i\leq n} \frac{1}{y_i}\sum_{j=1}^{n} m_{ij}y_j.$$

Both equalities hold if and only if $\mathbf{y}$ is an eigenvector of M corresponding to α.

The following lemma can be directly concluded from Lemma 4.

Lemma 5. *The spectral radius of the $\tilde{\mathcal{A}}$-matrix is bounded as*

$$\frac{2}{n}\mathcal{A}(G) \leq \vartheta_1 \leq \max_i \mathcal{A}(G)(i)$$

with equality at both sides if and only if $\mathcal{A}(G)(i)$ is the same for every node i of the graph.

Theorem 7. *Suppose $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$ are all $\tilde{\mathcal{A}}$-eigenvalues. Then the following hold.*
(i) $\sum_{i=1}^{n} \vartheta_i = 0$.
(ii) $\sum_{i=1}^{n} \vartheta_i^2 = \mathrm{tr}(\tilde{\mathcal{A}}^2) = 2\sum_{uv\in E} \frac{n_u+n_v-2}{n_u n_v}$.
In particular, if the graph G is bipartite, then

$$\sum_{i=1}^{n} \vartheta_i^2 = 2(n-2)\mathrm{RSz}(G).$$

(iii) If G has no pdb-edge, then

$$\mathcal{A}(G) \leq \sum_{e\in E(G)} \sqrt{\frac{n_u+n_v-2}{n_u+n_v}}. \tag{4.1}$$

(iv) If G is bipartite without pdb-edge, then

$$\mathcal{A}(G) \leq \frac{\sqrt{n-2}}{n} m. \tag{4.2}$$

(v) If G is bipartite, then

$$\mathcal{A}(G) \geq \begin{cases} \frac{2\sqrt{n-2}}{n} m & \text{if } 2 \mid n, \\ \frac{2\sqrt{n-2}}{n-1} m & \text{if } 2 \nmid n. \end{cases}$$

Proof. The proof of (i) follows directly from the definitions. For (ii), if G is bipartite, then for each edge $e = uv$, $n_u + n_v = n$, and the proof is straightforward. For (iii), since for each edge $e = uv$, we have $n_u + n_v \leq n_u n_v$, the inequality (4.1) holds. Also, if G is bipartite, then necessarily $n_u + n_v = n$, and thus (iv) holds.
For (v), if G is bipartite, then $n_u n_v \leq \left\lfloor \frac{n}{2} \right\rfloor^2$, and so

$$\sqrt{\frac{n_u + n_v - 2}{n_u n_v}} \geq \frac{2\sqrt{n-2}}{n}, \quad \text{if } n \text{ is even},$$

and

$$\sqrt{\frac{n_u + n_v - 2}{n_u n_v}} \geq \frac{2\sqrt{n-2}}{n-1}, \quad \text{if } n \text{ is odd}.$$

□

The Rayleigh quotient theorem [7] states: For any symmetric matrix $A \in \mathbb{R}^{n \times n}$ and vector $x \in \mathbb{R}^n$, the maximum value of the Rayleigh quotient $R(x) = \frac{x^T A x}{x^T x}$ occurs at the eigenvector corresponding to the largest eigenvalue $\lambda_{\max}$ of A.

Corollary 1. *If G is bipartite, then*

$$\vartheta_1 \geq \begin{cases} \dfrac{4\sqrt{n-2}}{n^2} m & \text{if } 2 \mid n, \\ \dfrac{4\sqrt{n-2}}{n(n-1)} m & \text{if } 2 \nmid n. \end{cases}$$

