

Not every graph can be reconstructed from its boundary distance matrix

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Abstract: A vertex v of a connected graph G is said to be a boundary vertex of G if for some other vertex u of G , no neighbor of v is further away from u than v . The boundary $\partial(G)$ of G is the set of all of its boundary vertices. The boundary distance matrix \hat{D}_G of a graph $G = ([n], E)$ is the square matrix of order κ , being κ the order of $\partial(G)$, such that for every $i, j \in \partial(G)$, $[\hat{D}_G]_{ij} = d_G(i, j)$. In a recent paper [doi.org/10.7151/dmgt.2567], it was shown that if a graph G is either a block graph or a unicyclic graph, then G is uniquely determined by the boundary distance matrix \hat{D}_G of G , and it was also conjectured that this statement holds for every connected graph G , whenever both the order n and the $|\partial(G)|$ (and thus also the boundary distance matrix) of G are prefixed (i.e., the set of boundary vertices is known as a subset of V , but not their identities). After proving that this conjecture is true for several graph families, such as being of diameter 2, having order at most $n = 6$ or being Ptolemaic, we show that this statement does not hold when considering, for example, either the family of split graphs of diameter 3 and order at least $n = 10$ or the family of distance-hereditary graphs of order at least $n = 8$.

Keywords: boundary, distance matrix, ptolemaic graphs, interval graphs.

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1. Introduction

Consider a connected, simple, and undirected graph G . The boundary of G , denoted by $\partial(G)$, is defined as the subset of vertices whose eccentricity is maximal among

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their neighbors. This concept was first introduced by Chartrand et al. in their foundational work [7], and since then, it has found numerous applications in graph theory, particularly in the study of convexity and metric localization in graphs.

In a recent study [6], the authors proposed a conjecture stating that the distance matrix of the boundary vertices, combined with the total number of vertices in the graph, is sufficient to reconstruct the entire graph. This is not the first time that a similar question is posed by far. Various approaches include reconstructing metric graphs from density functions [9], road networks from a set of trajectories [1], graphs utilizing shortest paths or distance oracles [13], labeled graphs from all r -neighborhoods [17], or reconstructing phylogenetic trees [3] has been proposed previously. Of particular note is the graph reconstruction conjecture [14, 22] which states the possibility of reconstructing any graph on at least three vertices (up to isomorphism) from the multiset of all unlabeled subgraphs obtained through the removal of a single vertex.

Going back to the conjecture proposed in [6], it holds true for trees, since the boundary coincides with the set of leaves. Additionally, the same work establishes the validity of the conjecture for block graphs and unicyclic graphs.

However, in this paper, we demonstrate that, unfortunately, the conjecture does not hold for general graphs. Despite this, the question of determining for which graph classes the conjecture remains valid is of significant theoretical and computational interest. We prove that for two well-known families –Ptolemaic graphs and 2-diameter graphs– the distance matrix of the boundary vertices, along with the vertex count, indeed provides enough information to reconstruct the original graph. On the other hand, we show that this property fails for bipartite graphs, distance-hereditary graphs and split graphs of diameter 3. The case of interval graphs presents a particularly intriguing challenge: although we are unable to definitively conclude whether the conjecture holds for this class, we provide a structural characterization of the boundary vertices, which may serve as a stepping stone for future investigations.

All the graphs considered are undirected, simple, finite, and (unless otherwise stated) connected. If $G = (V, E)$ is a graph of order n and size m , it means that $|V| = n$ and $|E| = m$. Unless otherwise specified, $V = [n] = \{1, \dots, n\}$.

Let v be a vertex of a graph G . The *open neighborhood* of v is $N(v) = \{w \in V(G) : vw \in E\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of v is $\deg(v) = |N(v)|$. The minimum degree (resp., maximum degree) of G is $\delta(G) = \min\{\deg(u) : u \in V(G)\}$ (resp., $\Delta(G) = \max\{\deg(u) : u \in V(G)\}$). If $\deg(v) = 1$, then v is said to be a *leaf*. Given $u, v \in V(G)$, they are called either *true twin* vertices or *false twin* vertices if either $N[u] = N[v]$ or $N(u) = N(v)$.

