

Coalition graphs of connected domination partitions in subcubic graphs

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Received: 18 September 2025; Accepted: 18 February 2026

Published Online: 25 April 2026

Abstract: A graph is subcubic if it is connected and its maximum vertex degree does not exceed 3. Two disjoint vertex subsets of a graph G form a connected coalition in G if neither of them is a connected dominating set but their union is a connected dominating set. A connected coalition partition of G is a partition of its vertices $\pi(G)$, such that each set of $\pi(G)$ is either a connected dominating set consisting of a single vertex or forms a coalition with some set of $\pi(G)$. The formation of connected coalitions is described by a coalition graph whose vertices correspond to the sets of π , and two vertices are adjacent if and only if the corresponding sets form a coalition in G . We characterize all coalition graphs of subcubic graphs.

Keywords: domination in graphs, coalition graph, subcubic graph.

AMS Subject classification: 05C69

1. Introduction

In this paper, we consider simple graphs $G(V, E)$ with the vertex set $V(G)$ and the edge set $E(G)$. The *order* of a graph is the number of its vertices. The set of vertices adjacent to a vertex v is denoted by $N(v)$. The closed neighbourhood of a vertex v is $N[v] = N(v) \cup \{v\}$. The *degree* $\deg(v)$ of a vertex v is the cardinality of $N(v)$. The maximum vertex degree in a graph G is denoted by $\Delta(G)$. A vertex v with $\deg(v) = n - 1$ in a graph G of order n is called a *full vertex*. A graph G is called *subcubic* if G is connected and $\Delta(G) \leq 3$. A subgraph $G[S]$ induced by $S \subseteq V$ is the

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subgraph with the vertex set S , where two vertices of S are adjacent in $G[S]$ if and only if they are adjacent in G . By K_n , P_n , C_n , S_n we denote the complete graph, the simple path, the simple cycle, and the star graph of order n , respectively. By $K_{p,q}$ we denote the complete bipartite graph with the parts of size p and q . A subset $D \subseteq V(G)$ is a *dominating set* if every vertex of $V(G) \setminus D$ is adjacent to at least one vertex of D . A dominating set $D \subseteq V(G)$ is a *connected dominating set* if $G[D]$ is connected. The domination theory in graphs has found important applications in facility location, analysis of transportation and communication networks, etc. For a detailed information of domination theory, we refer the reader to books [6, 15–18, 21] and articles [7, 19, 20]. Two disjoint vertex subsets of a graph G form a *connected coalition* in G if neither of them is a connected dominating set but their union is a connected dominating set. A connected coalition partition of G is a partition of its vertices $\pi(G) = \{V_1, V_2, \dots, V_k\}$, such that each V_i is either a connected dominating set consisting of a single vertex or forms a coalition with some set of $\pi(G)$. The *connected coalition number* $CC(G)$ of a graph G is the largest possible number of subsets in $\pi(G)$. The concept of coalitions in graphs has been introduced in [10]. Various properties of coalitions have been studied in [1–4, 9, 11–14]. A *coalition graph* is a means of describing the formation of coalitions [10]. Given a graph G and its connected coalition partition π , the coalition graph $CCG(G, \pi)$ is a graph with the vertex set $\{V_1, V_2, \dots, V_k\}$, where V_i and V_j are adjacent if and only if they form a coalition in G . One of the problems in the study of coalition partitions for graphs of various classes is the characterization of coalition graphs that arise here. Coalition graphs of paths, cycles and trees have been considered in [3, 5, 8, 13]. In this paper, we describe all coalition graphs based on connected coalition partitions in subcubic graphs.

2. Main result

Almost all coalition graphs defined by connected coalitions of subcubic graphs can be obtained as coalition graphs of Möbius ladders. Möbius ladder M_n is formed from an even n -cycle by adding edges connecting opposite pairs of vertices in the cycle. It can be also constructed by introducing a twist in a prism graph Pr_n of order n . Möbius ladders and prisms are vertex-transitive graphs. Two representations of M_n and a prism Pr_n are depicted in Fig. 1. The Möbius ladders M_n of order $n \leq 18$ define 21 coalition graphs of order at most 6. The number of coalition graphs for M_n is shown in Table 1. These numerical results have been obtained by computer calculations. Non-standard names of some graphs in the table are given in Fig. 2. Based on these computational data, we formulate the following result.

Theorem 1. *Subcubic graphs define the following coalition graphs: the infinite set of stars S_k for $k \geq 2$ and the finite set of 22 graphs of order at most 6: K_1 , $\overline{K}_2$, $\overline{K}_3$, $\overline{K}_4$, C_3 , $2K_2$, P_4 , C_4 , $C_3 + e$, $K_4 - e$, K_4 , $P_2 \cup P_3$, $S_{1,2}$, P_5 , $C_3 + e + e$, $C_3 + 2e$, $C_4 + e$, C_5 , $S_{2,2}$, $3K_2$, $K_{2,3}$, and $K_{3,3}$.*

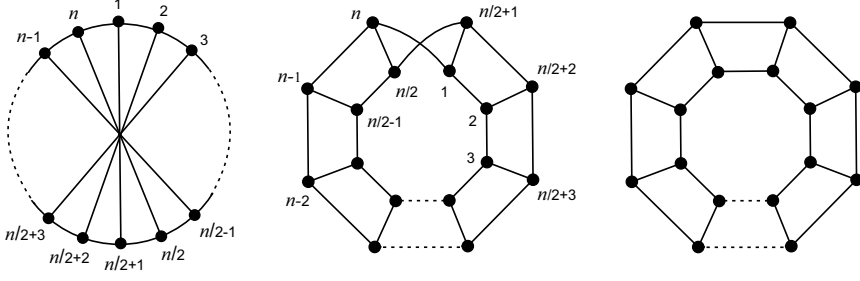


Figure 1. Möbius ladder M_n and prism Pr_n of order n .

Table 1. Number of coalition graphs CCG for Möbius ladder M_n , $6 \leq n \leq 18$.