Proof. By the Rayleigh quotient theorem, we have

$$\vartheta_1 \geq \frac{2\mathcal{A}(G)}{n},$$

and by Theorem 7 (v), the result follows. □

Example 3. Consider the unicycle graph $S_n + e$. Suppose $\alpha = \sqrt{\frac{n-3}{n-2}}$ and $\beta = \sqrt{\frac{n-2}{n-1}}$. Then

$$\tilde{\mathcal{A}}(S_n) = \begin{pmatrix} 0 & 0 & . & . & . & . & 0 & \alpha \\ 0 & 0 & . & . & . & . & 0 & \alpha \\ 0 & 0 & . & . & . & . & 0 & \beta \\ 0 & 0 & . & . & . & . & 0 & \beta \\ 0 & 0 & . & . & . & . & 0 & . \\ 0 & 0 & . & . & . & . & 0 & . \\ 0 & 0 & . & . & . & . & 0 & . \\ \alpha & \alpha & \beta & \beta & . & . & . & 0 \end{pmatrix},$$

and $\eta(\tilde{\mathcal{A}}) = n-2$. Now, by Theorem 7, we have:

$$\sum_{i=1}^{n} \vartheta_i = 0, \quad \text{and} \quad \sum_{i=1}^{n} \vartheta_i^2 = 2\left[2 \cdot \frac{n-3}{n-2} + (n-3) \cdot \frac{n-2}{n-1}\right] = 2(n-3)\left[\frac{n-2}{n-1} + \frac{2}{n-2}\right].$$

Furthermore, since the nullity is $n-2$, there are only two non-zero eigenvalues, which we denote by ϑ_1 and ϑ_2. From Theorem 7(i), the sum of all eigenvalues is zero. Since all other eigenvalues are zero, we must have $\vartheta_1 + \vartheta_2 = 0$, which implies $\vartheta_2 = -\vartheta_1$.
Let's denote the sum of squares of the eigenvalues by S:

$$S = \sum_{i=1}^{n} \vartheta_i^2 = \vartheta_1^2 + \vartheta_2^2 = 2(n-3)\left[\frac{2}{n-2} + \frac{n-2}{n-1}\right].$$

Therefore, the non-zero eigenvalues are explicitly found to be:

$$\vartheta_1 = \sqrt{\frac{S}{2}} = \sqrt{(n-3)\left[\frac{2}{n-2} + \frac{n-2}{n-1}\right]}$$

and $\vartheta_2 = -\vartheta_1$.

Example 4. Consider the cycle graph C_n on n vertices. Since C_n is edge-transitive by Note 2, we obtain

$$\vartheta_1(C_n) = \begin{cases} \dfrac{4\sqrt{n-2}}{n} & \text{if } 2 \mid n \\ \dfrac{4\sqrt{n-3}}{n-1} & \text{otherwise} \end{cases}.$$

Theorem 8. *Let ϑ_1 be the largest $\tilde{\mathcal{A}}$-eigenvalue of a graph G on n vertices.*

(a) If T is a tree on $n \geq 7$ vertices, then $\vartheta_1 \geq \frac{2\pi\sqrt{n-2}}{n}$.

(b) If G is a unicycle bipartite graph with $n \geq 8$, then

$$\vartheta_1 \geq \frac{4\sqrt{n-3}}{n-1}.$$

Proof. **(a):** If G is a tree by Theorem 1, $\mathcal{A}(T) \geq \mathcal{A}(P_n)$ and by Rayleigh quotient theorem, for any real n-vector $0 \neq x \in \mathbb{R}^n$, we have

$$\vartheta_1 \geq \frac{x^t \tilde{\mathcal{A}} x}{x^t x},$$

if $x = \mathbf{j}$, and n is large enough by [29] and [9, Corollary 2.2.], one can deduce that

$$\vartheta_1 \geq \frac{2\mathcal{A}(T)}{n} \geq \frac{2\mathcal{A}(P_n)}{n} = \frac{2\pi\sqrt{n-2}}{n}.$$

(b): By [9, Theorem 3.3.], the minimum value of $\mathcal{A}$ holds for the cycle graph C_n, where n is even. Hence, by the Rayleigh quotient theorem

$$\vartheta_1 \geq \frac{2\mathcal{A}(G)}{n} \geq \frac{2 \times 2\sqrt{n-3}n}{n(n-1)}.$$

□

Theorem 9. *Suppose G is a bipartite graph on n vertices, then*

$$\vartheta_1 \leq \sqrt{n-2},$$

with equality if and only if G is a complete bipartite graph.