Let $W \subseteq V(G)$ be a subset of vertices of G . The *closed neighborhood* of W is $N[W] = \cup_{v \in W} N[v]$. The subgraph of G induced by W , denoted by $G[W]$, has W as vertex set and $E(G[W]) = \{vw \in E(G) : v \in W, w \in W\}$. If $G[W]$ is a complete graph, then it is said to be a *clique* of G .

An abbreviated way to express the set $\{1, \dots, n\}$ will be $[n]$. This will be useful when defining subsets of vertices in a graph.

Given a pair of vertices u, v of a graph G , a $u - v$ *geodesic* is a $u - v$ shortest path, i.e., a path joining u and v of minimum order. Clearly, all $u - v$ geodesics have the same length, and it is called the *distance* between vertices u and v in G , denoted by $d_G(u, v)$, or simply by $d(u, v)$, when the context is clear. The *eccentricity* $\text{ecc}(v)$ of a vertex v is the distance to a farthest vertex from v . The *radius* and *diameter* of G are respectively, the minimum and maximum eccentricity of its vertices and are denoted as $\text{rad}(G)$ and $\text{diam}(G)$. A vertex $u \in V(G)$ is a *central* vertex of G if $\text{ecc}(u) = \text{rad}(G)$, and it is called a *peripheral* vertex of G if $\text{ecc}(u) = \text{diam}(G)$. The *distance matrix* D_G of a graph $G = (V, E)$ with n vertices is the square matrix of order n such that, for every $i, j \in [n]$, $[D_G]_{ij} = d(i, j)$.

Let S be a subset of vertices of order k of a graph $G = ([n], E)$. It is denoted by $D_{S,V}$ the submatrix of D_G of order $k \times n$ such that for every $i \in S$ and for every $j \in V$, $[D_{S,V}]_{ij} = d(i, j)$. Similarly, the so-called *S-distance matrix* of G , denoted by D_S , is the square submatrix of D_G of order k such that for every $i, j \in S$, $[D_S]_{ij} = d(i, j)$.

For additional details and information on basic graph theory, we refer the reader to [8]. The rest of this paper is organized as follows: in the next section it is established the conjecture with the definitions and facts that are related to it. Two subsections are devoted to Ptolemaic graphs and interval graphs. In Section 3, we provide various examples of graphs for which the conjecture is not true, providing the limits of the conjecture.

2. BDM graphs

A vertex v of a graph G is said to be a *boundary vertex* of a vertex u if no neighbor of v is further away from u than v , i.e., if for every vertex $w \in N(v)$, $d(u, w) \leq d(u, v)$. The set of boundary vertices of a vertex u is denoted by $\partial(u)$. Given a pair of vertices $u, v \in V(G)$ if $v \in \partial(u)$, then v is also said to be *maximally distant* from u . The *boundary* of G , denoted by $\partial(G)$, is the set of all of its boundary vertices, i.e., $\partial(G) = \cup_{u \in V(G)} \partial(u)$ [7]. The *boundary number* of G , denoted by $\kappa(G)$, is the cardinality of all of its boundary, i.e., $\kappa(G) = |\partial(G)|$. If $S = \partial(G)$, then the *S-distance matrix* D_S is also denoted by \hat{D}_G , being called the *boundary distance matrix* of G . The boundary of a graph exhibits a number of interesting properties, like being geodesic [4], a resolving set [12] and also a strong resolving set.

Theorem 1 ([19]). *The boundary $\partial(G)$ of every graph G is a strong resolving set of G .*

The concept of strong resolving set was first defined by Sebő and Tannier [20] in 2003, and later studied in [18]. They were interested in extending isometric embeddings of subgraphs into the whole graph and, to ensure that, they defined a *strong resolving set* of a graph G as a subset $S \subseteq V(G)$ such that for any pair $x, y \in V(G)$ there is an element $v \in S$ such that there exists either a $x - v$ geodesic that contains y or a $y - v$

geodesic containing x . As a consequence of the definition, it was implicitly mentioned in some papers [15, 19, 20] and proved in [5] that it only suffices to know the distances from the vertices of a strong resolving set to the rest of nodes, to uniquely determine the graph.

