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Research Article

Computing the precise total distance vertex irregularity strength of prism and web graphs

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Abstract: Given a graph $\Gamma(V, E)$ with a non-empty vertex set V and edge set E. A total d-labeling is assignment of positive integers to the set $V(\Gamma) \cup E(\Gamma)$. The labeling is called distance vertex irregular total d-labeling (DVITL) if any two different vertices in V have different weights. The weight of the vertex $u \in V$ is the aggregate of labels of the neighbor vertices and the labels of edges that incident at the vertex u. The least number d for which there exists a DVITL of Γ is called the total distance vertex irregularity strength of Γ symbolized by $\text{tdis}(\Gamma)$. The graph invariant $\text{tdis}(\Gamma)$ is known for paths, cycles, fans and wheels, and for some corona product graphs. In this research, we investigate more complex graph structures with regular or almost regular neighborhoods of vertices. The main results of the paper include determining the exact value of the total distance vertex irregularity strength for three families of graphs namely, prism graphs, web graphs and web graphs without center.

Keywords: distance vertex irregular total labeling, total distance vertex irregularity strength, prisms, web graphs, web graphs without center.

AMS Subject classification: 05C78, 05C90

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1. Introduction

A simple graph Γ is a couple $(V(\Gamma), E(\Gamma))$, in which $V(\Gamma)$ signifies a non empty vertex set, and $V(\Gamma)$ indicates an unordered set of two different vertices $u, v \in V(\Gamma)$. If every couple of different vertices in Γ has a path between them, then the graph is regarded as connected. The graph labeling is one a variety of problems that come up when studying graph theory and it has become increasingly significant recently, see [5]. Further, labelings play a significant role in many areas, such as electron microscopy, coding theory, x-ray analysis, sonar technology, circuit design and computer science [2, 3]. Each graph utilized in this research is simple, connected, and undirected. Labeling, defined here, is the process of assigning numbers (usually positive integers) to the edges or vertices (or both) of graphs. It is referred to as an edge labeling whenever labels are applied to edges and a vertex labeling whenever labels are applied to vertices, and a total labeling whenever labels are applied to both of them. In an edge labeling $\sigma : E(\Gamma) \to \{1, 2, \ldots, d\}$ with positive integer d the weight of a vertex u is found by summing the labels of all edges that connect to u. The idea of edge d-labeling, first proposed by Chartrand et al. in [4], guaranteeing that each pair of vertices of a graph has a different weights is referred to as irregular assignment. The lowest number d which permits a graph Γ to have an irregular assignment is known as the irregularity strength of Γ and is represented by s(Γ). Even for a simple graphs, evaluating the irregularity strength can be complicated. Bača et al. [1] subsequently constructed the idea of total labeling through the structure of graph Γ with an extra condition that every vertex in the graph has an uniquely specific weight. The weight of the vertex u equals the combination of its label and the labels of all edges that connect to u. For any graph $\Gamma = (V, E)$, a labeling $\sigma : V(\Gamma) \bigcup E(\Gamma) \to \{1, 2, \ldots, d\}$ is a total d-labeling. This labeling is known as vertex irregular total d-labeling if each pair of distinct vertices, u and v, possesses various weights, i.e. $wt_\sigma(u) \neq wt_\sigma(v)$, where

$$wt_\sigma(u) = \sigma(u) + \sum_{uv \in E(\Gamma)} \sigma(uv).$$

The least value of d, where such labeling occurs, is named the total vertex irregularity strength of graph Γ, indicated by tvs(Γ). Miller et al. [8] presented a distance vertex magic labeling, also referred to as 1-vertex-magic vertex labeling. In this labeling, the weight of each vertex $u \in V$ is aggregate of all labels of vertices that are in the neighborhood of u. Inspired by the notion of a distance magic labeling and irregular labeling, Slamin [9] proposes a new notion of irregular vertex labeling called a distance irregular vertex labeling. For any graph $\Gamma = (V, E)$, a vertex labeling $\sigma : V \to \{1, 2, \ldots, d\}$ is a distance vertex labeling whenever the weight of each vertex is unique. The weight of each vertex $u \in V$ according to σ is given as follows

$$wt_\sigma(u) = \sum_{v \in N(u)} \sigma(v),$$

where $N(u)$ is the set of u neighbors. The minimal value of d that permits a graph Γ to have a distance irregular vertex labeling with labels not exceeding d is termed the distance irregularity strength of Γ, indicated by $\text{dis}(\Gamma)$. Wijayanti et al. [14] provide the notion of distance vertex irregular total d-labeling (DVITL), which was motivated by distance irregular vertex labeling and vertex irregular total labeling, combining elements of both procedures.

Definition 1. Let $\Gamma(V, E)$ represent a simple finite graph which has its vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A distance vertex irregular total d-labeling (DVITL) is a mapping $\sigma : V(\Gamma) \cup E(\Gamma) \to \{1, 2, \ldots, d\}$ that ensures the weights of all vertices in Γ are unique. The weight of a vertex $u \in V$ according to the labeling σ is given as

$$wt_\sigma(u) = \sum_{v \in N(u)} \sigma(v) + \sum_{v \in N(u)} \sigma(uv),$$

where, $N(u)$ is the set of all neighbors of the vertex u.

The least number of d for which there exists a DVITL of Γ is called the total distance vertex irregularity strength of graph Γ and is indicated as $\text{tdis}(\Gamma)$. The precise value of total distance vertex irregularity strength for the join product of graphs, paths, cycles, fans, wheels and some corona product graphs has been extensively examined in published works [10–13, 15]. In this research, we investigate more complex graph structures with regular or almost regular neighborhoods of vertices.

In general, the following lemma gives a lower bound of the total distance vertex irregularity strength for a graph Γ as follows:

Lemma 1. [14] *Let $\Gamma(V, E)$ be a simple connected finite graph with a maximum degree Δ and minimum degree δ. Then* $\text{tdis}(\Gamma) \geq \left\lceil \frac{|V(\Gamma)|+2\delta-1}{2\Delta} \right\rceil$.

The rest of this paper is organized in the following way. In Section 2 we study properties of distance vertex irregular total d-labelings for the prisms D_p in six cases depending on the modularity of the parameter p and in each case we show that the vertex weights are pairwise different. Section 3 and Section 4 are devoted to the investigation of existence of distance vertex irregular total d-labelings for the web graphs $W(2, p)$ and web graphs without center $W_0(2, p)$ and to determining their precise values of the total distance vertex irregularity strength. The precise values of the graph invariant $\text{tdis}(\Gamma)$ for all three families of graphs prove the sharpness of the lower bound presented in Lemma 1.

For clarity, throughout this work, for numbers a and b, let $[a, b]$ denote the set of all integers t such that $a \leq t \leq b$.

2. DVITL of the prisms D_p

A prism D_p, where $p \geq 3$, is a 3-regular graph derived from a pair of cycles C_p by linking the t^{th} vertex of the first C_p with t^{th} vertex of the second C_p through an edge, see Figure 1. The prism D_p can be defined as the Cartesian product $C_p \times P_2$. The order and size of D_p are $2p$ and $3p$, respectively. The vertex set and the edge set of D_p, respectively are $V(D_p) = \{u_t, v_t : t \in [1,p]\}$ and $V(D_p) = \{u_t u_{t+1}, v_t v_{t+1}, u_t v_t : t \in [1,p]\}$. In this context, every index is considered modulo p.

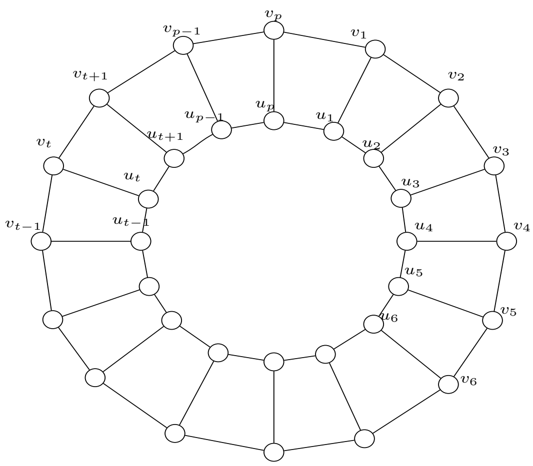

Figure 1. **The prism D_p.**

Theorem 1. *Let D_p be a prism of order $2p$ with $p \geq 3$. Then*

$$\operatorname{tdis}(D_p) = \begin{cases} 3, & \text{if } p = 3, \\ \left\lceil \frac{2p+5}{6} \right\rceil, & \text{otherwise.} \end{cases}$$

Proof. Let D_p be a prism with $p \geq 3$. Since D_p is a 3-regular graph then directly from Lemma 1 we get

$$\operatorname{tdis}(D_p) = d \geq \left\lceil \frac{2p+5}{6} \right\rceil.$$

Additionally, we demonstrate the equality by establishing the presence of DVITL of D_p, where $d = \left\lceil \frac{2p+5}{6} \right\rceil$. The suitable distance vertex irregular total d-labelings of D_p, for $3 \leq p \leq 7$, are given in Figure 2. Otherwise, we construct the total labeling $\sigma : V(D_p) \cup E(D_p) \to \left\{1, 2, \ldots, \left\lceil \frac{2p+5}{6} \right\rceil\right\}$ as follows.

