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Research Article

New bounds for Seidel energy of graphs

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Abstract: Let G be a graph and $S(G)$ be the Seidel matrix of G. Let $s_1 \geq s_2 \geq \cdots \geq s_n$ be the eigenvalues of $S(G)$. The spread of matrix $S(G)$ defined as $s(G) := max_{i,j}|s_i - s_j| = s_1 - s_n$. The Seidel energy of G, denoted by $SE(G)$, is defined to be the sum of the absolute value of all eigenvalues of the Seidel matrix of G. Willem Haemers conjectured that the Seidel energy of any graph with n vertices is at least $2n - 2$. Motivated by this conjecture, we prove that the conjecture is true if $s(G) \leq n$. Moreover, we present some new bounds for the Seidel energy and also we study some properties of the Seidel eigenvalues of G. Our results improve some known results.

Keywords: Seidel energy of graphs, Haemers's conjecture, Seidel eigenvalues.

AMS Subject classification: 05C07, 05C09, 05C31, 05C50

1. Introduction

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. The *adjacency matrix* $A(G)$ of G is defined by its entries as $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. In 1966, Van Lint and Seidel [17] introduced a $(-1, 0, 1)$-matrix for a graph G, called the Seidel matrix of G, defined as $S(G) = J - I - 2A(G)$, where J is a square matrix whose all entries are 1 and I is the identity matrix. Thus, $S(G) = [s_{ij}]$ has 0 on its diagonal and ± 1 on its off diagonal, where $s_{ij} = -1$ if v_i and v_j are adjacent and $s_{ij} = 1$, otherwise.

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Let $s_1 \geqslant s_2 \geqslant \cdots \geqslant s_n$ be the eigenvalues of the Seidel matrix $S(G)$ (we call these eigenvalues as Seidel eigenvalues of G). The Seidel spread of G, denoted by $s(G)$, is defined as $s(G) := \max_{i,j} |s_i - s_j| = s_1 - s_n$. The *Seidel energy* of the graph G introduced by Haemers [7] and defined as

$$SE(G) = \sum_{i=1}^{n} |s_i|.$$

By $S^k(G)$ (where G is a graph of order n) we mean $S^k(G) = \sum_{i=1}^{n} |s_i|^k$. In particular, $SE(G) = S^1(G)$. Haemers posed the following conjecture in [7].

Conjecture 1. Let G be a graph of order n. Then

$$SE(G) \geq 2n - 2.$$

This conjecture was first investigated by Haemers for $n \leq 10$ and then was settled for $n \leq 12$ in [6]. In [13] the Haemers's conjecture has been proved for every k-regular graph G of order n such that $k \neq \frac{n-1}{2}$ and G has no eigenvalue in $(-1, 0)$. In [5] it was proved that the conjecture is valid for all graphs G of order n such that $n - 1 \leq |\det(S(G))|$. This conjecture completely have been proved by Akbari et al. in [2]. Akbari et al. [1], presented some properties of eigenvalues of the Seidel matrix and obtained some bounds for the Seidel energy. In [7, 9, 13–16] there are some results about the Seidel eigenvalues and some bounds for Seidel energy graphs.

Motivated by above papers, we investigate the Haemers's conjecture in point view of spread of Seidel matrices. In this paper we first show that the Haemer's conjecture is true for any graph G of order n and with $s(G) \leq n$. Then we present some properties of eigenvalues of the Seidel matrix and obtain new bounds for the Seidel energy of a graph. Moreover, we improve some of the well-known published bounds for the Seidel energy of graphs.

2. Preliminaries and known results

In this section, we call up some bounds for the Seidel energy and some analytical inequalities that will be used in proofs of theorems. The following result appears in [7, 9] as well.

Lemma 1. *[7, 9] Let G be a graph of order n. Then*

$$SE(G) \leq n\sqrt{n-1}. \tag{2.1}$$

Next lemma plays a key role in this paper.