Theorem 2 ([5]). *Let S be a proper subset of vertices of a graph $G = (V, E)$. Then, S is a strong resolving set if and only if G is uniquely determined by the distance matrix $D_{S,V}$.*

Hence, as a direct consequence of these facts, the following result holds.

Corollary 1. *Let $G = (V, E)$ be a graph. Let $\partial(G)$ the set of boundary vertices of G . Then, G is uniquely determined by the distance matrix $D_{\partial(G),V}$ (distances from boundary to all vertices).*

Starting from this property, it was conjectured in [6] that every graph G of order n and boundary number κ could also be uniquely determined by the boundary distance matrix \hat{D}_G , provided that both the order n and the boundary $\partial(G)$ be prefixed. In the same paper, it was proved this claim to be true for trees, for unicyclic graphs and also for block graphs. In the next section, it is shown that there are graphs not satisfying this property. For this reason, we introduce the following definition.

Definition 1. A graph G of order n and boundary number κ is called **boundary-distance-matrix constructible**, also **BDM-constructible**, or simply a **BDM** graph, if it is uniquely determined by the distance matrix \hat{D}_G of the boundary $\partial(G)$ of G , provided that both n and $\partial(G)$ be prefixed.

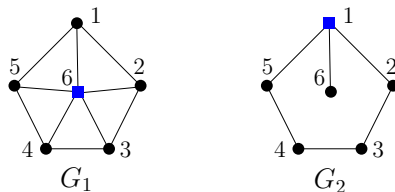


Figure 1. A pair of graphs of order $n = 6$, boundary number $\kappa = 5$, with the same [5]-distance matrix.

At this point, it is important to notice that in the previous definition, it is not enough to prefix both the order n and the boundary number κ . For example, for both of the graphs, G_1 and G_2 , displayed in Figure 1, $n = 6$ and $\kappa = 5$, they have the same [5]-distance matrix $D_{[5]}$, but $\partial(G_1) = [5]$ and $\partial(G_2) = \{2, 3, 4, 5, 6\}$. Hence, having the same κ -distance matrix D_κ does not fix which κ vertices constitute $\partial(G)$.

Next, we show that all graphs of diameter 2 are BDM graphs.

Lemma 1. *Let G be a graph of order n and boundary number κ .*

1. *If $v \in V(G)$ and $\text{ecc}(v) = \text{diam}(G)$, then $v \in \partial(G)$.*
2. *If $\text{rad}(G) = \text{diam}(G)$, then $\kappa = n$.*
3. *If $\text{diam}(G) = 2$, then $n - 1 \leq \kappa \leq n$. Moreover, $\kappa = n - 1$ if and only if G contains a unique central vertex.*

Proof. 1. If $w \in V(G)$ is an antipodal vertex of v , i.e., if $d(w, v) = \text{diam}(G)$, then $v \in \partial(w)$.

2. If every vertex of G is peripheral, then according to the previous item, $\partial(G) = V(G)$.

3. If every vertex of G is peripheral, then according to the previous item, $\kappa = n$. Let v a central vertex of G . Then every vertex of G other than v is peripheral, so $\partial(G) = V(G) - v$. Let v, w a pair of distinct central vertices of G . Then, $v \in \partial(w)$ and $w \in \partial(v)$. Hence, in this case, $\partial(G) = V(G)$. □

Theorem 3. *If $\text{diam}(G) = 2$, then G is a BDM graph.*

Proof. According to Lemma 1, $\kappa(G) = \kappa = |\partial(G)| = n - 1$ if and only if G contains a unique central vertex. Assume, w.l.o.g., that $\partial(G) = [n - 1]$. Let G' a graph of order n such that $\kappa(G') = [n - 1]$ and $\hat{D}_{G'} = \hat{D}_G$. In G and G' are not isomorphic, then the last column (resp., row) of the distance matrix $D_{G'}$ must be different from the last column (resp., row) of the distance matrix D_G of G . In other words, $N_G(n) = [n - 1]$ and for some vertex $k \in [n - 1]$, $kn \notin E(G')$. Thus, $d_{G'}(k, n) \geq 2$, which means that $n \in \partial(k)$, a contradiction. □

The previous result is far from being true for every graph $G = ([n], E)$ of diameter $\text{diam}(G) = 3$, even in the case $\kappa(G) = n - 1$. For example, if $G = ([n], E)$ denotes any of the graphs displayed in Figures 6,7,8 and 10, then $\text{diam}(G) = 3$, $\kappa(G) = n - 1$, and it is not a BDM graph.