N	CCG	M_6	M_8	M_{10}	M_{12}	M_{14}	M_{16}	M_{18}
1.	K_2	1	21	176	1089	5538	25575	111844
2.	P_3	6	238	3890	42282	361970	2711840	18731007
3.	C_3	.	184	1825	10830	54362	249280	1075418
5.	S_4	2	108	2500	23380	172004	1139776	7104534
4.	$2K_2$	.	50	905	9816	70371	416016	2198475
6.	P_4	.	288	2370	12624	52430	190016	634734
7.	$C_3 + e$	.	184	970	3048	8456	21984	54972
8.	C_4	9	37	110	207	329	476	648
9.	$K_4 - e$	.	32	10	12	14	16	18
10.	K_4	.	6	.	.	.	.	.
11.	S_5	.	2	70	948	9492	82800	665856
12.	$P_2 \cup P_3$	.	48	280	960	2604	6368	14796
13.	$S_{1,2}$	.	40	250	804	2114	5104	11790
14.	P_5	.	48	120	156	182	208	234
15.	$C_3 + 2e$	.	.	10	24	56	128	288
16.	$C_3 + e + e$	.	16	20	24	28	32	36
17.	$C_4 + e$	.	8	.	.	.	.	.
18.	C_5	.	8	2	.	.	.	.
19.	$K_{2,3}$	6	.	.	.	.	.	.
20.	$S_{2,2}$	.	4	5	6	7	8	9
21.	$K_{3,3}$	1	.	.	.	.	.	.

To prove this statement, we present the corresponding coalition partitions for the graphs of Theorem 1 and show that there are no other coalition graphs defined by subcubic graphs.

Theorem 2. *Coalition graphs K_1 , $\overline{K}_2$, $\overline{K}_3$, K_4 , $\overline{K}_4$, C_5 , $3K_2$, $K_{2,3}$, and $K_{3,3}$ are defined only by a finite number of subcubic graphs. Other coalition graphs of Theorem 1 can be defined by an infinite number of subcubic graphs.*

We will demonstrate that an infinite family of subcubic graphs of Theorem 2 is formed

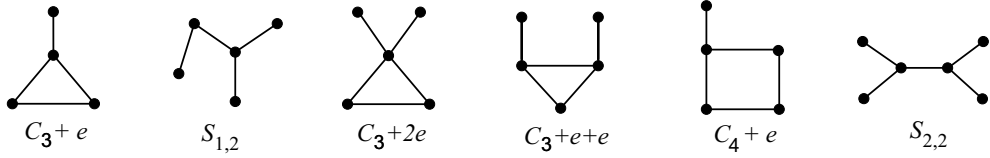


Figure 2. Notation of coalition graphs from Table 1.

by Möbius ladders and their modification. The proof is given in Sections 5 and 6.

3. Lemma on connected dominating sets in subcubic graphs

In this section, we state some useful results concerning the structure of connected dominating sets in subcubic graphs.

Lemma 1. *Let $D \subseteq V$ be a connected dominating set in G . Then $|D| \geq n/2 - 1$ and the induced subgraph $G[V \setminus D]$ has the maximum vertex degree at most 2. The equality $|D| = n/2 - 1$ holds if and only if all vertices in D have degree 3, the induced subgraph $G[D]$ is a tree, and each vertex in $V \setminus D$ is adjacent to exactly one vertex of D .*

Proof. Since every vertex in $V \setminus D$ is adjacent to a vertex of D , the maximum vertex degree of the subgraph $G[V \setminus D]$ does not exceed 2. The subgraph $G[D]$ is connected, so it has at least $|D| - 1$ edges and the sum of degrees of the vertices of D inside $G[D]$ is at least $2(|D| - 1)$. As G is subcubic, the sum of degrees of the vertices of D in G does not exceed $3|D|$. Therefore, the number of edges joining vertices of D with the vertices of $V \setminus D$ does not exceed $3|D| - 2(|D| - 1) = |D| + 2$. Thus, $|V \setminus D| \leq |D| + 2$ and $n = |D| + |V \setminus D| \leq 2|D| + 2$ imply that $|D| \geq n/2 - 1$. The arguments above show that the equality holds if and only if each vertex of D has degree 3 in G , each vertex of $V \setminus D$ is adjacent to exactly one vertex of D , and the number of edges in the subgraph $G[D]$ equals to $|D| - 1$ which implies that $G[D]$ is a tree. $\square$

Corollary 1. *Let $D_1, D_2 \subseteq V$ be two disjoint connected dominating sets in G . Then $|D_1| + |D_2| \geq n - 2$ and each of the induced subgraphs $G[D_1]$ and $G[D_2]$ is a path or a cycle. If $|D_1| + |D_2| = n - 2$, then $|D_1| = |D_2| = n/2 - 1$, both subgraphs $G[D_1]$ and $G[D_2]$ are paths, and D_1 and D_2 are joined by a matching of size $n/2 - 1$ in G .*

Proof. Lemma 1 implies that $|D_1| + |D_2| \geq (n/2 - 1) + (n/2 - 1) = n - 2$ and both subgraphs $G[D_1]$, $G[D_2]$ are connected and have the maximum vertex degree at most 2. Thus, each of these subgraphs is a path or a cycle. If $|D_1| + |D_2| = n - 2$, then $|D_1| = |D_2| = n/2 - 1$ and both subgraphs $G[D_1]$ and $G[D_2]$ are trees, so that they are paths. Finally, each vertex of D_2 is adjacent to exactly one vertex of D_1 , and each vertex of D_1 is adjacent to exactly one vertex of D_2 . This means that the vertex sets D_1 and D_2 are joined by a matching of size $n/2 - 1$ in G . $\square$

Corollary 2. *Let $D_1, D_2, D_3 \subseteq V$ be three disjoint connected dominating sets in G . If $n \geq 5$, then $|D_1| = |D_2| = |D_3| = 2$ and G is isomorphic to one of the graphs Pr_6 or $M_6 \cong K_{3,3}$.*

Proof. Since G is a subcubic graph of order $n \geq 5$, the size of a dominating set in G is at least 2. Thus, $|D_i| \geq 2$ for $i = 1, 2, 3$, and $n \geq |D_1| + |D_2| + |D_3| \geq 6$. By Lemma 1, it follows that $n \geq |D_1| + |D_2| + |D_3| \geq 3(n/2 - 1)$, which implies that $n \leq 6$. Thus, $n = 6$ and $|D_1| = |D_2| = |D_3| = 2$. By Corollary 1, the two vertices of D_i are joined by an edge for $i = 1, 2, 3$, and D_i is joined with D_j by a matching of size 2 in G for any pair of indices $i, j \in \{1, 2, 3\}$. It is easy to check that G is isomorphic to one of the graphs Pr_6 or M_6 . $\square$

4. Properties of coalition graphs

In this section, the structure and properties of coalition graphs are determined. In the following lemmas we assume that a subcubic graph G has a connected coalition partition $\pi(G) = \{V_1, V_2, \dots, V_k\}$ with the coalition graph $H = CCG(G, \pi)$. By $\alpha(H)$ we denote the maximal size of a matching in H .