Case 1. For $p \equiv 0 \pmod 6$.
The labels for the vertices are provided as follows:

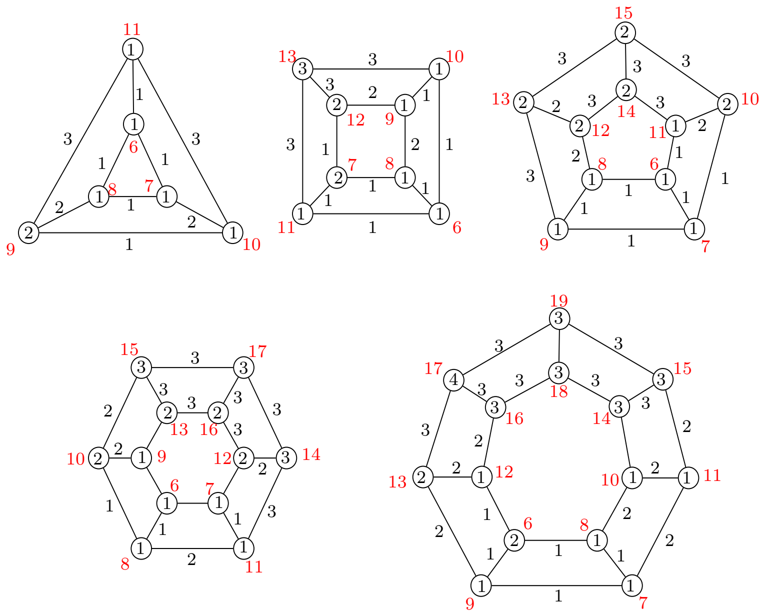

Figure 2. **The DVITL of D_p for $p = 3, 4, 5, 6, 7$.**

vertex label		
$\sigma(v_{t+1})$	for $t \in \left[0, \frac{p}{2} - 2\right]$	
	$d - \frac{2t}{3}$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
$\sigma(u_{t+1})$	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t-2}{3} - 1$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+5}{3}$	if $t \equiv 2 \pmod 3$
$\sigma(v_{t+1})$	for $t \in \left[\frac{p}{2} - 1, \frac{p}{2} + 1\right]$	
	2	if $t = \frac{p}{2} - 1$
	1	if $t = \frac{p}{2}, \frac{p}{2} + 1$
$\sigma(u_{t+1})$	1	
$\sigma(v_{t+1})$	for $t \in \left[\frac{p}{2} + 2, p - 2\right]$	
	$\frac{2t-p}{3} + 1$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p+1}{3}$	if $t \equiv 1 \pmod 3$
	$\frac{2t-p+5}{3}$	if $t \equiv 2 \pmod 3$
$\sigma(u_{t+1})$	$\frac{2t-p}{3}$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p-2}{3}$	if $t \equiv 1 \pmod 3$
	$\frac{2t-p+2}{3}$	if $t \equiv 2 \pmod 3$
$\sigma(v_p)$	d	
$\sigma(u_p)$	$d - 1$	

and we assign labels to edges in this manner

edge label		
$\sigma(v_{t+1}v_{t+2}) = \sigma(u_{t+1}u_{t+2})$	for $t \in \left[0, \frac{p}{2}-3\right]$	
	$d - \frac{2t}{3}$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
$\sigma\left(v_{t+1}v_{t+2}\right)$	for $t \in \left[\frac{p}{2}-2, \frac{p}{2}+1\right]$	
	2	if $t = \frac{p}{2}-2, \frac{p}{2}$
	1	if $t = \frac{p}{2}-1$
	3	if $t = \frac{p}{2}+1$
$\sigma\left(u_{t+1}u_{t+2}\right)$	1	
$\sigma(v_{t+1}v_{t+2}) = \sigma(u_{t+1}u_{t+2})$	for $t \in [\frac{p}{2}+2, p-2]$	
	$\frac{2t-p}{3}+1$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p+1}{3}$	if $t \equiv 1 \pmod 3$
	$\frac{2t-p+5}{3}$	if $t \equiv 2 \pmod 3$
$\sigma\left(v_pv_1\right) = \sigma\left(u_pu_1\right)$	d	

edge label		
$\sigma\left(v_{t+1}u_{t+1}\right)$	for $t \in \left[0, \frac{p}{2}\right]$	
	d	if $t = 0$
	$d - \frac{2t}{3}+2$	if $t \equiv 0 \pmod 3$, $3 \le t < \frac{p}{2}-1$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t-4}{3}$	if $t \equiv 2 \pmod 3$
	2	if $t = \frac{p}{2}-1$
	1	if $t = \frac{p}{2}$
$\sigma\left(v_{t+1}u_{t+1}\right)$	for $t \in \left[\frac{p}{2}+1, p-1\right]$	
	1	if $t = \frac{p}{2}+1$
	2	if $t = \frac{p}{2}+2$
	$\frac{2t-p}{3}+1$	if $t \equiv 0 \pmod 3$, $t \ge \frac{p}{2}+3$
	$\frac{2t-p+7}{3}$	if $t \equiv 1 \pmod 3$, $t \ge \frac{p}{2}+4$
	$\frac{2t-p+5}{3}$	if $t \equiv 2 \pmod 3$, $t \ge \frac{p}{2}+5$

The vertex weights associated with the given labeling σ are depicted in Table 1.

Table 1. **An overview of the vertex weights under the DVITL σ.**

t	$wt_\sigma\left(v_{t+1}\right)$	$wt_\sigma\left(u_{t+1}\right)$
$t = 0$	$2p+5$	$2p+4$
$t \in \left[1, \frac{p}{2}-3\right]$	$2p-4t+7$	$2p-4t+6$
$t = \frac{p}{2}-2$	15	13
$t = \frac{p}{2}-1$	10	9
$t = \frac{p}{2}$	8	6
$t = \frac{p}{2}+1$	11	7
$t = \frac{p}{2}+2$	14	12
$t \in \left[\frac{p}{2}+3, p-1\right]$	$4t-2p+5$	$4t-2p+4$

Case 2. For $p \equiv 1 \pmod 6$.
The labels for the vertices and edges are provided as follows:

vertex label		
	for $t \in \left[0, \frac{p-3}{2}\right]$	
$\sigma(v_{t+1})$	$d - \frac{2t}{3} - 1$	if $t \equiv 0, 3 \pmod 6$
	$d - \frac{2t+1}{3}$	if $t \equiv 1 \pmod 6$
	$d - \frac{2t+2}{3}$	if $t \equiv 2, 5 \pmod 6$
	$d - \frac{2t-2}{3}$	if $t \equiv 4 \pmod 6$
$\sigma(u_{t+1})$	$d - \frac{2t}{3} - 1$	if $t \equiv 0, 3 \pmod 6$
	$d - \frac{2t+1}{3}$	if $t \equiv 1, 4 \pmod 6$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 6$
	$d - \frac{2t+5}{3}$	if $t \equiv 5 \pmod 6$
	for $t \in [\frac{p-1}{2}, p-1]$	
$\sigma(v_{t+1})$	$\frac{2t-p+4}{3}$	if $t \equiv 0, 3 \pmod 6$
	$\frac{2t-p+2}{3}$	if $t \equiv 1, 4 \pmod 6$
	$\frac{2t-p}{3}$	if $t \equiv 2 \pmod 6$
	$\frac{2t-p}{3} + 1$	if $t \equiv 5 \pmod 6$
$\sigma(u_{t+1})$	2	if $t = \frac{p-1}{2}$
	$\frac{2t-p+1}{3}$	if $t \equiv 0 \pmod 6$, $t > \frac{p-1}{2}$
	$\frac{2t-p+2}{3}$	if $t \equiv 1, 4 \pmod 6$
	$\frac{2t-p}{3}$	if $t \equiv 2, 5 \pmod 6$
	$\frac{2t-p+4}{3}$	if $t \equiv 3 \pmod 6$, $t > \frac{p-1}{2}$

edge label		
	for $t \in [0, \frac{p-3}{2}]$	
$\sigma(v_{t+1}v_{t+2})$	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t+1}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
$\sigma(u_{t+1}u_{t+2})$	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t+4}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$, $t \neq \frac{p-3}{2}$
	1	if $t = \frac{p-3}{2}$
	for $t \in [\frac{p-1}{2}, p-1]$	
$\sigma(v_{t+1}v_{t+2})$	$\frac{2t-p+4}{3}$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p+2}{3}$	if $t \equiv 1 \pmod 3$, $t \neq \frac{p+1}{2}$
	2	if $t = \frac{p+1}{2}$
	$\frac{2t-p}{3} + 1$	if $t \equiv 2 \pmod 3$
$\sigma(u_{t+1}u_{t+2})$	$\frac{2t-p+4}{3}$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p+2}{3}$	if $t \equiv 1 \pmod 3$, $t \neq \frac{p+1}{2}$
	2	if $t = \frac{p+1}{2}$
	$\frac{2t-p}{3}$	if $t \equiv 2 \pmod 3$
	for $t \in [0, \frac{p-3}{2}]$	
$\sigma(v_{t+1}u_{t+1})$	$d - \frac{2t}{3}$	if $t \equiv 0 \pmod 6$
	$d - \frac{2t+1}{3}$	if $t \equiv 1, 4 \pmod 6$
	$d - \frac{2t+2}{3}$	if $t \equiv 2, 5 \pmod 6$
	$d - \frac{2t}{3} - 1$	if $t \equiv 3 \pmod 6$
	for $t \in [\frac{p-1}{2}, p-1]$	
$\sigma(v_{t+1}u_{t+1})$	$\frac{2t-p+4}{3}$	if $t \equiv 0, 3 \pmod 6$
	$\frac{2t-p+5}{3}$	if $t \equiv 1 \pmod 6$, $t \neq \frac{p+1}{2}$
	1	if $t = \frac{p+1}{2}$
	$\frac{2t-p}{3} + 2$	if $t \equiv 2, 5 \pmod 6$, $t \neq \frac{p+3}{2}$
	2	if $t = \frac{p+3}{2}$
	$\frac{2t-p+2}{3}$	if $t \equiv 4 \pmod 6$

Based on the definition of labeling σ the optimal weights for the vertices are depicted in Table 2.