As for the case $\kappa(G) = n - 1$, we have obtained a sufficient condition for a graph of this type to be a BDM graph, which is shown next.

Definition 2. Let $e = xy$ be an edge of a graph G . Consider the graph, denoted by $G - e$, obtained from G by deleting edge e . The edge e is called *x-irrelevant* if, for every vertex $z \in N(x)$, $d_{G-e}(y, z) \leq 2$. Otherwise, it is called *x-relevant*.

Theorem 4. *Let $G = ([n], E)$ be a graph such that $\partial(G) = [n - 1]$. If for every vertex $h \in N(n)$, the edge nh is n -relevant, then G is a BDM graph.*

Proof. Suppose that, for every vertex $h \in N(n)$, the edge $e_h = nh$ is n -relevant. W.l.o.g., we assume that $N(n) = [h]$ and $d_{G-e_h}(h, 1) \geq 3$. Thus, \hat{D}_{G-e_h} and \hat{D}_G are different, since $d_G(h, 1) \leq 2$ (see Figure 2 (a)).

Let $e_k = nk$, being $h + 1 \leq k \leq n - 1$. Consider the graph, denoted by $G + e_k$, obtained from G by adding edge e_k .

We distinguish two cases:

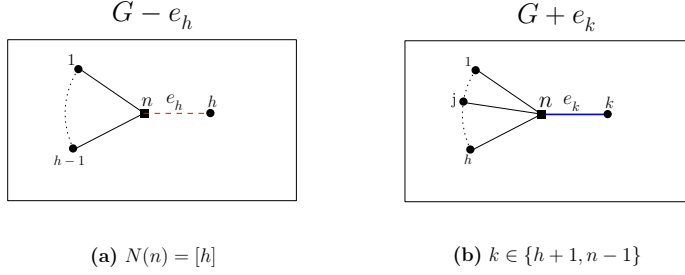


Figure 2. $G - e_h$ (resp., $G + e_k$) is the graph obtained from G by deleting (resp., adding) edge $e_h = nh$ (resp., $e_k = nk$).

1. Suppose that for every $i \in N(n) = [h]$, $d_G(k, i) \leq 2$. Then, vertex n is a boundary vertex of vertex k , since $d_G(k, n) \geq 2$, contradicting the fact that $\partial(G) = [n-1]$.
2. Assume that, for some vertex $j \in [h]$, $d_G(k, j) \geq 3$. Then, \hat{D}_G and the distance matrix of $\partial(G) = [n-1]$ in $G + e_k$ are different, since $d_{G+e_k}(k, j) = 2$ (see Figure 2 (b)).

□

Remark 1.

(1) Let $G = ([n], E)$ be a graph such that $\partial(G) = [n-1]$. If $e = nh$ is an n -irrelevant edge and we take the graph $G - e = ([n], E - e)$, then (i) vertex n is a boundary vertex of vertex h in $G - e$ and (ii) \hat{D}_G and the distance matrix of $[n-1]$ in $G - e$ are the same. Hence, $\partial(G - e) = [n]$.

(2) The converse of Theorem 4 is far from being true. For example, for each of the graphs displayed in Figure 3, $\partial(G) = [n-1]$, G is a BDM graph and every edge incident to vertex n is n -irrelevant.

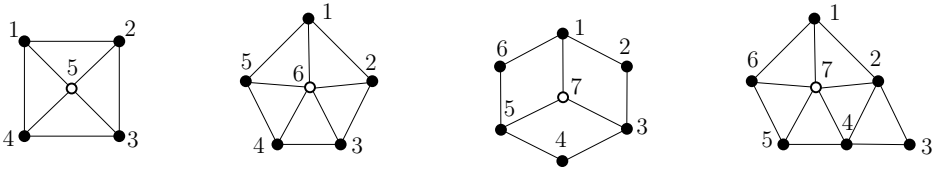


Figure 3. In all cases, the boundary is the set of black vertices, i.e., $\partial(G) = [n-1]$.