Lemma 2. *[8] Let G be a graph of order n. The following hold:*
(i) $S^2(G) = \sum_{i=1}^{n} s_i^2 = n(n-1)$,
(ii) For every integer $k \geq 2$, $n\sqrt{(n-1)^k} \leq S^k(G) \leq n(n-1)^k + n - 1$.

In this paper we apply the following algebraic inequalities.

Lemma 3. *[4] Let $a_1, \ldots, a_n$ be some positive real numbers. Then the following inequality holds*

$$\sqrt{\frac{a_1^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + \cdots + a_n}{n} + \frac{1}{4n} \sum_{i=1}^{n} \frac{(na_i^2 - (a_1^2 + \cdots + a_n^2))^2}{n^2 a_i^4 + (a_1^2 + \cdots + a_n^2)^2} a_i. \tag{2.2}$$

Lemma 4. *[3] Let $x_1, \ldots, x_n$ be some non-negative numbers and s, k be some positive integers where $s \leq k$. Then*

$$\left(\sum_{i=1}^{n} x_i^s\right)^{\frac{k}{s}} \leq n^{\frac{k}{s}-1} \left(\sum_{i=1}^{n} x_i^k\right). \tag{2.3}$$

Lemma 5. *[11] Let $p = (p_i)_{i=1}^n$ be a sequence of non-negative real numbers and $a = (a_i)_{i=1}^n$ be a sequence of positive real numbers. Then for every $r \in \mathbb{R} \setminus (0,1)$,*

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geq \left(\sum_{i=1}^{n} p_i a_i\right)^r, \tag{2.4}$$

and the equality holds if and only if either $r = 0$ or $r = 1$ or $a_1 = a_2 = \cdots = a_n$.

Lemma 6. *[10] Let $a_1, \ldots, a_n$ be some non-negative numbers. If $X = \frac{1}{n}\sum_{i=1}^{n} a_i$ and $Y = \left(\prod_{i=1}^{n} a_i\right)^{1/n}$, then*

$$\frac{1}{n(n-1)} \sum_{i<j} \left(\sqrt{a_i} - \sqrt{a_j}\right)^2 \leq X - Y \leq \frac{1}{n} \sum_{i<j} \left(\sqrt{a_i} - \sqrt{a_j}\right)^2.$$

3. Main results

In this section, we establish new bounds for the Seidel energy that improves some of known results. Among these results we prove that the Haemers's conjecture is true if $s(G) \leq n$. We begin with the next result which gives a new bound on $SE(G)$ in terms of the spread of the Seidel matrix.

Theorem 1. *Let G be a graph of order n. Then*

$$SE(G) \geq \frac{2n(n-1)}{s(G)}. \tag{3.1}$$

The equality holds if and only if G has at most three distinct eigenvalues, which are s_1 with multiplicity k, 0 with multiplicity t_0 and s_n with multiplicity $n - t_0 - k$.

Proof. Since $\sum_{s_i \neq 0} s_i = 0$, by the first part of Lemma 2, we have

$$\begin{aligned} n(n-1) = \sum_{s_i \neq 0} s_i^2 &= \frac{1}{2} \Big| \sum_{s_i \neq 0} (2s_i - s_1 - s_n)\, s_i \Big| \\ &\leq \frac{1}{2} \sum_{s_i \neq 0} \Big| (2s_i - s_1 - s_n)\, s_i \Big| \qquad (3.2) \\ &= \frac{1}{2} \sum_{s_i \neq 0} \Big| 2s_i - s_1 - s_n \Big| \Big| s_i \Big|. \end{aligned}$$

Since for $i = 1, 2, \ldots, n$, $s_n \leq s_i \leq s_1$, one can easily see that

$$\Big| 2s_i - s_1 - s_n \Big| |s_i| \leq (s_1 - s_n)|s_i|. \qquad (3.3)$$

Combining (3.2) and (3.3) we obtain that

$$\begin{aligned} \frac{1}{2} s(G) SE(G) &= \frac{1}{2} (s_1 - s_n) \sum_{s_i \neq 0} |s_i| \\ &= \frac{1}{2} \sum_{s_i \neq 0} (s_1 - s_n)\, |s_i| \\ &\geq n(n-1). \qquad (3.4) \end{aligned}$$