Table 2. **An overview of the vertex weights of D_P.**

t	$wt_\sigma\ (v_{t+1})$	$wt_\sigma(u_{t+1})$
$t=0$	$2p+5$	$2p+4$
$t\in[1,\frac{p-3}{2}]$	$2p-4t+7$	$2p-4t+6$
$t=\frac{p-1}{2}$	9	6
$t=\frac{p+1}{2}$	7	8
$t\in[\frac{p+3}{2},p-1]$	$4t-2p+5$	$4t-2p+4$

Case 3. For $p \equiv 2 \pmod 6$.
The labels for the vertices and edges are provided as follows

vertex label		
	for $t\in[0,\frac{p}{2}+1]$	
$\sigma\ (v_{t+1})$	$d-\frac{2t}{3}-1$	if $t\equiv 0 \pmod 3, t\neq \frac{p}{2}-1$
	$d-\frac{2t-2}{3}$	if $t\equiv 1 \pmod 3, t\neq \frac{p}{2}$
	$d-\frac{2t+2}{3}$	if $t\equiv 2 \pmod 3, t\neq \frac{p}{2}-2, \frac{p}{2}+1$
	3	if $t=\frac{p}{2}-2$
	2	if $t=\frac{p}{2}-1$
	1	if $t=\frac{p}{2}, \frac{p}{2}+1$
$\sigma(u_{t+1})$	$d-\frac{2t}{3}$	if $t\equiv 0 \pmod 3, t\neq \frac{p}{2}-1$
	$d-\frac{2t-2}{3}$	if $t\equiv 1 \pmod 3, t\neq \frac{p}{2}$
	$d-\frac{2t+2}{3}$	if $t\equiv 2 \pmod 3, t\neq \frac{p}{2}+1$
	1	if $t=\frac{p}{2}-1, \frac{p}{2}, \frac{p}{2}+1$
	for $t\in[\frac{p}{2}+2,p-1]$	
$\sigma(v_{t+1})$	$\frac{2t-p+2}{3}$	if $t\equiv 0 \pmod 3$
	$\frac{2t-p}{3}+2$	if $t\equiv 1 \pmod 3$
	$\frac{2t-p+1}{3}$	if $t\equiv 2 \pmod 3$
$\sigma(u_{t+1})$	1	if $t=\frac{p}{2}+2$
	$\frac{2t-p+2}{3}$	if $t\equiv 0 \pmod 3, t\neq \frac{p}{2}+2$
	$\frac{2t-p}{3}+2$	if $t\equiv 1 \pmod 3$
	$\frac{2t-p+4}{3}$	if $t\equiv 2 \pmod 3$

edge label		
$\sigma(v_{t+1}v_{t+2})$	for $t\in[0,\frac{p}{2}-1]$	
	$d-\frac{2t}{3}-1$	if $t\equiv 0 \pmod 3, t\neq \frac{p}{2}-1$
	$d-\frac{2t+1}{3}$	if $t\equiv 1 \pmod 3$
	$d-\frac{2t-1}{3}$	if $t\equiv 2 \pmod 3, t\neq \frac{p}{2}-2$
	2	if $t=\frac{p}{2}-2, \frac{p}{2}-1$
	for $t\in[\frac{p}{2},p-1]$	
	$\frac{2t-p-1}{3}$	if $t\equiv 0 \pmod 3, t\neq \frac{p}{2}+2$
	$\frac{2t-p}{3}+1$	if $t\equiv 1 \pmod 3$
	$\frac{2t-p+1}{3}$	if $t\equiv 2 \pmod 3, t\neq \frac{p}{2}+1$
	2	if $t=\frac{p}{2}+1, \frac{p}{2}+2$

	edge label	
$\sigma\left(u_{t+1}u_{t+2}\right)$	for $t \in [0, \frac{p}{2}-1]$	
	$d-\frac{2t}{3}-1$	if $t\equiv 0 \pmod 3, t\neq \frac{p}{2}-1$
	$d-\frac{2t+1}{3}$	if $t\equiv 1 \pmod 3, t\neq \frac{p}{2}-3$
	$d-\frac{2t-1}{3}$	if $t\equiv 2 \pmod 3, t\neq \frac{p}{2}-2$
	2	if $t=\frac{p}{2}-3, \frac{p}{2}-2$
	1	if $t=\frac{p}{2}-1$
	for $t \in [\frac{p}{2}, p-1]$	
	$\frac{2t-p-1}{3}$	if $t\equiv 0 \pmod 3$
	$\frac{2t-p}{3}+1$	if $t\equiv 1 \pmod 3$
	$\frac{2t-p+1}{3}$	if $t\equiv 2 \pmod 3$

	edge label	
$\sigma\left(v_{t+1}u_{t+1}\right)$	for $t \in [0, \frac{p}{2}-1]$	
	$d-\frac{2t}{3}-1$	if $t\equiv 0 \pmod 3, t\neq \frac{p}{2}-1$
	$d-\frac{2t+1}{3}$	if $t\equiv 1 \pmod 3$
	$d-\frac{2t+5}{3}$	if $t\equiv 2 \pmod 3, t\neq \frac{p}{2}-2$
	2	if $t=\frac{p}{2}-2, \frac{p}{2}-1$
	for $t \in [\frac{p}{2}, p-1]$	
	$\frac{2t-p+5}{3}$	if $t\equiv 0 \pmod 3$
	$\frac{2t-p}{3}+1$	if $t\equiv 1 \pmod 3$
	$\frac{2t-p+1}{3}$	if $t\equiv 2 \pmod 3, t\neq \frac{p}{2}+1$
	2	if $t=\frac{p}{2}+1$

Under the labeling σ, the optimal weights of the vertices are depicted in Table 3.

Table 3. **An overview of the vertex weights under the DVITL σ.**

t	$wt_\sigma\left(v_{t+1}\right)$	$wt_\sigma\left(u_{t+1}\right)$	
$t=0$	$2p+5$	$2p+4$	
$t\in[1, \frac{p}{2}-1]$	$2p-4t+7$	$2p-4t+6$	$t\equiv 0 \pmod 3$, $t=\frac{p}{2}-3, \frac{p}{2}-2$
	$2p-4t+6$	$2p-4t+7$	$t\equiv 1,2 \pmod 3$, $t\neq\frac{p}{2}-3, \frac{p}{2}-2$
$t=\frac{p}{2}$	8	6	
$t=\frac{p}{2}+1$	9	7	
$t\in[\frac{p}{2}+2, p-1]$	$4t-2p+5$	$4t-2p+4$	$t\equiv 2 \pmod 3$, $t=\frac{p}{2}+2, \frac{p}{2}+3$
	$4t-2p+4$	$4t-2p+5$	$t\equiv 0,1 \pmod 3$, $t\neq\frac{p}{2}+2, \frac{p}{2}+3$