2.1. Ptolemaic graphs

A graph G is called *distance-hereditary* if every induced path is isometric. A *chordal graph* is a graph where there is no induced cycle of length at least four. A distance-hereditary graph is called *Ptolemaic* if it is also chordal.

Theorem 5 ([2]). *A graph G is Ptolemaic if it can be constructed from a single vertex by the following operations:*

- (1) A leaf is added to an existing vertex.
- (2) A vertex is duplicated to obtain two true twins.
- (3) A vertex whose neighborhood is a clique is duplicated to get a false twin.

We will use this construction to prove the following result.

Lemma 2. *The boundary $\partial(G)$ of a Ptolemaic graph G consists of all its vertices except the cut-vertices.*

Proof. We prove the above claim by induction in the sequence G_1, G_2, \dots, G_n that leads to the graph G according to the characterization shown in Theorem 5. Obviously, the claim is true for $G_1 = K_1$. Let us assume that it is also true for all G_i with $1 \leq i \leq k-1$ and we next prove it for G_k . For simplicity, $d_k(x, y)$ denotes the distance between vertices x and y in G_k , and $\partial_k(x)$ for the boundary of x in G_k .

Suppose first that G_k is obtained by adding a leaf v to an existing vertex $u \in V(G_{k-1})$. Let $w, w' \in V(G_{k-1})$ such that $w, w' \neq u$. Since the distances from w to the neighbors of w' are the same in G_{k-1} and in G_k we have that $w' \in \partial_{k-1}(w)$ if and only if $w' \in \partial_k(w)$. Let $w \in V(G_{k-1})$ such that $w \in \partial_{k-1}(u)$. Now, it is clear that $w \in \partial_k(v)$. Moreover, if $w \notin \partial_{k-1}(u)$ and since the distances from u to the neighbors of w have not changed, then $w \notin \partial_k(u)$ and also $w \notin \partial_k(v)$ due to the fact that $d_k(v, x) = d_k(u, x) + 1$ for any $x \in N[w]$. Finally, it is obvious that $v \in \partial_k(u)$. Hence, $\partial(G_k) = (\partial(G_{k-1}) \setminus \{u\}) \cup \{v\}$.

Suppose now, that G_k contains one more vertex v than G_{k-1} which is a true twin of an existing vertex u . Note that $u \in \partial_k(v)$ and $v \in \partial(u)$, and that if u is a cut-vertex in G_{k-1} , then it is not in G_k . The distances from u to any other vertex $w \in V(G_{k-1})$ have not changed. Moreover, the distances among the rest of vertices are the same in G_{k-1} as in G_k , and so for all $w, w' \in V(G_{k-1}) \setminus \{u\}$ we have that $w \in \partial(G_{k-1})$ if and only if $w \in \partial(G_k)$.

Finally, let us consider the case in which G_k is obtained by adding a false twin v to a vertex u such that $N[u]$ is a clique. Then clearly, u cannot be a cut-vertex in G_{k-1} and $u \in \partial(G_{k-1})$, therefore u is not a cut-vertex in G_k and $u \in \partial(G_k)$ as well as v . Here again, the distances among $w, w' \in V(G_{k-1}) \setminus \{u\}$ have not changed because u cannot be in a shortest path between two other vertices, and consequently $w \in \partial(G_{k-1})$ if and only if $w \in \partial(G_k)$. Hence, the proof is completed. \square

With this in mind, we are going to reconstruct a Ptolemaic graph G from $\hat{D}(G)$ in two phases: first, we build its *block-cutpoint tree* which is defined as the tree whose vertex set is the union of the set of blocks and the set of cutpoints of G , and in which two vertices are adjacent only if one corresponds to a block B of G , and the other to a cutpoint c of G and $c \in B$ in G (see [11]). The second step would be then reconstructing all the blocks of the graph and adding the cut-vertices. The whole process will take $O(|\partial(G)|) \subseteq O(n^2)$.