Now suppose that the equality holds in (3.1). Then in the inequalities (3.2) and (3.3) the equality holds. If $s_i = 0$ for some i, then the equality occurs in (3.3). If $s_i \neq 0$, then the equality holds in (3.3) if and only if $s_i = s_1$ or $s_i = s_n$. Therefore, the equality holds in (3.1) if and only if every eigenvalue of $S(G)$ is 0 or s_1 or s_n. In other words, the Seidel spectrum of G is

$$\left\{ s_1^{t_k}, 0^{t_0}, s_n^{n - t_0 - t_k} \right\},$$

for some t_0 and t_k. □

Remark 1. Applying Theorem 1 for any graph with $s(G) \leq n$, we obtain that $SE(G) \geq 2(n-1)$. Hence, the Haemers's conjecture [7] is true for any graph G of order n with $s(G) \leq n$.

In the next result, we present an upper bound for the Seidel energy of graphs in terms of order and the sum of powers of the eigenvalues of the Seidel matrix.

Theorem 2. *Let G be a graph of order n and $|s'_1| \geqslant |s'_2| \geqslant \cdots \geqslant |s'_n|$ be the absolute eigenvalues values of the Seidel matrix. Then*

$$SE(G) \leq n\sqrt{n-1} - \frac{|s'_n| \left(\sum_{i=1}^{n} s_i^4 - n(n-1)^2\right)}{4\left(s_1^4 + (n-1)^2\right)}. \tag{3.5}$$

Proof. If we take $a_i = |s_i|$ for $i = 1, 2, \ldots, n$ in inequality (2.2), we get

$$\begin{aligned}
\sqrt{\frac{s_1^2 + \cdots + s_n^2}{n}} &\geq \frac{|s_1| + \cdots + |s_n|}{n} + \frac{1}{4n} \sum_{i=1}^{n} \frac{(ns_i^2 - (s_1^2 + \cdots + s_n^2))^2}{n^2 s_i^4 + (s_1^2 + \cdots + s_n^2)^2} |s_i| \\
&\geq \frac{|s_1| + \cdots + |s_n|}{n} + \frac{|s'_n|}{4n} \sum_{i=1}^{n} \frac{(s_i^2 - (n-1))^2}{s_i^4 + (n-1)^2} \\
&\geq \frac{|s_1| + \cdots + |s_n|}{n} + \frac{|s'_n|}{4n} \sum_{i=1}^{n} \frac{(s_i^2 - (n-1))^2}{s_1^4 + (n-1)^2} \\
&= \frac{|s_1| + \cdots + |s_n|}{n} + \frac{|s'_n|}{4n\left(s_1^4 + (n-1)^2\right)} \sum_{i=1}^{n} (s_i^2 - (n-1))^2 \\
&= \frac{|s_1| + \cdots + |s_n|}{n} + \frac{|s'_n| \left(\sum_{i=1}^{n} s_i^4 - n(n-1)^2\right)}{4n\left(s_1^4 + (n-1)^2\right)}.
\end{aligned}$$

The last inequality leads to the desired bound. □

Remark 2. Note that by Lemma 2, we have $\sum_{i=1}^{n} s_i^4 - n(n-1)^2 \geq n(n-1)^2 - n(n-1)^2 = 0$, this means that inequality (3.5) is stronger than (2.1).

In the next result, we provide a general upper bound for the Seidel energy of graphs. (we recall that $S^k(G)$ is $\sum_{i=1}^{n} |s_i|^k$, where $s_1 \geq \cdots \geq s_n$ are the Seidel eigenvalues of G).