Case 4. For $p\equiv 3 \pmod 6$.
The labels for the vertices and edges are provided as follows

vertex label		
	for $t \in [0, \frac{p-1}{2}]$	
$\sigma(v_{t+1})$	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3, t \neq \frac{p-3}{2}$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p-1}{2}$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3, t \neq \frac{p-5}{2}$
	3	if $t = \frac{p-5}{2}$
	2	if $t = \frac{p-3}{2}$
	1	if $t = \frac{p-1}{2}$
$\sigma(u_{t+1})$	$d - \frac{2t}{3}$	if $t \equiv 0 \pmod 3, t \neq \frac{p-3}{2}$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p-1}{2}$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
	1	if $t = \frac{p-3}{2}, \frac{p-1}{2}$
	for $t \in [\frac{p+1}{2}, p-1]$	
$\sigma(v_{t+1})$	1	if $t = \frac{p+1}{2}$
	2	$t = \frac{p+3}{2}$
	$\frac{2t-p}{3}$	if $t \equiv 0 \pmod 3, t \neq \frac{p+3}{2}$
	$\frac{2t-p+1}{3}$	if $t \equiv 1 \pmod 3$
	$\frac{2t-p+5}{3}$	if $t \equiv 2 \pmod 3, t \neq \frac{p+1}{2}$
$\sigma(u_{t+1})$	1	if $t = \frac{p+1}{2}, \frac{p+3}{2}, \frac{p+5}{2}$
	$\frac{2t-p}{3} + 1$	if $t \equiv 0 \pmod 3, t \neq \frac{p+3}{2}$
	$\frac{2t-p+1}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p+5}{2}$
	$\frac{2t-p+5}{3}$	if $t \equiv 2 \pmod 3, t \neq \frac{p+1}{2}$

edge label		
	for $t \in [0, \frac{p-1}{2}]$	
$\sigma(v_{t+1}v_{t+2})$	$d - \frac{2t}{3}$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p-1}{2}$
	$d - \frac{2t-4}{3}$	if $t \equiv 2 \pmod 3, t \neq \frac{p-5}{2}$
	3	if $t = \frac{p-5}{2}$
	1	if $t = \frac{p-1}{2}$
$\sigma(u_{t+1}u_{t+2})$	$d - \frac{2t}{3}$	if $t \equiv 0 \pmod 3, t \neq \frac{p-3}{2}$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p-7}{2}, \frac{p-1}{2}$
	$d - \frac{2t-4}{3}$	if $t \equiv 2 \pmod 3, t \neq \frac{p-5}{2}$
	3	if $t = \frac{p-7}{2}, \frac{p-5}{2}$
	1	if $t = \frac{p-3}{2}, \frac{p-1}{2}$

edge label		
	for $t \in [\frac{p+1}{2}, p-1]$	
$\sigma(v_{t+1}v_{t+2})$	$\frac{2t-p}{3} + 1$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p+1}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p+5}{2}$
	$\frac{2t-p+5}{3}$	if $t \equiv 2 \pmod 3$
	3	if $t = \frac{p+5}{2}$
$\sigma(u_{t+1}u_{t+2})$	1	$t = \frac{p+1}{2}$
	$\frac{2t-p}{3} + 1$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p+1}{3}$	if $t \equiv 1 \pmod 3$
	$\frac{2t-p+5}{3}$	if $t \equiv 2 \pmod 3, t \neq \frac{p+1}{2}$

	edge label	
$\sigma(v_{t+1}u_{t+1})$	for $t \in [0, \frac{p-1}{2}]$	
	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3, t \neq \frac{p-3}{2}$
	$d - \frac{2t+1}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+5}{3}$	if $t \equiv 2 \pmod 3, t \neq \frac{p-5}{2}$
	2	if $t = \frac{p-5}{2}$
	3	if $t = \frac{p-3}{2}$
	for $t \in [\frac{p+1}{2}, p-1]$	
	2	if $t = \frac{p+1}{2}$
	3	if $t = \frac{p+3}{2}$
	$\frac{2t-p}{3}$	if $t \equiv 0 \pmod 3, t \neq \frac{p+3}{2}$
	$\frac{2t-p+4}{3}$	if $t \equiv 1 \pmod 3$
	$\frac{2t-p+2}{3}$	if $t \equiv 2 \pmod 3, t \neq \frac{p+1}{2}$

Under the labeling σ, the optimal weights of the vertices are depicted in Table 4.

Table 4. An overview of the vertex weights under the DVITL σ.

t	$wt_\sigma(v_{t+1})$	$wt_\sigma(u_{t+1})$	
$t = 0$	$2p + 5$	$2p + 4$	
$t \in [1, \frac{p-3}{2}]$	$2p - 4t + 7$	$2p - 4t + 6$	$t \equiv 0 \pmod 3$, $t = \frac{p-7}{2}, \frac{p-5}{2}$
	$2p - 4t + 6$	$2p - 4t + 7$	$t \equiv 1, 2 \pmod 3$, $t \neq \frac{p-7}{2}, \frac{p-5}{2}$
$t = \frac{p-1}{2}$	8	6	
$t = \frac{p+1}{2}$	9	7	
$t \in [\frac{p+3}{2}, p-1]$	$4t - 2p + 5$	$4t - 2p + 4$	$t \equiv 0 \pmod 3$ $t = \frac{p+5}{2}, \frac{p+7}{2}$
	$4t - 2p + 4$	$4t - 2p + 5$	$t \equiv 1, 2 \pmod 3$ $t \neq \frac{p+5}{2}, \frac{p+7}{2}$

Case 5. For $p \equiv 4 \pmod 6$.
Here, the vertices are labeled as follows

	vertex label	
$\sigma(v_{t+1})$	for $t \in [0, \frac{p}{2}]$	
	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p}{2} - 1$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
	2	if $t = \frac{p}{2} - 1$
$\sigma(u_{t+1})$	for $t \in [0, \frac{p}{2}]$	
	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t+1}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p}{2} - 1$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
	1	if $t = \frac{p}{2} - 1$
$\sigma(v_{t+1})$	for $t \in [\frac{p}{2} + 1, p - 1]$	
	1	if $t = \frac{p}{2} + 1$
	$\frac{2t-p+4}{3}$	if $t \equiv 0 \pmod 3, t \neq \frac{p}{2} + 1$
	$\frac{2t-p+2}{3}$	if $t \equiv 1 \pmod 3$
	$\frac{2t-p}{3} + 1$	if $t \equiv 2 \pmod 3, t \neq \frac{p}{2} + 3$
	2	if $t = \frac{p}{2} + 3$
$\sigma(u_{t+1})$	1	if $t = \frac{p}{2} + 1$
	$\frac{2t-p+4}{3}$	if $t \equiv 0 \pmod 3, t \neq \frac{p}{2} + 1$
	$\frac{2t-p+2}{3}$	if $t \equiv 1 \pmod 3$
	$\frac{2t-p}{3}$	if $t \equiv 2 \pmod 3$

and for the edges, the labels are given in the following way

edge label		
$\sigma(v_{t+1}v_{t+2}) = \sigma(u_{t+1}u_{t+2})$	for $t \in [0, \frac{p}{2}]$	
	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3, t \neq \frac{p}{2} - 2$
	$d - \frac{2t+4}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
	3	if $t = \frac{p}{2} - 2$

edge label		
$\sigma(v_{t+1}v_{t+2})$	for $t \in [\frac{p}{2} + 1, p - 1]$	
	3	if $t = \frac{p}{2} + 2, \frac{p}{2} + 3$
	$\frac{2t-p+4}{3}$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p+2}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p}{2} + 2$
	$\frac{2t-p}{3}$	if $t \equiv 2 \pmod 3, t \neq \frac{p}{2} + 3$
$\sigma(u_{t+1}u_{t+2})$	1	if $t = \frac{p}{2} + 1$
	3	if $t = \frac{p}{2} + 2$
	$\frac{2t-p+4}{3}$	if $t \equiv 0 \pmod 3, t \neq \frac{p}{2} + 1$
	$\frac{2t-p+2}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p}{2} + 2$
	$\frac{2t-p}{3}$	if $t \equiv 2 \pmod 3$

edge label		
$\sigma(v_{t+1}u_{t+1})$	for $t \in [0, \frac{p}{2}]$	
	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t+1}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
	for $t \in [\frac{p}{2}, p - 1]$	
	$\frac{2t-p+4}{3}$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p+2}{3}$	if $t \equiv 1 \pmod 3, t \neq \frac{p}{2} + 2$
	$\frac{2t-p}{3} + 2$	if $t \equiv 2 \pmod 3, t \neq \frac{p}{2} + 3$
	3	if $t = \frac{p}{2} + 2, \frac{p}{2} + 3$

Under the labeling σ, the optimal weights of the vertices are depicted in Table 5.

Table 5. **An overview of the vertex weights under the DVITL σ.**

t	$wt_\sigma(v_{t+1})$	$wt_\sigma(u_{t+1})$	
$t = 0$	$2p + 5$	$2p + 4$	
$t \in [1, \frac{p}{2}]$	$2p - 4t + 7$	$2p - 4t + 6$	$t \equiv 0, 2 \pmod 3$
	$2p - 4t + 6$	$2p - 4t + 7$	$t \equiv 1 \pmod 3$
$t \in [\frac{p}{2} + 1, p - 1]$	$4t - 2p + 5$	$4t - 2p + 4$	$t \equiv 0, 1 \pmod 3$
	$4t - 2p + 4$	$4t - 2p + 5$	$t \equiv 2 \pmod 3, t \neq \frac{p}{2} + 3$
	17	16	$t = \frac{p}{2} + 3$