Lemma 2. *The coalition graph H satisfies $\alpha(H) \leq 3$. If $\alpha(H) = 3$, then one of the following statements hold:*

- (i) $G \cong H \cong M_6 \cong K_{3,3}$, that is, G is a self-coalition graph;
- (ii) $G \cong Pr_6$ and $H \cong 3K_2$.

Proof. Suppose that H contains a matching $(V_1, V_2), (V_3, V_4), (V_5, V_6)$. Then $D_1 = V_1 \cup V_2$, $D_2 = V_3 \cup V_4$, and $D_3 = V_5 \cup V_6$ are three disjoint connected dominating sets of size at least 2 in G . By Corollary 2, we have $|D_1| = |D_2| = |D_3| = 2$, and G is isomorphic to one of the graphs Pr_6 or M_6 . Hence $n = k = 6$, $\alpha(H) = 3$, and $V_i = \{v_i\}$ for $i = 1, 2, \dots, 6$. If $G \cong M_6$, then the set $V_i \cup V_j$ is a coalition for every edge (v_i, v_j) of G . Thus, $H \cong G \cong M_6$. If $G \cong Pr_6$, then only $V_1 \cup V_2$, $V_3 \cup V_4$, and $V_5 \cup V_6$ are coalitions in G . For any other pair of indices i, j , the set $V_i \cup V_j$ is either disconnected or is not dominating. Thus, $H \cong 3K_2$. $\square$

Lemma 3. *If H contains an isolated vertex, then $n = k \leq 4$ and $G \cong K_n$, $H \cong \overline{K}_n$.*

Proof. Let V_i be an isolated vertex in H . Then $V_i = \{v_i\}$ for some full vertex v_i of G . So we have $n = d(v_i) + 1 \leq 4$. If $G \cong K_n$, then all vertices of G are full and hence $H \cong \overline{K}_n$. Assume that G is not complete. Since G has a full vertex, it is isomorphic to one of the graphs P_3 , S_4 , $C_3 + e$ or $K_4 - e$. However, none of these graphs has a connected coalition partition because the set of all non-full vertices of G is disconnected, hence it can not contain any connected coalition of G . $\square$

Lemma 4. *If H contains a vertex V_i of degree at least 4, then V_i is a dominating set in G , and the induced subgraph $G[V_i]$ is disconnected.*

Proof. Suppose that V_i is adjacent to vertices V_1, V_2, V_3, V_4 in H . Assume that V_i is not a dominating set in G . Consider a vertex $v \in V$ such that $N[v] \cap V_i = \emptyset$. Since $V_i \cup V_j$ is a connected dominating set in G for $j = 1, 2, 3, 4$, we have $N[v] \cap V_j \neq \emptyset$. Therefore, $d(v) = 3$ and $N[v]$ contains exactly one vertex from each set V_1, V_2, V_3, V_4 . Assume, without loss of generality, that $v \in V_1$ and $N(v) \subseteq V_2 \cup V_3 \cup V_4$. It follows that v is an isolated vertex in the subgraph $G[V_1 \cup V_i]$. This contradicts the assumption that $V_1 \cup V_i$ is a connected dominating set in G . Hence V_i is a dominating set in G but the subgraph $G[V_i]$ is disconnected. $\square$

Lemma 5. *If $\alpha(H) = 1$, then H is isomorphic to one of the graphs K_3 or S_k , where $k \leq \lfloor \frac{n+7}{3} \rfloor$.*

Proof. By Lemma 3 and the equality $\alpha(\overline{K}_n) = 0$, it follows that H has no isolated vertices. As $\alpha(H) = 1$, this implies that H is connected. Let V_i be a vertex of the maximum degree d in H . Clearly, $d \geq 1$. If $d = 1$, then $H \cong K_2 \cong S_2$. Suppose that $d \geq 2$ and let $V_1, V_2, \dots, V_d$ be the neighbors of V_i in H . If H contains only the edges $(V_i, V_1), (V_i, V_2), \dots, (V_i, V_d)$, then $k = d + 1$ and $H \cong S_k$. If $k \leq 3$, then $k \leq \lfloor \frac{k+7}{3} \rfloor \leq \lfloor \frac{n+7}{3} \rfloor$. If $k = 4$, then it is easy to check that $n \geq 5$ and hence $k \leq \lfloor \frac{n+7}{3} \rfloor$. Suppose that $k \geq 5$, $d \geq 4$. By Lemma 4, the subgraph $G[V_i]$ is disconnected. Let K be an arbitrary component of $G[V_i]$ with m vertices. Similarly to the proof of Lemma 1, one can show that K is joined with the set $V_1 \cup V_2 \cup \dots \cup V_d$ by at most $m + 2$ edges. Since every subgraph $G[V_i \cup V_j]$ is connected for $j = 1, 2, \dots, d$, K is joined by an edge with each of the sets $V_1, V_2, \dots, V_d$. Hence $d \leq m + 2$ and $m \geq d - 2 = k - 3$. As the subgraph $G[V_i]$ has at least two components, we get $n \geq 2(k - 3) + d = 3k - 7$. Thus, $k \leq \lfloor \frac{n+7}{3} \rfloor$.

Assume that H has an edge $e \notin \{(V_i, V_1), (V_i, V_2), \dots, (V_i, V_d)\}$. Since $\alpha(H) = 1$, the edge e must be incident with all vertices $V_1, V_2, \dots, V_d$. Therefore, $d = 2$, $k = 3$, and $e = (V_1, V_2)$. Thus, $H \cong K_3$. $\square$

Lemma 6. *If $k \leq 4$, then H is isomorphic to one of the graphs $K_1, K_2, \overline{K}_2, P_3, K_3, \overline{K}_3, 2K_2, S_4, P_4, C_4, C_3 + e, K_4 - e, K_4, \overline{K}_4$.*

Proof. If $\alpha(H) \leq 1$, then by Lemmas 3 and 5, H is isomorphic to one of the graphs $\overline{K}_1 \cong K_1, \overline{K}_2, \overline{K}_3, \overline{K}_4, S_2 \cong K_2, S_3 \cong P_3, S_4$ or K_3 . So we are left with the case $\alpha(H) = 2$, $k = 4$. Clearly, H is isomorphic to one of the graphs $2K_2, P_4, C_4, C_3 + e, K_4 - e$ or K_4 . $\square$

Denote by $2C_3 + e$ a graph which is obtained from $K_4 - e$ by identifying its vertex of degree 3 with a vertex of K_2 (see Fig. 3).