Theorem 3. *Let G be a graph of order $n \geq 3$ and with Seidel eigenvalues $s_1 \geq \cdots \geq s_n$. Then for any real $k \geq 2$,*

$$SE(G) \leq s_1 + (n-1)^{\frac{k-2}{k}} \left(S^{\frac{k}{2}}(G) - s_1^{\frac{k}{2}}\right)^{\frac{2}{k}}. \tag{3.6}$$

Proof. Setting $x_i = \sqrt{|s_i|}$ for $i = 2, 3, \ldots, n$ and $s = 2$ in Inequality (2.3) we have

$$\left(\sum_{i=2}^{n} |s_i|\right)^{\frac{k}{2}} \leq (n-1)^{\frac{k}{2}-1} \left(\sum_{i=2}^{n} |s_i|^{\frac{k}{2}}\right)$$

that is,

$$\sum_{i=2}^{n}|s_i| \le \left((n-1)^{\frac{k-2}{2}}\sum_{i=2}^{n}|s_i|^{\frac{k}{2}}\right)^{\frac{2}{k}}$$
$$= (n-1)^{\frac{k-2}{k}}\left(\sum_{i=2}^{n}|s_i|^{\frac{k}{2}}\right)^{\frac{2}{k}}.$$

This completes the proof.

□

Remark 3. Note that $|s_1| \ge |n-1-\frac{4m}{n}|$ [12], hence for $k=4$, the inequality (3.6) becomes

$$SE(G) \le |s_1| + \sqrt{(n-1)\left(n^2-n-s_1^2\right)}.$$

Since the function $f(x) = x+\sqrt{(n-1)\left(n^2-n-x_1^2\right)}$ decreases on the interval $|n-1-\frac{4m}{n}| \le x \le \sqrt{n^2-n}$, therefore we have

$$SE(G) \le |n-1-\frac{4m}{n}| + \sqrt{(n-1)\left(n^2-n-\left(n-1-\frac{4m}{n}\right)^2\right)}.$$

The above inequality was proven in [12]. This means that these results are corollaries of (3.6).

Since the proof of the next result is similar to Theorem 3, we omit its proof.

Theorem 4. *Let G be a graph of order $n \ge 3$. Then for any real $k \ge 2$,*

$$SE(G) \le n^{\frac{k-2}{k}}\left(S^{\frac{k}{2}}(G)\right)^{\frac{2}{k}}. \tag{3.7}$$

Remark 4. If we take $k=4$ in (3.7), then the inequality (2.1) is obtained. This means that (2.1) is a result of (3.7).

The next result gives a relationship between the Seidel energy of a graph G and $S^k(G)$.

Theorem 5. *Let G be a graph of order n and $r \in \mathbb{R}\setminus(0,1)$. Assume that n is even. Then*

$$SE(G) \le \sqrt[r]{(n(n-1))^{r-1}\,S^{2-r}(G)} \tag{3.8}$$

and the equality holds if and only if G is a conference graph.

Proof. Since n is even, all Seidel eigenvalues of G are non-zero [1]. Assume that $s_1, \ldots, s_n$ are Seidel eigenvalues of G. Thus by the first part of Lemma 2, $s_1^2 + \cdots + s_n^2 = n(n-1)$. Let $p_i = s_i^2$ and $a_i = \frac{1}{|s_i|}$, for $i = 1, \ldots, n$. Using inequality (2.4) of Lemma 5, we obtain that

$$(n(n-1))^{r-1} \sum_{i=1}^{n} |s_i|^{2-r} \geq \left(\sum_{i=1}^{n} |s_i| \right)^r \tag{3.9}$$

and the equality holds if and only if $r = 0$ or $r = 1$ or $|s_1| = \cdots = |s_n|$. This shows that

$$SE(G) \leq \sqrt[r]{(n(n-1))^{r-1} \, S^{2-r}(G)}$$

and the equality holds if and only if $|s_1| = \cdots = |s_n|$. Now we investigate the equality. Suppose thta all eigenvalues of $S(G)$ have equal absolute value. Then the spectrum of $S(G)$ can be written as

$$\{\underbrace{\alpha, \ldots, \alpha}_{k}, \underbrace{-\alpha, \ldots, -\alpha}_{n-k}\}, \qquad \alpha > 0.$$