Case 6. For $p \equiv 5 \pmod 6$.
In this case, the vertices are labeled as follows

vertex label		
$\sigma(v_{t+1})$	for $t \in [0, \frac{p-1}{2}]$	
	$d - \frac{2t}{3}$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t-1}{3}$	if $t \equiv 2 \pmod 3), t \neq \frac{p-1}{2}$
	1	if $t = \frac{p-1}{2}$
$\sigma(u_{t+1})$	$d - \frac{2t}{3}$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t-2}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
$\sigma(v_{t+1})$	for $t \in [\frac{p+1}{2}, p-1]$	
	$\frac{2t-p+2}{3}$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p}{3} + 2$	if $t \equiv 1 \pmod 3), t \neq \frac{p+3}{2}$
	$\frac{2t-p+4}{3}$	if $t \equiv 2 \pmod 3$
	2	if $t = \frac{p+3}{2}$
$\sigma(u_{t+1})$	$\frac{2t-p+2}{3}$	if $t \equiv 0 \pmod 3$
	$\frac{2t-p}{3} + 1$	if $t \equiv 1 \pmod 3), t \neq \frac{p+3}{2}$
	$\frac{2t-p+4}{3}$	if $t \equiv 2 \pmod 3$
	1	if $t = \frac{p+3}{2}$

and for the edges, the labels are given in the following way

edge label		
$\sigma(v_{t+1}v_{t+2})$	for $t \in [0, \frac{p-1}{2}]$	
	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t+4}{3}$	if $t \equiv 1 \pmod 3), t \neq \frac{p-3}{2}$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
	2	if $t = \frac{p-3}{2}$
$\sigma(u_{t+1}u_{t+2})$	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t+4}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$

edge label		
$\sigma(v_{t+1}v_{t+2}) =$ $\sigma(u_{t+1}u_{t+2})$	for $t \in [\frac{p+1}{2}, p-1]$	
	1	if $t = \frac{p+1}{2}$
	3	if $t = \frac{p+3}{2}$
	$\frac{2t-p-1}{3}$	if $t \equiv 0 \pmod 3), t \neq \frac{p+1}{2}$
	$\frac{2t-p}{3} + 1$	if $t \equiv 1 \pmod 3), t \neq \frac{p+3}{2}$
	$\frac{2t-p+1}{3}$	if $t \equiv 2 \pmod 3$

edge label		
$\sigma(v_{t+1}u_{t+1})$	for $t \in [0, \frac{p-1}{2}]$	
	$d - \frac{2t}{3} - 1$	if $t \equiv 0 \pmod 3$
	$d - \frac{2t+1}{3}$	if $t \equiv 1 \pmod 3$
	$d - \frac{2t+2}{3}$	if $t \equiv 2 \pmod 3$
	for $t \in [\frac{p+1}{2}, p-1]$	
	1	if $t = \frac{p+1}{2}, \frac{p+3}{2}$
	$\frac{2t-p+5}{3}$	if $t \equiv 0 \pmod 3), t \neq \frac{p+1}{2}$
	$\frac{2t-p}{3} + 1$	if $t \equiv 1 \pmod 3), t \neq \frac{p+3}{2}$
	$\frac{2t-p+1}{3}$	if $t \equiv 2 \pmod 3$

In the Table 6. We will show that each vertex in D_p, has a different vertex weight under the labeling σ.

Table 6. **An overview of the vertex weights under the DVITL σ.**

t	$wt_\sigma(v_{t+1})$	$wt_\sigma(u_{t+1})$	
$t=0$	$2p+5$	$2p+4$	
$t \in [1, \frac{p-1}{2}]$	$2p-4t+7$	$2p-4t+6$	$t \equiv 0,1 \pmod 3, t = \frac{p-1}{2}$
	$2p-4t+6$	$2p-4t+7$	$t \equiv 2 \pmod 3, t \neq \frac{p-1}{2}$
$t \in [\frac{p+1}{2}, p-1]$	$4t-2p+5$	$4t-2p+4$	$t \equiv 0,2 \pmod 3$
	$4t-2p+4$	$4t-2p+5$	$t \equiv 1 \pmod 3$

With respect to the labelings σ in all previous cases we noticed that neither the vertex nor the edge labels exceed $\lceil \frac{2p+5}{6} \rceil$. Further, the vertex weights of D_p range from 6 to $2p+5$. Thus, $\mathrm{tdis}(D_p) = \lceil \frac{2p+5}{6} \rceil$. $\square$

Example 1. Figure 3 displays an example of a distance vertex irregular total 7-labeling of the prism D_{16}.

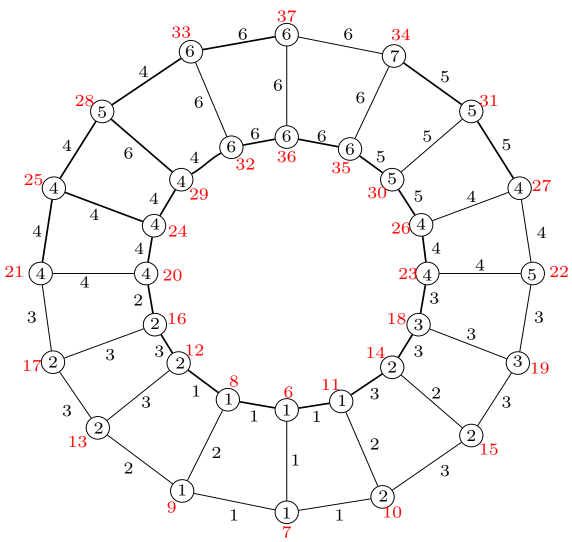

Figure 3. **A DVITL of D_{16} with $d=6$.**

3. DVITL of the web graphs $W(2,p)$

The web graph $W(2,p)$ was initially presented by Koh et al. [7] whenever a graph created by connecting the p pendent vertices of a helm graph to form a cycle, followed by addition of a single pendent to every vertex of this outward cycle. See Figure 4. The order and size of $W(2,p)$ are $3p+1$ and $5p$ respectively. The vertex set and the edge set of $W(2,p)$ will be regarded as $V(W(2,p)) = \{u_t, v_t, y_t, v_0 : t \in [1,p]\}$ and $E(W(2,p)) = \{y_t y_{t+1}, v_t v_{t+1}, u_t y_t, y_t v_t, v_t v_0 : t \in [1,p]\}$. Note that, every index is considered modulo p.

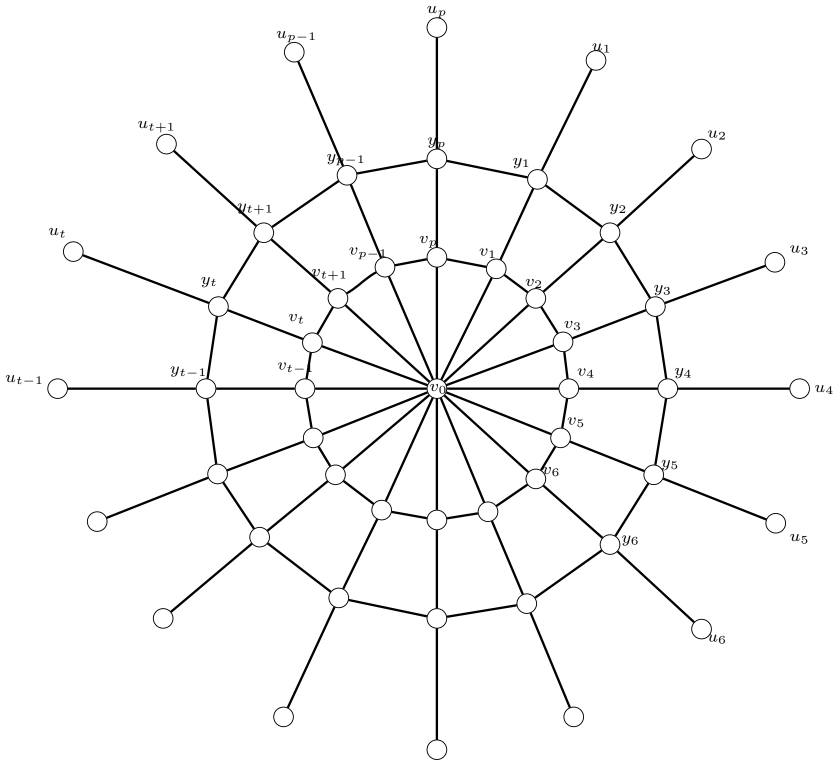

Figure 4. **The web graph** $W(2,p)$

Theorem 2. *Let* $W(2,p)$ *be a web graph of order* $3p+1$*, with* $p \geq 3$*. Then* $\mathrm{tdis}(W(2,p)) = \left\lceil \frac{p+1}{2} \right\rceil$.

Proof. Suppose $W(2,p)$ is a web graph with $p \geq 3$ vertices of degree 1, $2p$ vertices of degree 4 and one vertex of degree p. With respect to Lemma 1 we have $\mathrm{tdis}(W(2,p)) \geq \left\lceil \frac{3p+2}{2p} \right\rceil$. However, if we consider a distance vertex irregular total labeling of $W(2,p)$ and enumerate the weights of the vertices the smallest weight among all vertices of $W(2,p)$ is at least 2 and the largest weight of a vertex of degree 2 is at least $p+1$. Since the weight of any vertex of degree 2 is the sum of two positive integers, then at least one label is at least $\left\lceil \frac{p+1}{2} \right\rceil$.

The largest value among the weights of vertices of degree 2 and 4 is at least $3p+1$ and this weight is the sum of at most eight integers. Hence the largest label contributing to this weight has to be at least $\left\lceil \frac{3p+1}{8} \right\rceil$.

Moreover, the largest value among the weights of vertices of degree 2, 4 and p is at least $3p+2$ and this weight is the sum of at most $2p$ integers. Hence the largest label contributing to this weight must be at least $\left\lceil \frac{3p+2}{2p} \right\rceil$.