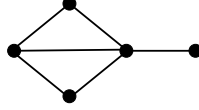


Figure 3. Graph $2C_3 + e$.

Lemma 7. *The coalition graph H does not contain any of the following subgraphs:*

- (i) $K_2 \cup K_3$;
- (ii) $K_2 \cup S_4$;
- (iii) $2C_3 + e$.

Proof. (i) Suppose that H contains a subgraph $K_2 \cup K_3$ with the vertices $V_1, V_2, \dots, V_5$ and edges $(V_1, V_2), (V_3, V_4), (V_4, V_5), (V_5, V_3)$. Then $V_3 \cup V_4, V_4 \cup V_5$, and $V_5 \cup V_3$ are connected dominating sets of G . This implies that every vertex in $D = V_1 \cup V_2$ is adjacent to vertices of at least two of the sets V_3, V_4, V_5 . Thus, $G[D]$ is a connected subgraph with the maximum degree 1 which yields $G[D] \cong K_2, V_1 = \{v_1\}, V_2 = \{v_2\}$. Lemma 1 implies that $n \leq 2|D| + 2 = 6$. Thus, $|V_3 \cup V_4 \cup V_5| \leq n - |D| \leq 4$. Consider two possible cases.

Case 1. $|V_3 \cup V_4 \cup V_5| = 3, G[V_3 \cup V_4 \cup V_5] \cong K_3, V_i = \{v_i\}, i = 3, 4, 5$. Observe that every vertex v_3, v_4, v_5 is adjacent to at most one of the vertices v_1 or v_2 . So either v_1 or v_2 is adjacent to at most one of the vertices v_3, v_4, v_5 . Without loss of generality, assume that v_1 is not adjacent to v_4 and v_5 . Then $V_4 \cup V_5$ is not a dominating set in G , a contradiction.

Case 2. $|V_3 \cup V_4 \cup V_5| = 4, n = 6, V_3 = \{v_3\}, V_4 = \{v_4\}, V_5 = \{v_5, v_6\}$, and $(v_3, v_4) \in E$. Since both sets $V_1 \cup V_2$ and $V_3 \cup V_4$ are dominating, the edges (v_1, v_2) and (v_3, v_4) do not belong to triangles in G . Without loss of generality, assume that $(v_1, v_3), (v_1, v_5), (v_2, v_4), (v_2, v_6) \in E$ and either $(v_3, v_5), (v_4, v_6) \in E$ or $(v_3, v_6), (v_4, v_5) \in E$. In both cases, the connectedness of $G[V_3 \cup V_5]$ implies that $(v_5, v_6) \in E$. Thus, V_5 is a connected dominating set in G , a contradiction.

(ii) Suppose that H contains a subgraph $K_2 \cup S_4$ with the vertices $V_1, V_2, \dots, V_6$ and edges $(V_1, V_2), (V_3, V_4), (V_3, V_5), (V_3, V_6)$. Since $V_1 \cup V_2$ is a connected dominating set in G , the subgraph $G[V_3 \cup V_4 \cup V_5 \cup V_6]$ has the maximum degree at most 2. Let F be an arbitrary connected component of the subgraph $G[V_3]$. Then F is a path or a cycle. Since the subgraphs $G[V_3 \cup V_4], G[V_3 \cup V_5]$, and $G[V_3 \cup V_6]$ are connected, F must be joined by an edge to each of the sets V_4, V_5, V_6 . However, this is impossible because the maximum degree of $G[V_3 \cup V_4 \cup V_5 \cup V_6]$ does not exceed 2 and F is a path or a cycle.

(iii) Suppose that H contains a subgraph $2C_3 + e$ with the vertices $V_1, V_2, \dots, V_5$ and edges $(V_1, V_2), (V_1, V_3), (V_1, V_4), (V_1, V_5), (V_2, V_3), (V_2, V_4)$. By Lemma 4, V_1 is a dominating set in G but the subgraph $G[V_1]$ is disconnected. So the maximum degree of the subgraph $G[V_2 \cup V_3 \cup V_4 \cup V_5]$ does not exceed 2. This means that $G[V_2 \cup V_3 \cup V_4 \cup V_5]$ (and any of its subgraphs) is a collection of paths and cycles.

Let $F_1, F_2, \dots, F_t$ be all connected components of the subgraph $G[V_2]$. Since $G[V_2 \cup V_3]$ is a connected subgraph whose maximum degree does not exceed 2, it follows that $G[V_2 \cup V_3]$ is a path or a cycle containing all components of $G[V_2]$ and $G[V_3]$. Hence $G[V_2 \cup V_3]$ contains a path $(F_1, P_1, F_2, P_2, \dots, P_{t-1}, F_t)$, where $P_1, P_2, \dots, P_{t-1}$ are components of $G[V_3]$. If $t \geq 3$, then the vertices of F_2 are joined in $G[V_2 \cup V_3 \cup V_4 \cup V_5]$ by edges only to the vertices of V_3 (namely, to the end vertices of P_1 and P_2). Thus, F_2 is a connected component in the subgraph $G[V_2 \cup V_4]$. Therefore, $G[V_2 \cup V_4]$ is disconnected, a contradiction.

Assume that $t = 2$. Then $G[V_2 \cup V_3]$ contains a path (F_1, P_1, F_2) . Similarly, the subgraph $G[V_2 \cup V_4]$ is connected and contains a path (F_1, Q_1, F_2) , where Q_1 is a component of $G[V_4]$. Therefore, the subgraph $G[V_2 \cup V_3 \cup V_4]$ is a cycle $C = (F_1, P_1, F_2, Q_1)$, where $P_1 = G[V_3]$ and $Q_1 = G[V_4]$. Note that C is a connected component of the subgraph $G[V_2 \cup V_3 \cup V_4 \cup V_5]$. So the vertices of $V_2 \cup V_3$ are not adjacent to the vertices of V_5 . Thus, the set $V_2 \cup V_3$ is not dominating in G , a contradiction.