Theorem 6. *Every Ptolemaic graph is a BDM graph*

Proof. Let G be a Ptolemaic graph with n vertices and let \hat{D}_G be its boundary distance matrix. According to Lemma 2, that distance matrix contains the distance between all the vertices except the cut-vertices of the graph.

Algorithm 1 Reconstructing-Ptolemaic

Require: A matrix \hat{D}_G of a Ptolemaic graph G .

Ensure: The graph $G = (V, E)$.

- 1: Separate the vertices into maximal subsets of boundary vertices contained in the same block;
 - 2: Compute the distances between any two subsets S and S' . Since $S, S' \subseteq \partial(G)$, $d(S, S') = \min\{\hat{D}_G(u, v) : u \in S, v \in S'\}$;
 - 3: Construct a tree where any subset is a vertex and the distances between components are the same as in previous step; The resulted tree is the block-cutpoint tree of our graph and it is unique according to [21];
 - 4: Rebuild blocks of the graph G by considering the vertices of each subset S , and by adding the adjacencies shown in \hat{D}_G ;
 - 5: Add cut-vertices and adjacencies whenever the distance between boundary vertices is 2.
-

First, we are going to divide the vertices of $\partial(G)$ in equivalence classes S_1, \dots, S_p according to the following rules: $v \in S_i$ if and only if there exists $u \in S_i$ such that $\hat{D}_G(v, u) = 1$; if no such u exists, then v forms its own class of equivalence. By Lemma 2, this partition corresponds exactly to blocks with cut-vertices removed.

It only remains to construct the so called block-cutpoint tree of G . That tree is uniquely determined from the inter-class distances. The task can be done by apply the algorithm for trees given in [6] with a slight modification to consider non-leaves vertices. Once the tree is constructed, the vertices corresponding to each class S_i are substituted by the vertices in $G[S_i]$ and the graph is reconstructed. \square

2.2. Interval graphs

A graph is called an *interval graph* if each of its vertices can be associated with an interval on the real line in such a way that two vertices are adjacent if and only if the associated intervals have a non-empty intersection.

For any interval graph G , it is possible to give an enumeration C_1, C_2, \dots, C_p of the cliques of G such that if x is a vertex such that $x \in C_i \cap C_k$, then $x \in C_j$ for all cliques with $i < j < k$ (see [10]).

Prior to use this characterization, we prove the following result of general graphs:

Lemma 3. *Given a connected graph G , if a vertex x has a complete neighborhood, then $x \in \partial(G)$.*

Proof. Simply, pick a neighbor u of x . All the vertices of $N[x]$ are adjacent to u , in other words, $x \in \partial(u) \subseteq \partial(G)$. \square

Now the main result of this section, using the characterization of interval graphs. Recall that a 3-fan is $K_1 + P_3$.

Theorem 7. *Let G be an interval graph. Then, $u \in \partial(G)$ if and only if u is neither a cut-vertex nor the central vertex of an induced 3-fan.*

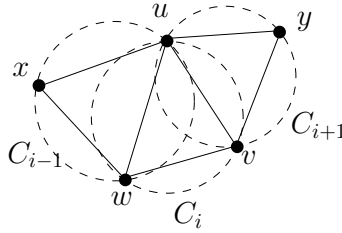


Figure 4. How a 3-fan K_1+P_3 is constructed in an interval graph. Here $K_1 = \{u\}$ and $P_3 = \langle x, w, v, y \rangle$.

Proof. First, let $u \notin \partial(G)$. By Lemma 3, its closed neighborhood cannot be complete, so it should belong to at least two cliques, C_i and C_{i+1} (see Figure 4). Assume u is not a cut-vertex, so there exists at least $v \in C_i \cap C_{i+1}$. If $N[u] \subseteq C_i \cup C_{i+1}$, then $u \in \partial(v) \subseteq \partial(G)$ contradicting our assumptions. Hence, there is a neighbor w of u belonging to C_{i-1} or C_{i+2} (suppose the first, without loss of generality) and C_i . Let $x \in C_{i-1} \setminus C_i$ different from w , and $y \in C_{i+1} \setminus C_i$. Then, $\langle u, v, w, x, y \rangle$ induces a 3-fan having u as central vertex, which leads to a contradiction.