Consequently, the largest label of a vertex or an edge of $W(2,p)$ is at least $\max\left\{ \left\lceil \frac{p+1}{2} \right\rceil, \left\lceil \frac{3p+1}{8} \right\rceil, \left\lceil \frac{3p+2}{2p} \right\rceil \right\} = \left\lceil \frac{p+1}{2} \right\rceil$ for $p \geq 3$. Thus

$$\mathrm{tdis}(W(2,p)) = d \geq \left\lceil \frac{p+1}{2} \right\rceil. \tag{3.1}$$

Therefore, to demonstrate the converse inequality, it suffices to establish the existence of $\mathrm{tdis}(W(2,p))$ with $d = \left\lceil \frac{p+1}{2} \right\rceil$. For $p = 3, 4, 5, 6, 7$ and 8 the suitable distance vertex irregular total $\left\lceil \frac{p+1}{2} \right\rceil$-labelings for $W(2,p)$ are illustrated in Figure 5. For $p \geq 9$, we

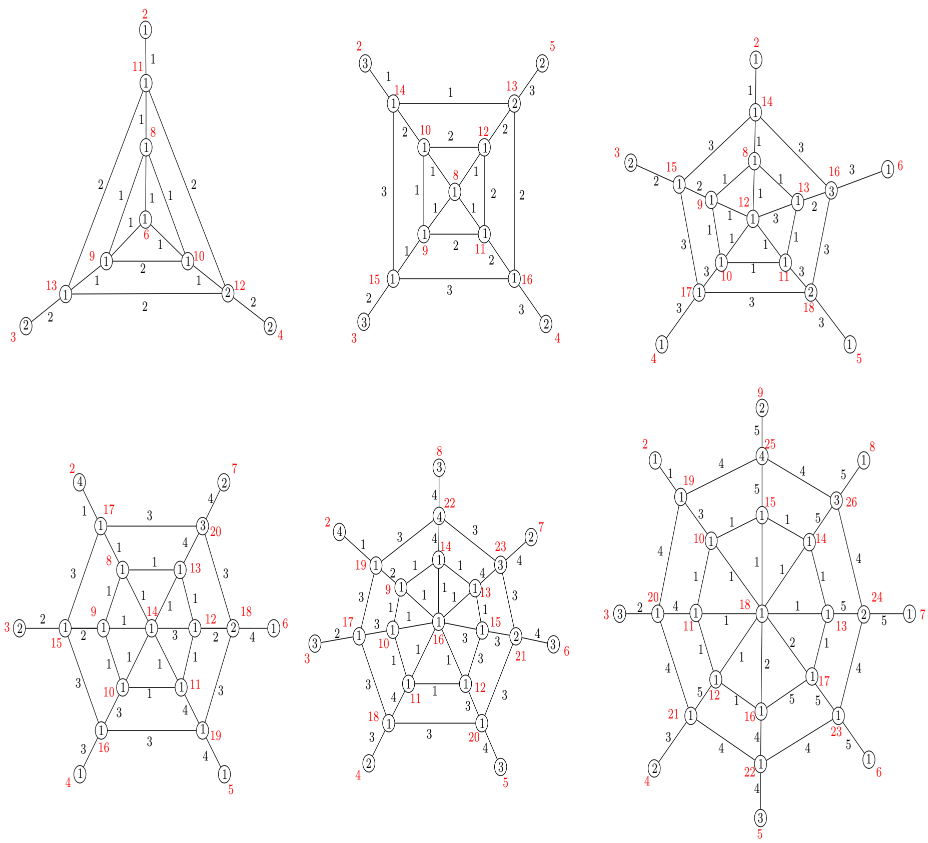

Figure 5. **The DVITL of $W(2,p)$ for $p = 3, 4, 5, 6, 7, 8$.**

construct the total labeling, namely $\sigma : V(W(2,p)) \cup E(W(2,p)) \to \left\{1, 2, \dots, \left\lceil \frac{p+1}{2} \right\rceil \right\}$, hence the labels for the vertices are provided as follows

vertex label			
$\sigma(u_t)$	for $t \in [1, p]$		
	t	if $t \in [1, d-2]$	
	$d-3$	if $t = d-1$	
	$d-2$	if $t \in [d, p-2]$	
	$\lceil \frac{d-1}{2} \rceil - 2$	if $t = p-1$	p is even
	$\lceil \frac{d}{2} \rceil - 2$		p is odd
	1	if $t = p$	
$\sigma(y_t)$	$d-t$	if $t \in [1, d-2]$	
	1	if $t \in [d-1, p-1]$	
	d	if $t = p$	
$\sigma(v_t)$	1		
$\sigma(v_0)$	d		

meanwhile, the labels of edges are labeled in the following manner.
Case 1. For p is even.

edge label		
$\sigma(u_t y_t)$	for $t \in [1, p]$	
	$d-1$	if $t \in [1, d-2]$
	$2d-t-2$	if $t \in [d-1, p-1]$
	$d-1$	if $t = p$
$\sigma(y_t y_{t+1})$	1	
$\sigma(v_t v_{t+1})$	$d-1$	
$\sigma(y_t v_t)$	1	if $t = p-1$
	$d - 2\lfloor \frac{d-1}{2} \rfloor + 2$	otherwise
$\sigma(v_t v_0)$	$d-1$	if $t \in [1, d-2]$
	$2d-t-2$	if $t \in [d-1, p-2]$
	$d - 2\lfloor \frac{d-1}{2} \rfloor + 2$	if $t = p-1$
	$d-1$	if $t = p$

Case 2. For p is odd.

edge label		
$\sigma(u_t y_t)$	for $t \in [1, p]$	
	d	if $t \in [1, d-2]$
	$2d-t-1$	if $t \in [d-1, p-1]$
	d	if $t = p$
$\sigma(y_t y_{t+1})$	1	
$\sigma(v_t v_{t+1})$	d	
$\sigma(y_t v_t)$	1	$t = p-1$
	$d - 2\lfloor \frac{d}{2} \rfloor + 3$	otherwise
$\sigma(v_t v_0)$	$d-1$	if $t \in [1, d-2]$
	$2d-t-2$	if $t \in [d-1, p-2]$
	$d - 2\lfloor \frac{d}{2} \rfloor + 2$	if $t = p-1$
	$d-1$	if $t = p$

Furthermore, the Table. 7 shows that the weights of vertices under the labeling σ are distinct.

Table 7. **An overview of the vertex weights of $W(2,p)$.**

for $t \in [1,p]$	p is even	p is odd	
$wt_\sigma(u_t)$	$2d-t-1$	$2d-t$	if $t \in [1,p-1]$
	$2d-1$	$2d$	if $t = p$
$wt_\sigma(y_t)$	$4d-2\lfloor\frac{d-1}{2}\rfloor-t+4$	$4d-2\lfloor\frac{d}{2}\rfloor-t+6$	if $t \in [1,d-2]$
	$4d-2\lfloor\frac{d-1}{2}\rfloor-t+3$	$4d-2\lfloor\frac{d}{2}\rfloor-t+5$	if $t \in [d-1,p-2]$
	$d+\lceil\frac{d-1}{2}\rceil+4$	$d+\lceil\frac{d}{2}\rceil+4$	if $t = p-1$
	$3d-2\lfloor\frac{d-1}{2}\rfloor+5$	$3d-2\lfloor\frac{d}{2}\rfloor+7$	if $t = p$
$wt_\sigma(v_t)$	$6d-2\lfloor\frac{d-1}{2}\rfloor-t+1$	$6d-2\lfloor\frac{d}{2}\rfloor-t+4$	if $t \in [1,p-2]$
	$4d-2\lfloor\frac{d-1}{2}\rfloor+4$	$4d-2\lfloor\frac{d}{2}\rfloor+6$	if $t = p-1$
	$6d-2\lfloor\frac{d-1}{2}\rfloor+1$	$6d-2\lfloor\frac{d}{2}\rfloor+4$	if $t = p$
$wt_\sigma(v_0)$	$\frac{3d^2+d+2}{2}-2\lfloor\frac{d-1}{2}\rfloor-1$	$\frac{3d^2+d}{2}-2\lfloor\frac{d}{2}\rfloor+2$	

From previous two cases, the vertex weights $\{wt_\sigma(u_t), wt_\sigma(y_t), wt_\sigma(v_t) : t \in [1,p]\}$ of $W(2,p)$ vary from 2 up to $3p+1$, additionally the vertex v_0 has a weight greater than $3p+1$. Then really $\text{tdis}(W(2,p)) = \lceil\frac{p+1}{2}\rceil$, for $p \geq 3$. □

Example 2. Figure 6 illustrates a distance vertex irregular total 7-labeling of $W(2,12)$.