Finally, assume that $t = 1$. Then $G[V_2] = F_1$ and the subgraph $G[V_2 \cup V_3]$ contains a path (F_1, P_1) . Similarly, the subgraph $G[V_2 \cup V_4]$ contains a path (F_1, Q_1) . Hence the subgraph $G[V_2 \cup V_3 \cup V_4]$ is a path (or a cycle) $P = (P_1, F_1, Q_1)$, where $P_1 = G[V_3]$ and $Q_1 = G[V_4]$. Let $a \in P_1$ and $b \in Q_1$ be the end vertices of P . Observe that $V_2 \cup V_3$ can be joined with V_5 only by one edge going from a . As $V_2 \cup V_3$ is a dominating set, we get $V_5 = \{v_5\}$ and $(a, v_5) \in E$. Similarly, we obtain $(b, v_5) \in E$. Thus, v_5 can be adjacent to only one vertex of V_1 . Since the subgraph $G[V_1]$ is disconnected, the subgraph $G[V_1 \cup V_5]$ is also disconnected, which is a contradiction. $\square$

Lemma 8. *If $k = 5$, then H is isomorphic to one of the graphs $P_2 \cup P_3$, $S_{1,2}$, S_5 , P_5 , $C_3 + e + e$, $C_3 + 2e$, $C_4 + e$, C_5 , $K_{2,3}$.*

Proof. By Lemma 3, H has no isolated vertex. If H is a forest, then H is isomorphic to one of the graphs $P_2 \cup P_3$, $S_{1,2}$, S_5 or P_5 . Suppose that H is not a forest. Let $C = (V_1, V_2, \dots, V_m)$ be the shortest cycle in H . Then C has no chords. Clearly, if $m = k = 5$, then $H = C \cong C_5$. Suppose that $m = 4$. Since V_5 is not an isolated vertex, we can assume that V_5 is adjacent to V_1 . If V_5 is not adjacent to other vertices of H , then $H \cong C_4 + e$. If V_5 is adjacent to V_2 or V_4 , then H contains a subgraph $K_2 \cup K_3$, which contradicts Lemma 7(i). If V_5 is adjacent to V_3 , then $H \cong K_{2,3}$.

Finally, assume that $m = 3$. By Lemma 7(i), we have $(V_4, V_5) \notin E(H)$. So each vertex V_4, V_5 is adjacent to a vertex of C . If V_4 or V_5 is adjacent to at least two vertices of C , then H contains one of the subgraphs $K_2 \cup K_3$ or $2C_3 + e$, which contradicts Lemma 7(i),(iii). Thus, each vertex V_4, V_5 is adjacent to exactly one vertex of C . If they are adjacent to the same vertex of C , then $H \cong C_3 + 2e$, otherwise, $H \cong C_3 + e + e$. $\square$

Lemma 9. *If $k \geq 6$, then H is isomorphic to one of the graphs M_6 , $3K_2$, $S_{2,2}$, S_k .*

Proof. By Lemma 3, H has no isolated vertex. If $\alpha(H) = 1$, then by Lemma 5, we have $H \cong S_k$. If $\alpha(H) = 3$, then by Lemma 2, H is isomorphic to one of the graphs

M_6 or $3K_2$. So we are left with the case $\alpha(H) = 2$. Let (V_1, V_2) and (V_3, V_4) be edges of H . Corollary 1, applied to the sets $D_1 = V_1 \cup V_2$ and $D_2 = V_3 \cup V_4$, gives $|D_1| + |D_2| = |V_1| + |V_2| + |V_3| + |V_4| \geq n - 2$. Hence G contains at most two vertices v_5, v_6 outside $D_1 \cup D_2$. Since $k \geq 6$, we get $|D_1| + |D_2| = n - 2$, $k = 6$, $V_5 = \{v_5\}$, $V_6 = \{v_6\}$. By Corollary 1, both subgraphs $G[D_1]$ and $G[D_2]$ are paths with $n/2 - 1$ vertices and D_1 is joined with D_2 by a matching of size $n/2 - 1$ in G .

The condition $\alpha(H) = 2$ implies that $(V_5, V_6) \notin E(H)$. Thus, each vertex V_5, V_6 is adjacent to one of the vertices V_1, V_2, V_3, V_4 in H . Assume that V_5 is adjacent to V_1 . If V_6 is adjacent to V_2 or V_1 , then either $\alpha(H) = 3$ or H contains the subgraph $K_2 \cup S_4$, which contradicts Lemma 7(ii). So we can assume that V_6 is adjacent to V_3 . Since $\alpha(H) = 2$ and H does not contain $K_2 \cup K_3$ and $K_2 \cup S_4$, the vertices V_2, V_4, V_5, V_6 are not incident with the edges other than (V_2, V_1) , (V_5, V_1) , (V_4, V_3) , (V_6, V_3) in H . So H can have only one edge (V_1, V_3) other than (V_2, V_1) , (V_5, V_1) , (V_4, V_3) , (V_6, V_3) .

We now prove that $(V_1, V_3) \in E(H)$. Corollary 1, applied to the sets $D_3 = V_1 \cup V_5$ and $D_4 = V_3 \cup V_6$, shows that $V_2 = \{v_2\}$, $V_4 = \{v_4\}$, and both subgraphs $G[D_3]$ and $G[D_4]$ are paths with $n/2 - 1$ vertices. By Lemma 1, v_2 is adjacent to only one vertex of the set $D_3 = V_1 \cup V_5$ and hence v_2 is adjacent to only one vertex of V_1 . Therefore, v_2 is an end vertex of the path $G[D_1] = G[V_1 \cup V_2]$. Similarly, v_5 is an end vertex of the path $G[D_3] = G[V_1 \cup V_5]$. Since $D_2 = V_3 \cup V_4$ is a dominating set of G , the subgraph $G[V_1 \cup V_2 \cup V_5]$ has the maximum degree at most 2. This implies that $G[V_1 \cup V_2 \cup V_5]$ is a path (v_2, P_1, v_5) , where $P_1 = G[V_1]$. Similarly, the subgraph $G[V_3 \cup V_4 \cup V_6]$ is a path (v_4, P_3, v_6) , where $P_3 = G[V_3]$. Thus, $V_1 \cup V_3$ is a dominating set in G and the subgraph $G[V_1 \cup V_3]$ has at most two components. Consider an arbitrary vertex $v \in V_1$. Note that v has degree 2 in $G[V_1 \cup V_2 \cup V_5]$. As $V_3 \cup V_4$ and $V_3 \cup V_6$ are dominating sets of G , v is either adjacent to a vertex of V_3 or to both vertices v_4 and v_6 . In the latter case, v has degree at least 4 in G , which is a contradiction. So v is adjacent to a vertex of V_3 . Hence the subgraph $G[V_1 \cup V_3]$ is connected, and $V_1 \cup V_3$ is a connected dominating set in G . Therefore, $(V_1, V_3) \in E(H)$ and $H \cong S_{2,2}$. $\square$

5. Coalition graphs defined by a finite number of subcubic graphs

To prove Theorem 2, we demonstrate that a part of coalition graphs is defined by a finite number of subcubic graphs while the other part of coalition graphs is defined by an infinite number of Möbius ladders and their modifications.