Conversely, it is clear that if u is a cut-vertex or the central vertex in a 3-fan, it does not belong to the boundary of G . □

Unlikely other graph families, having a characterization of the boundary of an interval graph does not lead immediately to be BDM-constructible (as an instance of the contrary, see Subsection 2.1). As an example of the difficulties of finding an algorithm to reconstruct an interval graph, consider the graph G of Figure 5. It is an interval graph whose boundary consists of $\partial(G) = \{1, 2, 5, 6\}$ as one can expect from Lemma 7. However it is very difficult to detect the two 3-fans that are contained in the graph from the information given by \hat{D}_G .

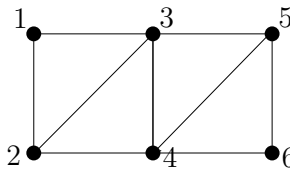


Figure 5. Recognizing a 3-fan in an interval graph is not easy.

Conjecture 1. Every interval graph is a BDM graph.

3. Some non-BDM graphs

Through a brute-force algorithm applied to the list of all connected graphs of order at most 10 (see [16]), we have obtained the following results where we used the canonical

set of boundary vertices computed from the definition.

Proposition 1. *A graph G on n vertices is a BDM graph if at least one of the following conditions holds.*

1. $n \leq 7$ and $G \notin \{G_1, G_2, G_3, G_4\}$ (See Figure 6).
2. G is bipartite, $n \leq 8$ and $G \notin \{G_5, G_6\}$ (See Figure 7).
3. G is distance-hereditary, $n \leq 8$ and $G \notin \{G_7, G_8\}$ (See Figure 8).
4. G is chordal and $n \leq 9$.
5. G is an interval graph and $n \leq 10$.

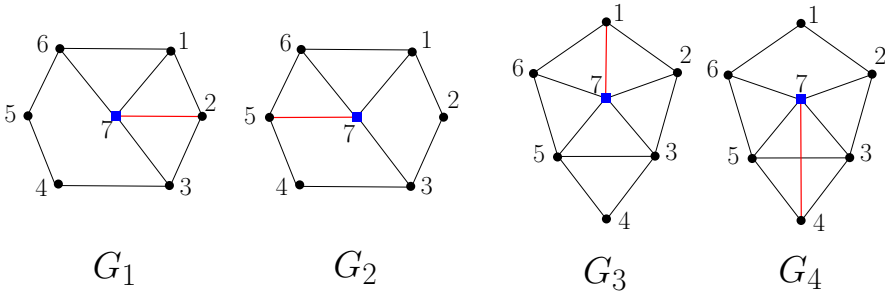


Figure 6. These are the unique pairs of graphs of order 7 having the same boundary distance matrix (see Table 1). In all cases, $\partial(G_i) = [6]$.

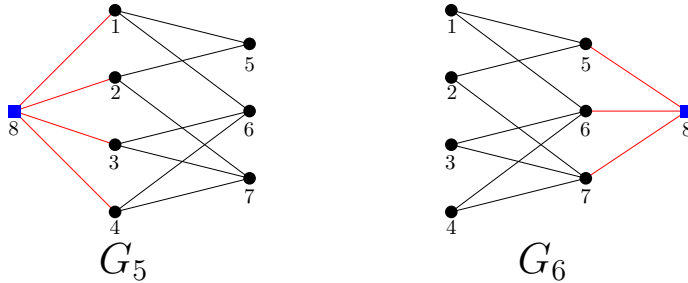


Figure 7. This is the unique pair of bipartite graphs of order 8 having the same boundary distance matrix (see Table 1). In both cases, $\partial(G_i) = [7]$.

A *split graph* $G = (V, E)$ of order n is a connected graph in which its vertex set V can be partitioned into a clique K_h and a (maximum) independent set \bar{K}_{n-h} . It is denoted by $G = K_h \otimes \bar{K}_{n-h}$ (see Figure 9). Every split graph G has diameter at most $\text{diam}(G) = 3$. Certainly, as proved in Theorem 3. Nevertheless, as shown in Figure 10, not every split graph of diameter 3 is a BDM graph.