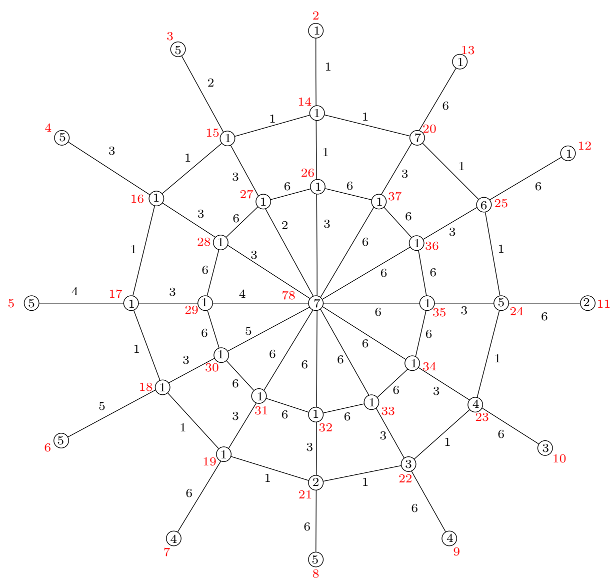

Figure 6. **A DVITL of $W(2,12)$ with $d=7$.**

4. DVITL of the web graphs $W_0(2,p)$

The web graph $W_0(2,p)$ is constructed from the graph $W(2,p)$ by removing the center vertex, see Figure 7. The web graphs without center were investigated in regard to magic labeling by Hegde and Shetty in [6]. In this section, we investigate them for the study of DVITL. The vertex set and the edge set of $W_0(2,p)$ will be regarded as follows: $V(W_0(2,p)) = \{u_t, v_t, y_t : t \in [1,p]\}$ and $E(W_0(2,p)) = \{y_t y_{t+1}, v_t v_{t+1}, u_t y_t, y_t v_t : t \in [1,p]\}$. Note that, every index is considered modulo p.

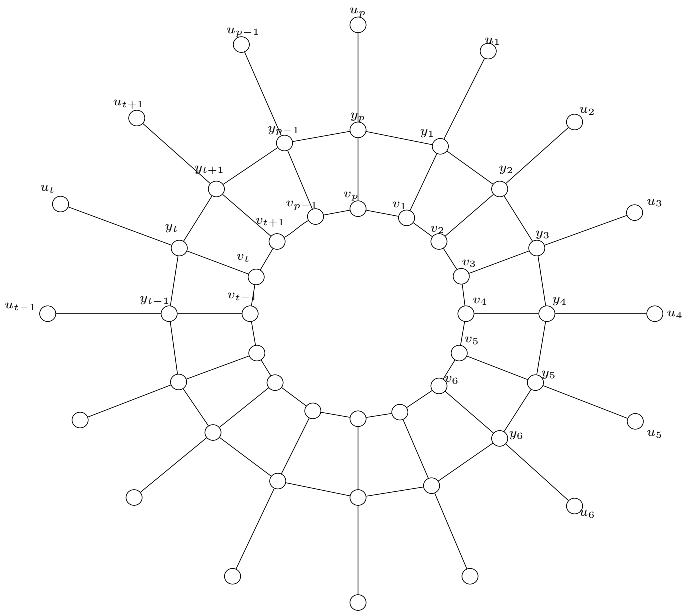

Figure 7. **The web graph** $W_0(2,p)$.

Theorem 3. *Let $W_0(2,p)$ be a web graph of order $3p$, with $p \geq 3$. Then*

$$\text{tdis}(W_0(2,p)) = \left\lceil \frac{p+1}{2} \right\rceil.$$

Proof. Let $W_0(2,p)$ be a web graph without center, with $p \geq 3$ vertices of degree 1, p vertices of degree 3 and p vertices of degree 4. From Lemma 1 it follows that $\text{tdis}(W_0(2,p)) \geq \left\lceil \frac{3p+1}{8} \right\rceil$. On the other hand, if we have a distance vertex irregular total labeling of $W_0(2,p)$ and consider the weights of the vertices then the smallest weight among all vertices of $W_0(2,p)$ is at least 2 and the largest weight of a vertex of degree 2 is at least $p+1$. The weight of any vertex of degree 2 is the sum of two positive integers therefore at least one label is at least $\left\lceil \frac{p+1}{2} \right\rceil$.

The largest value among the weights of vertices of degree 2 and 3 is at least $2p+1$ and this weight is the sum of at most six integers. Clearly the largest label contributing to this weight has to be at least $\lceil \frac{2p+1}{6} \rceil$.

Moreover, the largest value among the weights of vertices of degree 2, 3 and 4 is at least $3p+1$ and this weight is the sum of at most eight integers. Obviously the largest label contributing to this weight must be at least $\lceil \frac{3p+1}{8} \rceil$.

Thereafter the largest label of a vertex or an edge of $W_0(2,p)$ is at least $\max\left\{\lceil \frac{p+1}{2} \rceil, \lceil \frac{2p+1}{6} \rceil, \lceil \frac{3p+1}{8} \rceil\right\} = \lceil \frac{p+1}{2} \rceil$ for $p \geq 3$. Hence for $p \geq 3$ we get

$$\operatorname{tdis}(W_0(2,p)) = d \geq \left\lceil \frac{p+1}{2} \right\rceil. \tag{4.1}$$

Furthermore, we validate the equality by confirming the existence of DVITL of $W_0(2,p)$, where $d = \lceil \frac{p+1}{2} \rceil$. Figure 8 illustrates appropriate distance vertex irregular total $\lceil \frac{p+1}{2} \rceil$-labelings for $W_0(2,p)$, $p = 3, 4$ and 5.

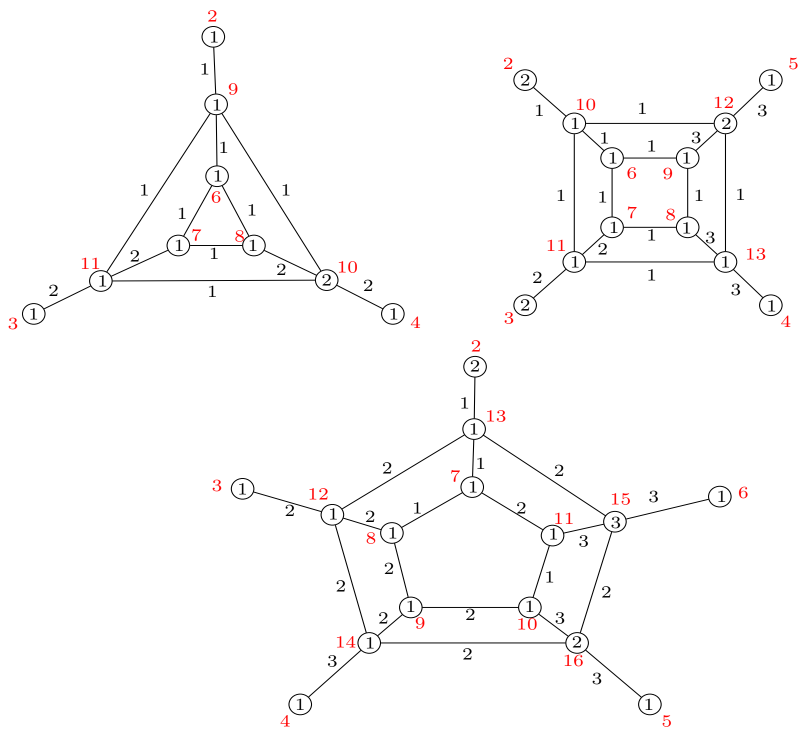

Figure 8. **The DVITL of $W_0(2,p)$ for $p = 3, 4, 5$.**

To accomplish this for $p \geq 6$, we establish the total labeling denoted as $\sigma : V(W_0(2,p)) \cup E(W_0(2,p)) \to \{1, 2, \ldots, \lceil \frac{p+1}{2} \rceil\}$ in the following way.

Case 1. For $p \equiv 0 \pmod 4$.
The labels used for the vertices are presented as follows.

vertex label		
	for $t \in [1, p]$	
$\sigma(u_t)$	$t+1$	if $t \in [1, d-3]$
	$d-2$	if $t \in [d-2, p-2]$
	$\frac{d-1}{2}$	if $t = p-1$
	1	if $t = p$
$\sigma(y_t)$	$d-t-1$	if $t \in [1, d-2]$
	1	if $t \in [d-1, p-1]$
	$d-1$	if $t = p$
$\sigma(v_t)$	$d-2$	

and the edge labels are represented as follows.

edge label			
	for $t \in [1, p]$		
$\sigma(u_t y_t)$	d	if $t \in [1, d-2]$	
	$2d-t-2$	if $t \in [d-1, p-1]$	
	d	if $t = p$	
$\sigma(y_t v_t)$	d	if $t \in [1, d-2]$	
	$2d-t-2$	if $t \in [d-1, p-1]$	
	d	if $t = p$	
$\sigma(v_t v_{t+1})$	1		
$\sigma(y_t y_{t+1})$	$\frac{d-1}{2}$	if $t \in [1, d-2]$	
	$\frac{t}{2}+1$	if $t \in [d-1, p-2]$	and $t \equiv (d-1) \pmod 2$
	$\frac{t+3}{2}-1$		and $t \equiv d \pmod 2$
	$\frac{d-1}{2}$	if $t = p-1, p$	

Moreover, the Table 8. shows that the weights of vertices under the labeling σ are distinct.