Proposition 1. *Let H be one of the graphs $K_1, \overline{K}_2, \overline{K}_3, K_4, \overline{K}_4, C_5, 3K_2, K_{2,3}$ or $K_{3,3}$. Then H is a coalition graph for at least one subcubic graph G , and any such a graph G has order $n \leq 10$. Hence H is defined only by a finite number of subcubic graphs.*

Proof. If H is isomorphic to one of the graphs $K_1, \overline{K}_2, \overline{K}_3$ or $\overline{K}_4$, then $n \leq 4$ and G is isomorphic to K_1, K_2, K_3 or K_4 , respectively, by Lemma 3. The realizations

of coalition graphs K_4 , C_5 , $3K_2$, $K_{2,3}$, and $K_{3,3}$ are shown in Fig. 4. Let us prove the upper bound $n \leq 10$. If H is isomorphic to $3K_2$ or $K_{3,3} \cong M_6$, then $n = 6$ by Lemma 2. Assume that $H = (V_1, V_2, \dots, V_5) \cong C_5$. Corollary 1, applied to the sets $D_1 = V_2 \cup V_3$ and $D_2 = V_4 \cup V_5$, implies that $|V_1| \leq 2$. Similarly, $|V_i| \leq 2$ for $i = 2, 3, 4, 5$. Hence $n \leq 10$.

Assume that $H \cong K_4$. Let m_i be the number of edges in the subgraph $G[V_i]$ for $i = 1, 2, 3, 4$ and $m = m_1 + m_2 + m_3 + m_4$. For every pair of indices $i, j \in \{1, 2, 3, 4\}$, denote by m_{ij} the number of edges joining V_i with V_j in G . Then the number of edges of the subgraph $G[V_i \cup V_j]$ is equal to $m_i + m_j + m_{ij}$, and the number of edges of G is equal to $|E| = m + \sum_{i,j} m_{ij}$. Since $V_2 \cup V_3$, $V_2 \cup V_4$, and $V_3 \cup V_4$ are dominating sets in G , every vertex of V_1 is adjacent to vertices of at least two sets V_2, V_3, V_4 . So the maximum vertex degree of the subgraph $G[V_1]$ does not exceed 1, and the edge set of $G[V_1]$ is a matching. Hence $m_1 \leq |V_1|/2$. Similarly, $m_i \leq |V_i|/2$ for $i = 2, 3, 4$. Therefore, $m \leq n/2$. Since the subgraph $G[V_i \cup V_j]$ is connected, we have $m_i + m_j + m_{ij} \geq |V_i| + |V_j| - 1$. Summarizing these inequalities over all pairs of indices i, j gives

$$3m + \sum_{i,j} m_{ij} = 2m + |E| \geq 3(|V_1| + |V_2| + |V_3| + |V_4|) - 6 = 3n - 6.$$

As G is subcubic, we have $|E| \leq 3n/2$. Thus, $2m \geq 3n - 6 - |E| \geq 3n/2 - 6$. Combining this inequality with $m \leq n/2$ implies that $n \leq 12$, $n \neq 11$, and if $n = 12$, then $m = n/2 = 6$. So if $n = 12$, then the edge set of every subgraph $G[V_i]$ is a perfect matching for $i = 1, \dots, 4$. Thus, all the numbers $|V_1|, \dots, |V_4|$ are even. Since $|V_1| + \dots + |V_4| = 12$, at least two of the numbers $|V_1|, \dots, |V_4|$ are equal to 2. Let $|V_1| = |V_2| = 2$. By Lemma 1 for the set $D = V_1 \cup V_2$, it follows that $|D| = |V_1| + |V_2| \geq 12/2 - 1 = 5$, which is a contradiction. Thus, $n \leq 10$.

Finally, assume that $H \cong K_{2,3}$ and H has edges $(V_1, V_3), (V_1, V_4), (V_1, V_5), (V_2, V_3), (V_2, V_4), (V_2, V_5)$. Corollary 1, applied to the sets $D_1 = V_1 \cup V_4$ and $D_2 = V_2 \cup V_5$, implies that $|V_3| \leq 2$. Similarly, $|V_i| \leq 2$ for $i = 4, 5$. Let F be an arbitrary connected component of the subgraph $G[V_1]$. Since $V_2 \cup V_3$, $V_2 \cup V_4$, and $V_2 \cup V_5$ are dominating sets in G , every vertex of F is either adjacent to a vertex of V_2 or is adjacent to vertices of all three sets V_3, V_4 , and V_5 . In the latter case, F is a single vertex. In the former case, F is a path or a cycle, which is joined by at most two edges with the set $V_3 \cup V_4 \cup V_5$. Therefore, one of the subgraphs $G[V_1 \cup V_3]$, $G[V_1 \cup V_4]$ or $G[V_1 \cup V_5]$ is disconnected, a contradiction. So every component of $G[V_1]$ is a single vertex v , which is adjacent to vertices of V_3, V_4 , and V_5 . Thus, v has degree 1 in the subgraph $G[V_1 \cup V_3]$. Corollary 1, applied to the sets $D_3 = V_1 \cup V_3$ and $D_4 = V_2 \cup V_4$, implies that $G[V_1 \cup V_3]$ is a path or a cycle. Hence $G[V_1 \cup V_3]$ has at most two vertices of degree 1 and so $|V_1| \leq 2$. Similarly, we have $|V_2| \leq 2$. Thus, $n = |V_1| + |V_2| + \dots + |V_5| \leq 5 \cdot 2 = 10$. The upper bound of Proposition 1 is sharp because the coalition graph C_5 is defined by the ladder M_{10} and the coalition graph $K_{2,3}$ is defined by a cubic graph of order 10 (see Fig. 4). $\square$