Proof. Let G be a bipartite graph. For each vertex $v_i \in V(G)$, let $d_i = \deg(v_i)$ denote its degree. Define the vector $x = (x_1, \ldots, x_n)^T$ by $x_i = \sqrt{d_i}$, for all $i = 1, \ldots, n$ and the i-th component of the vector $\tilde{\mathcal{A}}(G)x$ by $(\tilde{\mathcal{A}}(G)x)_i$. Then

$$\begin{aligned}
(\tilde{\mathcal{A}}(G)x)_i &= \sum_{v_j \in N(v_i)} \sqrt{\frac{n_i + n_j - 2}{n_i n_j}} \sqrt{d_j} \\
&\leq \sum_{v_j \in N(v_i)} \sqrt{\frac{n-2}{d_i d_j}} \sqrt{d_j} \\
&= \sum_{v_j \in N(v_i)} \sqrt{\frac{n-2}{d_i}} \\
&= d_i \sqrt{\frac{n-2}{d_i}} \\
&= \sqrt{d_i}\sqrt{n-2}.
\end{aligned}$$

Now, we investigate the equality condition. Let $K_{a,b}$ be a complete bipartite graph, then for every arbitrary edge $e = uv$, we have $n_u = a$ and $n_v = b$, so

$$\sqrt{\frac{n_u + n_v - 2}{n_u n_v}} = \sqrt{\frac{a+b-2}{ab}} = \sqrt{\frac{n-2}{ab}},$$

hence

$$\vartheta_1 = \sqrt{\frac{n-2}{ab}}\lambda_1 = \sqrt{\frac{n-2}{ab}}\sqrt{ab} = \sqrt{n-2}.$$

Conversely, suppose that $\vartheta_1 = \sqrt{n-2}$. This equality can only hold if all inequalities in the derivation above are equalities. This requires that for every edge $v_i v_j \in E$, we must have $n_i + n_j = n$ and $n_i n_j = d_i d_j$. This is precisely the definition of a complete bipartite graph. □

5. Conclusion

This article offers an in-depth exploration of the Graovac-Ghorbani matrix and its spectral behavior across a variety of graph families. Through a detailed examination, it becomes evident that the spectral characteristics of the Graovac-Ghorbani matrix vary significantly depending on the underlying structure of the graph. These variations highlight the matrix's sensitivity to graph topology and its potential as a tool for structural graph analysis. In particular, evaluating $\mathcal{A}(G)$ for specific categories of graphs such as NDB graphs, vertex-transitive graphs, edge-transitive graphs, and bipartite edge-transitive graphs provides meaningful insight into the behavior and implications of the Graovac-Ghorbani index.

The analysis further extends to establishing bounds on $\mathcal{A}(G)$ for trees. These bounds offer a deeper understanding of how hierarchical and acyclic structures impact the spectral radius of the matrix. Additionally, by investigating the rank of the Laplacian version of the Graovac-Ghorbani matrix, denoted as $\tilde{\mathcal{A}}$-matrix, we underscore the usefulness of this index in uncovering fundamental spectral properties of graphs.

Several theorems and corollaries are introduced in this study, each contributing new perspectives on the nature and structure of the Graovac-Ghorbani matrix. One of the key results demonstrates that a graph with diameter d must have at least $d+1$ distinct eigenvalues in its Graovac-Ghorbani spectrum. This result connects the spectral diversity of the matrix directly to a basic topological feature of the graph. Moreover, a classification is provided for graphs that possess exactly one eigenvalue, establishing a foundational understanding of such extreme spectral cases.

In contrast, it is also shown that no connected graph exists with exactly two distinct eigenvalues in the Graovac-Ghorbani spectrum. This negative result further clarifies the landscape of spectral possibilities and rules out a certain class of potential graphs.

The findings presented here contribute to advancing the study of the Laplacian $\tilde{\mathcal{A}}$-matrix and its spectral properties, an area that continues to attract considerable attention in graph theory. Moreover, the paper opens several directions for future research, including investigations into how the Graovac-Ghorbani index relates to other well-known indices, such as the Szeged index and the product variant of the Graovac-Ghorbani index. These connections may offer a unified perspective on distance-based and symmetry-based graph invariants.

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