Since $\mathrm{tr}(S(G)) = 0$, we obtain that $(2k - n)\alpha = 0$. Thus $k = \frac{n}{2}$. Since $\sum_{i=1}^{n} s_i^2 = n(n-1)$, we find that $\alpha^2 = n - 1$. Hence the spectrum of $S(G)$ is exactly

$$\{\underbrace{\sqrt{n-1}, \ldots, \sqrt{n-1}}_{n/2}, \underbrace{-\sqrt{n-1}, \ldots, -\sqrt{n-1}}_{n/2}\}.$$

Since $S(G)$ is a real symmetric matrix, it is diagonalizable. Its minimal polynomial divides $x^2 - (n-1)$, so $S(G)^2 = (n-1)I$. By definition, a symmetric $\{0, \pm 1\}$-matrix S with zero diagonal and satisfying $S^2 = (n-1)I$ is a conference matrix. Therefore $S(G)$ is the Seidel matrix of a conference graph.
Conversely, if G is a conference graph, then $S(G)^2 = (n-1)I$ and every eigenvalue satisfies $\lambda^2 = n - 1$, so $s_i = \pm\sqrt{n-1}$. Thus all eigenvalues have equal absolute value. This completes the proof.

□

Remark 5. Note that Inequality (2.1) is an immediate consequence of Theorem 5, for $r = 2$.

Corollary 1. *(Theorem 3.9. [1] also Theorem 4.1. [9]) For any graph G of order $n \geq 2$,*

$$SE(G) \geq \sqrt{n(n-1) + n(n-1)\left(|\det S(G)|\right)^{2/n}}. \tag{3.10}$$

In the next theorem , we determine a lower bound on the $SE(G)$ in terms of the order of G and $|\det S(G)|$.

Theorem 6. *Let G be a graph of order $n \geq 3$. Then*

$$SE(G) \geq \sqrt{n(n-1)+n(n-1)\left(|\det S(G)|\right)^{2/n}+\frac{4}{n^2-n-2}\sum_{i<j,k<l \text{ and } i<k \text{ or } i=k \text{ and } j<l}\left(\sqrt{|s_i||s_j|}-\sqrt{|s_k||s_l|}\right)^2}.$$

Proof. First note that

$$SE(G)^2=\sum_{i=1}^{n}|s_i|^2+2\sum_{i<j}|s_i||s_j|. \tag{3.11}$$

Setting $N=\frac{n(n-1)}{2}$ and

$$(a_1,a_2,\ldots,a_N)=(|s_1||s_2|,|s_1||s_3|,\ldots,|s_1||s_n|,\ldots,|s_2||s_n|,\ldots,|s_{n-1}||s_n|)$$

in Lemma 6, we obtain

$$\sum_{1\leq i<j\leq n}|s_i||s_j| \geq \frac{n(n-1)}{2}\left(\prod_{i=1}^{n}|s_i|\right)^{2/n} + \frac{2}{n^2-n-2}\sum_{i<j,k<l \text{ and } i<k \text{ or } i=k \text{ and } j<l}\left(\sqrt{|s_i||s_j|}-\sqrt{|s_k||s_l|}\right)^2$$

yielding

$$2\sum_{1\leq i<j\leq n}|s_i||s_j| \geq n(n-1)\left(\det S(G)\right)^{2/n} + \frac{4}{n^2-n-2}\sum_{i<j,k<l \text{ and } i<k \text{ or } i=k \text{ and } j<l}\left(\sqrt{|s_i||s_j|}-\sqrt{|s_k||s_l|}\right)^2.$$