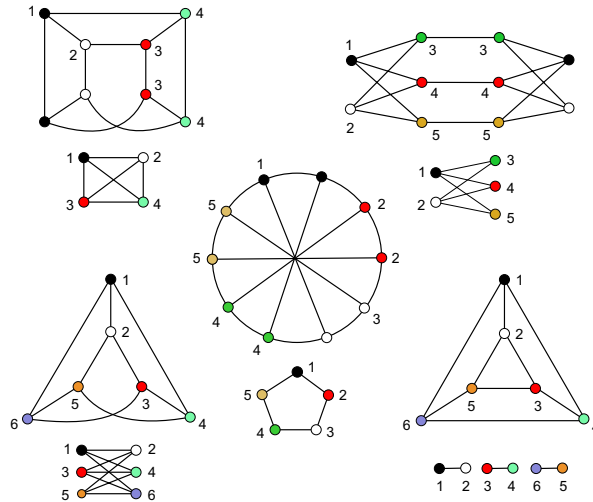


Figure 4. Coalition graphs K_4 , $K_{2,3}$, C_5 , $K_{3,3}$, and $3K_2$.

6. Coalition graphs defined by an infinite number of subcubic graphs

In this section, we determine coalition graphs that are generated by an infinite number of Möbius ladders and their modifications.

Proposition 2. *Coalition graphs C_3 , $2K_2$, P_4 , C_4 , $C_3 + e$, $K_4 - e$, $P_2 \cup P_3$, $S_{1,2}$, P_5 , $C_3 + e + e$, $C_3 + 2e$, $C_4 + e$, $S_{2,2}$, and S_k , $k \geq 2$, are defined by an infinite number of subcubic graphs.*

A graphical proof of this statement is presented in Figs. 5, 6. Stars S_k are generated by the subcubic graph F_1 , where $F_1[V_1]$ and $F_1[V_2]$ are arbitrary connected subgraphs. The corresponding partition of F_1 is $\pi = \{V_1 \cup V_2, \{v_1\}, \{v_2\}, \dots, \{v_{k-1}\}\}$, and the set $V_1 \cup V_2 \cup \{v_i\}$ forms a connected coalition for $i = 1, 2, \dots, k-1$. All other coalition graphs of Proposition 2, except $C_4 + e$, can be defined by infinite families of Möbius ladders of order $n \geq 10$. The corresponding coalition graphs are shown in Figs. 5, 6 below the ladders. Vertices of each set of π have the same color and are marked by the number of the corresponding coalition set. The sets V_1 and V_2 of π consist of black and white vertices, respectively (only one vertex of these sets is marked). To obtain the coalition graph $C_4 + e$, the structure of a Möbius ladder has been modified (see graph F_2 in Fig. 6). Observe that Lemmas 6, 8, 9 and Propositions 1, 2 imply Theorems 1 and 2.

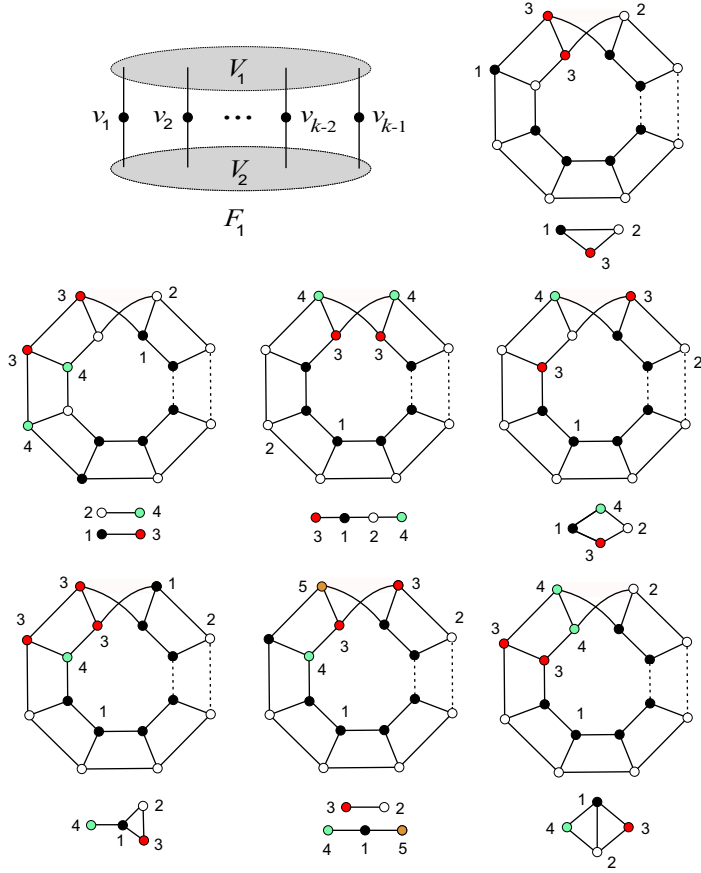


Figure 5. Coalition graphs for Möbius ladder M_n of order $n \geq 10$.

7. Coalition graphs for prisms

Since Möbius ladders can be obtained from prisms only by switching two edges, it is interesting to compare their sets of coalition graphs. Table 2 contains the number of coalition graphs for prisms Pr_n of order $n \leq 18$. These prisms define only 14 coalition graphs only one of which is not generated by Möbius ladders. Namely, Lemma 2 implies that $3K_2$ cannot be a coalition graph for M_n . Note that the smallest prism Pr_6 defines significantly more coalition graphs than the ladder M_6 .

8. Connected coalition number of subcubic graphs

Trees and cycles are the simplest classes of subcubic graphs. It was shown that $CC(T) = 2$ for any tree T of order $n \geq 3$ without a full vertex [1]. For any cycle of order n , $CC(C_4) = 4$ and $CC(C_n) = 3$ if $n \neq 4$ [9].