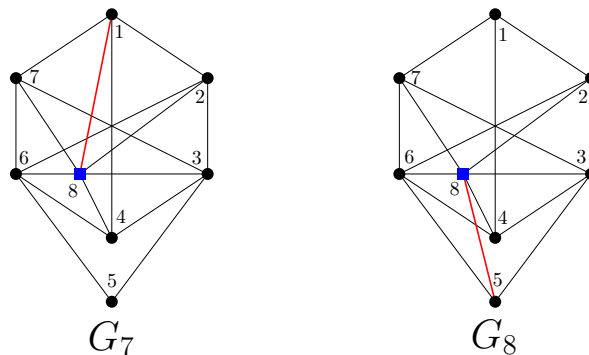


Figure 8. This is the unique pair of distance-hereditary graphs of order 8, having the same boundary distance matrix (see Table 1). In both cases, $\partial G_i = [7]$.



Figure 9. A split graph of order $n = 7$, independence number 4 and size $m = 11$, drawn in two different ways.

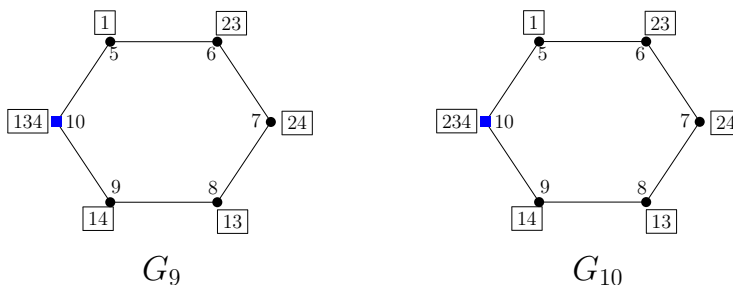


Figure 10. A pair of split graphs of order 10, having the same boundary distance matrix (see Table 1) where $\partial G_i = [9]$.

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G_i	$\partial(G_i)$	\hat{D}_{G_i}
G_1, G_2	$\{1, 2, 3, 4, 5, 6\}$	$\begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 3 & 2 \\ 2 & 1 & 0 & 1 & 2 & 2 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 3 & 2 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 & 0 \end{pmatrix}$
G_3, G_4	$\{1, 2, 3, 4, 5, 6\}$	$\begin{pmatrix} 0 & 1 & 2 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 & 2 \\ 2 & 1 & 0 & 1 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 2 & 1 & 0 \end{pmatrix}$
G_5, G_6	$\{1, 2, 3, 4, 5, 6, 7\}$	$\begin{pmatrix} 0 & 2 & 2 & 2 & 1 & 1 & 3 \\ 2 & 0 & 2 & 2 & 1 & 3 & 1 \\ 2 & 2 & 0 & 2 & 3 & 1 & 1 \\ 2 & 2 & 2 & 0 & 3 & 1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 & 2 \\ 1 & 3 & 1 & 1 & 2 & 0 & 2 \\ 3 & 1 & 1 & 1 & 2 & 2 & 0 \end{pmatrix}$
G_7, G_8	$\{1, 2, 3, 4, 5, 6, 7\}$	$\begin{pmatrix} 0 & 1 & 2 & 1 & 3 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 2 & 1 & 2 \\ 3 & 2 & 1 & 2 & 0 & 1 & 2 \\ 2 & 1 & 2 & 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 2 & 2 & 1 & 0 \end{pmatrix}$
G_9, G_{10}	$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$	$\begin{pmatrix} 0 & 3 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\ 3 & 0 & 2 & 2 & 2 & 1 & 1 & 2 & 2 \\ 2 & 2 & 0 & 2 & 2 & 1 & 2 & 1 & 2 \\ 2 & 2 & 2 & 0 & 2 & 2 & 1 & 2 & 1 \\ \hline 1 & 2 & 2 & 2 & 0 & 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 & 0 & 1 & 1 & 1 \\ 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$

Table 1. The boundary distance matrices of the examples in Section 3. In all the cases the rows and columns are ordered by the same boundary vertex order for each pair.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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