Table 8. **An overview of the vertex weights of $W_0(2, p)$**

for $t \in [1, p]$		
$wt_\sigma(u_t)$	$2d-t-1$	if $t \in [1, p-1]$
	$2d-1$	if $t = p$
$wt_\sigma(y_t)$	$6d-t-4$	if $t \in [1, d-2]$
	$6d-t-5$	if $t \in [d-1, p-1]$
	$5d-3$	if $t = p$
$wt_\sigma(v_t)$	$4d-t-3$	if $t \in [1, p-2]$
	$2d$	if $t = p-1$
	$4d-3$	if $t = p$

Case 2. For $p \equiv 1 \pmod 4$

The labels for the vertices are provided as follows

vertex label		
	for $t \in [1, p]$	
$\sigma(u_t)$	t	if $t \in [1, d-2]$
	$d-2$	if $t \in [d-1, p-2]$
	$\frac{d-1}{2} - 1$	if $t = p-1$
	1	if $t = p$
$\sigma(y_t)$	$d-t$	if $t \in [1, d-2]$
	1	if $t \in [d-1, p-1]$
	d	if $t = p$
$\sigma(v_t)$	$d-2$	

and the label for edges are

edge label			
	for $t \in [1, p]$		
$\sigma(u_t y_t)$	d	if $t \in [1, d-2]$	
	$2d-t-1$	if $t \in [d-1, p-1]$	
	d	if $t = p$	
$\sigma(y_t v_t)$	d	if $t \in [1, d-2]$	
	$2d-t-1$	if $t \in [d-1, p-1]$	
	$d-1$	if $t = p$	
$\sigma(v_t v_{t+1})$	1	if $t \in [1, p-1]$	and t is odd
	2		and t is even
	2	if $t = p$	
$\sigma(y_t y_{t+1})$	$\frac{d+1}{2}$	if $t \in [1, d-1]$	
	$\frac{t-3}{2} + 3$	if $t \in [d, p-2]$	and $t \equiv d \pmod 2$
	$\frac{t}{2} + 1$		and $t \equiv (d+1) \pmod 2$
	$\frac{d+1}{2}$	if $t = p-1, p$	

Under the labeling σ, the optimal weights of the vertices are depicted in Table 9.

Table 9. **An overview of the vertex weights of $W_0(2, p)$.**

for $t \in [1, p]$		
$wt_\sigma(u_t)$	$2d-t$	if $t \in [1, p-1]$
	$2d$	if $t = p$
$wt_\sigma(y_t)$	$6d-t-1$	if $t \in [1, d-1]$
	$6d-t-2$	if $t \in [d, p-1]$
	$5d-1$	if $t = p$
$wt_\sigma(v_t)$	$4d-t-1$	if $t \in [1, p-1]$
	$4d-1$	if $t = p$

Case 3. For $p \equiv 2 \pmod 4$
The labels for the vertices and edges are provided as follows

vertex label		
$\sigma(u_t)$	for $t \in [1,p]$	
	$t+2$	if $t \in [1, d-3]$
	$d-1$	if $t \in [d-2, p-2]$
	$\frac{d}{2}+1$	if $t = p-1$
	2	if $t = p$
$\sigma(y_t)$	$d-t-1$	if $t \in [1, d-2]$
	1	if $t \in [d-1, p-1]$
	$d-1$	if $t = p$
$\sigma(v_t)$	$d-2$	

edge label			
$\sigma(u_t y_t)$	for $t \in [1,p]$		
	d	if $t \in [1, d-2]$	
	$2d-t-2$	if $t \in [d-1, p-1]$	
	d	if $t = p$	
$\sigma(y_t v_t)$	d	if $t \in [1, d-2]$	
	$2d-t-2$	if $t \in [d-1, p-1]$	
	d	if $t = p$	
$\sigma(v_t v_{t+1})$	1		
$\sigma(y_t y_{t+1})$	$\frac{d}{2}-1$	if $t \in [1, d-2]$	
	$\frac{t+1}{2}$	if $t \in [d-1, p-2]$	and $t \equiv (d-1) \pmod 2$
	$\frac{t}{2}$		and $t \equiv d \pmod 2$
	$\frac{d}{2}-1$	if $t = p-1, p$	

According to the concept of the labeling σ, the optimal vertex weights are illustrated in Table 10.

Table 10. **An overview of the vertex weights of $W_0(2,p)$.**

for $t \in [1,p]$		
$wt_\sigma(u_t)$	$2d-t-1$	if $t \in [1, p-1]$
	$2d-1$	if $t = p$
$wt_\sigma(y_t)$	$6d-t-4$	if $t \in [1, d-2]$
	$6d-t-5$	if $t \in [d-1, p-1]$
	$5d-3$	if $t = p$
$wt_\sigma(v_t)$	$4d-t-3$	if $t \in [1, p-1]$
	$4d-3$	if $t = p$

Case 4. For $p \equiv 3 \pmod 4$

The labels for the vertices and edges are provided as follows

vertex label		
$\sigma(u_t)$	for $t \in [1,p]$	
	$t+1$	if $t \in [1, d-2]$
	$d-1$	if $t \in [d-1, p-2]$
	$\frac{d}{2}$	if $t = p-1$
	2	if $t = p$
$\sigma(y_t)$	$d-t$	if $t \in [1, d-2]$
	1	if $t \in [d-1, p-1]$
	d	if $t = p$
$\sigma(v_t)$	$d-2$	

edge label			
$\sigma(u_t y_t)$	for $t \in [1,p]$		
	d	if $t \in [1, d-2]$	
	$2d-t-1$	if $t \in [d-1, p-1]$	
	d	if $t = p$	
$\sigma(y_t v_t)$	d	if $t \in [1, d-2]$	
	$2d-t-1$	if $t \in [d-1, p-1]$	
	$d-1$	if $t = p$	
$\sigma(v_t v_{t+1})$	1	if $t \in [1, p-1]$	and t is odd
	2		and t is even
	2	if $t = p$	
$\sigma(y_t y_{t+1})$	$\frac{d}{2}$	if $t \in [1, d-1]$	
	$\frac{t}{2}+1$	if $t \in [d, p-2]$	and $t \equiv d \pmod 2$
	$\frac{t-1}{2}+1$		and $t \equiv (d+1) \pmod 2$
	$\frac{d}{2}$	if $t = p-1, p$	

The ideal weights of the vertices according to the labeling σ are specified in Table 11 as follows:

Table 11. **An overview of the vertex weights of $W_0(2,p)$.**

for $t \in [1,p]$		
$wt_\sigma(u_t)$	$2d-t$	if $t \in [1, p-1]$
	$2d$	if $t = p$
$wt_\sigma(y_t)$	$6d-t-1$	if $t \in [1, d-1]$
	$6d-t-2$	if $t \in [d, p-1]$
	$5d-1$	if $t = p$
$wt_\sigma(v_t)$	$4d-t-1$	if $t \in [1, p-1]$
	$4d-1$	if $t = p$

In all cases, for $t \in [1,p]$, it can be verified that the vertex weights $\{wt_\sigma(u_t), wt_\sigma(y_t), wt_\sigma(v_t)\} = \{2, 5, ..., 3p+1\}$. Additionally, the weights of vertices of $W_0(2,p)$ under the labeling σ are pairwise distinct. Hence, $\text{tdis}(W_0(2,p)) = \lceil \frac{p+1}{2} \rceil$. □

Example 3. Figure 9 displays a distance vertex irregular total 7-labeling for $W_0(2, 13)$.

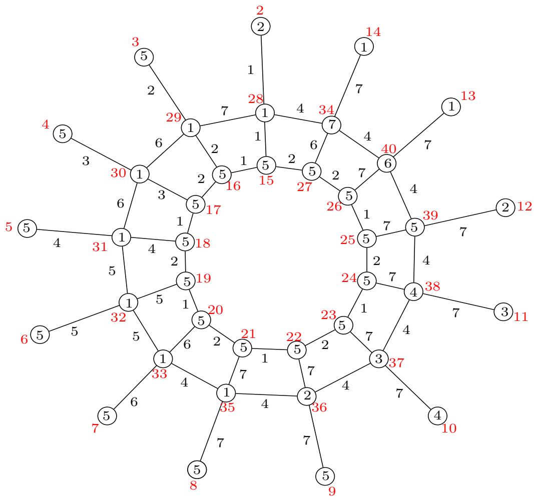

Figure 9. **A DVITL of $W_0(2, 13)$ with $d = 7$.**

5. Conclusion

This study aims to analyze the distance vertex irregular total d-labeling (DVITL), ascertain its lower bound, and find the exact quantity of the total distance vertex irregularity strength for three specific families of graphs: the prisms D_p, web graphs $W(2, p)$, and web graphs $W_0(2, p)$. We obtain that $\text{tdis}(D_3) = 3$ and $\text{tdis}(D_p) = \lceil \frac{2p+5}{6} \rceil$, for $p \geq 4$. Also, we get that $\text{tdis}(W(2, p)) = \text{tdis}(W_0(2, p)) = \lceil \frac{p+1}{2} \rceil$.

These theoretical results on the total distance vertex irregularity strength obtained for the investigated families of graphs can have meaningful implications in real-world graph models, communication networks or information flows of algorithmic processes.

If the reference model of a technical (working) system of objects is represented by a corresponding graph model with regular or almost regular neighborhoods of vertices, then its vertex irregularity means that the distance vertex irregular total d-labelings of the graph model ensure the resolution of vertices (each vertex has a distinct overall transmission load, a distinct length of technological operation or a different energy consumption, etc.) at certain edge capacities (material flows, data flows or energy flows, etc.). Thus, the distance vertex irregular total d-labelings should help in managing and controlling the real-world graph models.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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