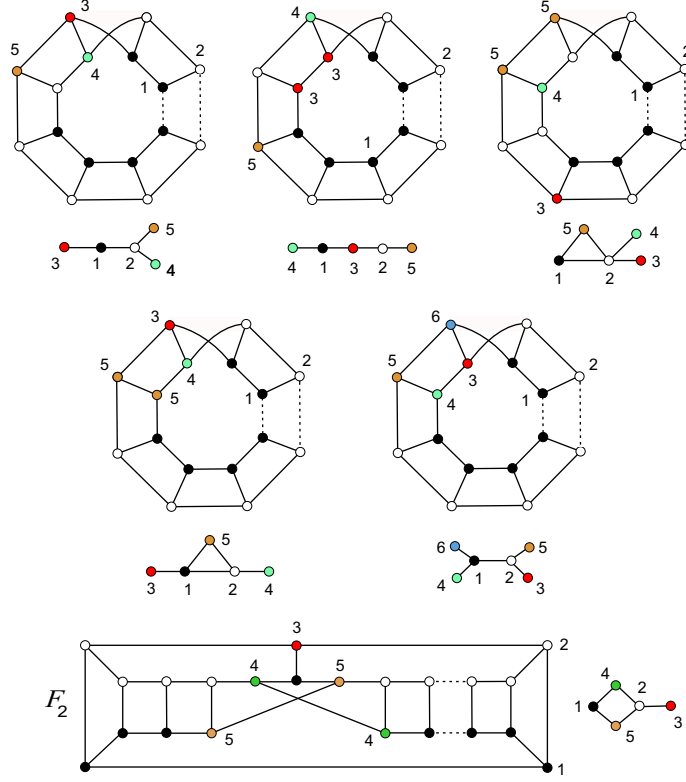


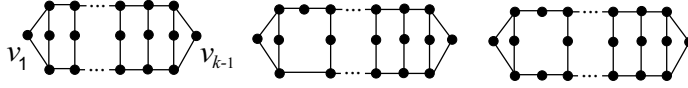
Figure 6. Coalition graphs for Möbius ladder M_n and graph F_2 of order $n \geq 10$.

Table 2. Number of coalition graphs CCG for prism Pr_n , $6 \leq n \leq 18$.

N	CCG	Pr_6	Pr_8	Pr_{10}	Pr_{12}	Pr_{14}	Pr_{16}	Pr_{18}
1.	K_2	3	28	190	1112	5572	25622	111906
2.	P_3	18	312	4140	42988	363636	2716026	18741699
3.	C_3	8	216	1870	10826	54201	248748	1074777
4.	S_4	6	140	2660	24268	177562	1169328	7240476
5.	$2K_2$	3	63	775	9913	84875	586045	3542679
6.	P_4	18	264	2350	14208	68726	286084	1079694
7.	$C_3 + e$	.	240	1180	4740	15260	44032	119196
8.	C_4	3	15	45	219	560	1140	2043
9.	$K_4 - e$	6	48	165	246	343	456	585
10.	K_4	.	14	.	.	.	.	.
11.	S_5	.	6	130	1362	12068	97648	746784
12.	$P_2 \cup P_3$	6	.	.	.	.	.	.
13.	$S_{1,2}$	6	.	.	.	.	.	.
14.	$3K_2$	1	.	.	.	.	.	.

Table 3. Number of cubic graphs of order n with connected coalition number k .

$k \setminus n$	4	6	8	10	12	14
2	.	.	.	.	.	.
3	.	.	.	1	.	1
4	1	.	.	1	9	36
5	.	.	4	15	32	67
6	.	2	1	2	44	268
7	.	.	.	.	.	137
total	1	2	5	19	85	509

**Figure 7.** The smallest subcubic graphs for which the exact upper bound is achieved.

It is known that the coalition number $C(G)$, based on dominating sets in graphs, is bounded by the maximum vertex degree of G . Namely, $C(G) \leq (\Delta(G) + 3)^2/4$ for any graph G [16]. For cubic graphs, this bound yields $C(G) \leq 9$. It was proved that the coalition number of cubic graphs of order $n \leq 10$ is at most 8, and the smallest cubic graphs with $C(G) = 9$ have 16 vertices [2, 4]. Table 3 represents the number of cubic graphs of order n having the connected coalition number k for each even $n = 4, 6, \dots, 14$ and for each $k = 2, 3, \dots, 7$. Based on the results of the previous sections, we prove the following upper bound for the connected coalition number of subcubic graphs.

Theorem 3. For the connected coalition number of a subcubic graph G , the following inequality holds

$$CC(G) \leq \max \left\{ 6, \left\lfloor \frac{n+7}{3} \right\rfloor \right\}$$

where the bound is sharp for $n \in \{6, 8\}$ and for $n \geq 10$. If $CC(G) = k > 6$, then the only coalition graph of G of order k is S_k .

Proof. If the coalition graph $H \cong S_k$, then by Lemma 5, we have $k \leq \lfloor \frac{n+7}{3} \rfloor$. Lemma 9 implies that if $H \not\cong S_k$, then $k \leq 6$, and if $CC(G) = k > 6$, then the only coalition graph of G of order k is S_k . If $n \in \{6, 8, 10\}$, then $\max \{6, \lfloor \frac{n+7}{3} \rfloor\} = 6$ and the upper bound is sharp because the coalition graph $3K_2$ is defined by the prism Pr_6 while the coalition graph $S_{2,2}$ is defined by the ladders M_8 and M_{10} (see Fig. 4). If $n \geq 11$, then $\max \{6, \lfloor \frac{n+7}{3} \rfloor\} = \lfloor \frac{n+7}{3} \rfloor$ and the bound is sharp because the coalition graph S_k for $k \geq 6$ can be defined by subcubic graphs of order $3k - 7$, $3k - 6$, and $3k - 5$ (see Fig. 7). $\square$

9. Conclusion

In this paper, all coalition graphs of connected coalition partitions of subcubic graphs are described. The upper bound on the coalition number of subcubic graphs is established in terms of the graph order.

A graph of some family is called universal if it defines all coalition graphs that are generated by all graphs of the family. For families of all paths or all cycles, it is known that there is no universal graph in the case of standard domination [13]. Therefore, the following problem arises: find a graph defining the maximal number of coalition graphs (mc-graph) for a given family. If such an mc-graph exists, then what is the minimal order of the graph? It is known that each path P_n of order $n \geq 10$ is an mc-graph for the family of all paths, and each cycle C_{3k} for $k \geq 5$ is an mc-graph for the family of all cycles [5, 8, 13]. Our considerations imply that there is no universal graph for connected domination in subcubic graphs. Therefore, it would be interesting to find a subcubic mc-graph.

Acknowledgement: The authors would like to thank the anonymous referee for suggestions for improving the paper. This work was supported by the state contract of the Sobolev Institute of Mathematics (project number FWNF-2026-0011).

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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