Combining the above inequality with (3.11) leads to the desired inequality. □

Remark 6. Since $\left(\sqrt{|s_i||s_j|}-\sqrt{|s_k||s_l|}\right)^2 \geq 0$, we have

$$\begin{aligned}SE(G) &\geq \sqrt{n(n-1)+n(n-1)\left(|\det S(G)|\right)^{2/n}+\frac{4}{(n+1)(n-2)}\sum_{i<j,k<l \text{ and } i<k \text{ or } i=k \text{ and } j<l}\left(\sqrt{|s_i||s_j|}-\sqrt{|s_k||s_l|}\right)^2} \\ &\geq \sqrt{n(n-1)+n(n-1)\left(|\det S(G)|\right)^{2/n}}.\end{aligned}$$

Therefore the bound of Theorem 6 is better than the bound of Corollary 1.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] S. Akbari, J. Askari, and K.C. Das, *Some properties of eigenvalues of the Seidel matrix*, Linear Multilinear Algebra **70** (2022), no. 11, 2150–2161. https://doi.org/10.1080/03081087.2020.1790481.

[2] S. Akbari, M. Einollahzadeh, M.M. Karkhaneei, and M.A. Nematollahi, *Proof of a conjecture on the Seidel energy of graphs*, European J. Combin. **86** (2020), 103078. https://doi.org/10.1016/j.ejc.2019.103078.

[3] D.S. Bernstein, *Matrix mathematics: Theory, facts and formulas*, Princeton University Press, Princeton, USA, 2009.

[4] S. Filipovski, *Improved Cauchy-Schwarz inequality and its applications*, Turkish J. Ineq. **3** (2019), no. 2, 8–12.

[5] E. Ghorbani, *On eigenvalues of Seidel matrices and Haemers' conjecture*, Des. Codes Cryptogr. **84** (2017), no. 1, 189–195. https://doi.org/10.1007/s10623-016-0248-x.

[6] G. Greaves, J.H. Koolen, A. Munemasa, and F. Szöllősi, *Equiangular lines in Euclidean spaces*, J. Combin. Theory Ser. A **138** (2016), 208–235. https://doi.org/10.1016/j.jcta.2015.09.008.

[7] W.H. Haemers, *Seidel switching and graph energy*, MATCH Commun. Math. Comput. Chem. **68** (2012), 653–659.

[8] A. Iranmanesh and J.A. Farsangi, *Upper and lower bounds for the power of eigenvalues in Seidel matrix*, J. Appl. Math. Informatics **33** (2015), no. (5-6), 627–633. https://doi.org/10.14317/jami.2015.627.

[9] M.R. Kanna, R.P. Kumar, and M.R. Farahani, *Milovanovic bounds for Seidel energy of a graph*, Advances in Theoretical and Applied Mathematics **10** (2016), no. 1, 37–44.

[10] H. Kober, *On the arithmetic and geometric means and the Hölder inequality*, Proc. Am. Math. Soc. **59** (1958), 452–459.

[11] D.S. Mitrinović, J.E. Pecaric, and A.M. Fink, *Classical and new inequalities in analysis*, Kluwer Academic Publishers, Dordrecht, 2013.

[12] P. Nageswari and P.B. Sarasija, *Seidel energy and its bounds*, Int. J. Math. Anal. **8** (2014), no. 57, 2869–2871. http://dx.doi.org/10.12988/ijma.2014.410309.

[13] M.R. Oboudi, *Energy and seidel energy of graphs*, MATCH Commun. Math. Comput. Chem. **75** (2016), no. 2, 291–303.

[14] ———, *Seidel energy of complete multipartite graphs*, Spec. Matrices **9** (2021), no. 1, 212–216.

[15] ———, *On the seidel estrada index of graphs*, Linear Multilinear Algebra **72** (2024), no. 10, 1625–1632. https://doi.org/10.1080/03081087.2023.2192459.

[16] M.R. Oboudi and M.A. Nematollahi, *Improving a lower bound for Seidel energy*

of graphs, MATCH Commun. Math. Comput. Chem. **89** (2023), no. 2, 489–502. https://doi.org/10.46793/match.89-2.489O.

[17] J.H. Van Lint and J.J. Seidel, *Equilateral point sets in elliptic geometry*, Indag. Math. **28** (1966), 335–348.