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Research Article

A class of Unit $\mathbb{Z}_r$-Simplex codes

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Abstract: In this paper, we determine the parameters of the unit $\mathbb{Z}_r$-Simplex code under the homogeneous weight metric and show that it is an $\left[\frac{\varrho^{k-1}}{\varrho-1}, k, \varrho^{k-1}((p-1)p^{m-2})\right]$ Z_r-Simplex code when $m = 2$, and $\left[\frac{\varrho^{k-1}}{\varrho-1}, k, \varrho^{k-2}((p-2)p^{2m-2}+2p^{m-1})\right]$ when $m > 2$, where $r = p^m$ and the code has rank k. Furthermore, we derive the weight distribution of the $\mathbb{Z}_r$-Simplex code under the homogeneous weight metric for the specific case of rank $k = 2$.

Keywords: $\mathbb{Z}_r$-linear code; unit $\mathbb{Z}_r$-Simplex code; homogeneous weight; weight distribution

AMS Subject classification: 94B05, 11T71, 94B60

1. Introduction

A fascinating field of study within the field of coding theory, which deals with error-correcting codes, is linear codes over finite chain rings. These rings provide an interesting framework for the development of linear codes with unique characteristics and uses. A major contribution to coding theory is the study of codes over finite rings, which has produced several families of nonlinear codes by using Gray images of linear codes over finite rings [8, 9, 14, 24]. Over the past few decades, a number of coding

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theorists have been researching codes over $\mathbb{Z}_4$ [1–3, 6] and Hammons et al. [17] have made a significant contribution to codes over $\mathbb{Z}_4$.

The concept of a homogeneous weight metric is a generalisation of the traditional Hamming weight metric. It is designed to extend the concept of distance from codes over finite fields to codes over finite rings and more general algebraic structures. In 1950, Richard Hamming introduced the Hamming weight [16], which counts the number of nonzero entries in a vector over a finite field. In contrast, the homogeneous weight offers a natural way to define a meaningful weight function on codes over finite rings. The primary goal was to assign weights to ring elements in a manner that conserves desirable attributes, such as linearity and compatibility with Gray maps. Homogeneous weights preserve distances in code equivalencies by frequently aligning with Gray isometries, which are mappings between various algebraic structures [18]. Homogeneous weights on integer residue rings were first introduced by Constantinescu [5] and Heise et al. [25], who later extended them to finite Frobenius rings. Carlet, in [4] expanded on this idea to $\mathbb{Z}_{2^k}$ with homogeneous weight, which helped to identify the generalized Kerdock codes. These codes were nonlinear binary codes with significant minimum distances. The idea of Gray maps was also expanded upon by Greferath et al and Ling et al., in [13, 21, 26], to include more general rings with particular homogeneous weights defined on them. In [21], Ling and Blackford notably presented a Gray map from $\mathbb{Z}^n_{p^{k+1}}$ to $\mathbb{Z}_p^{n(p^k)}$. Additionally, in [32], the author described a novel combinatorial approach for the coordinate-wise construction of the homogeneous weight preserving Gray map for Galois rings using fundamental tools from Affine-Geometries. Notably, the Gray image of the extended cyclic codes over $\mathbb{Z}_4$ and $\mathbb{Z}_{\mathrm{p}^s}$ yielded several new families of nonlinear codes.

In the discipline of coding theory, the study of weight distribution holds significant prominence since it serves as an intriguing research area offering insights into the error detection and correction capabilities of codes. Being able to calculate the error probabilities associated with error detection and correction through weight distribution Manish Gupta et al. in [15] investigated on punctured versions of Simplex codes, specifically focusing on types α and β. Additionally, binary linear codes, MDS codes and convolutional codes are some popular linear codes having their weight distributions discovered by several authors [7, 12, 19, 22, 29–31]. Recent research includes the examination of $[n, k, d]$ $\mathbb{Z}_q$-Simplex codes under the Hamming distance for prime power values of q in [27], with subsequent extensions presented in [11], considering any natural number $q \geq 2$, and obtaining $\{A_j\}_{j=1}^n$ for $\mathbb{Z}_q$-Simplex codes when $q = \mathrm{p}^m$ or p is a product of primes and $k = 2$. Durairajan et al. [10] further analyzed the parameters of punctured $\mathbb{Z}_q$-Simplex codes for $k = 2$ and examined $\{A_j\}_{j=1}^n$ of punctured $\mathbb{Z}_q$-Simplex codes in the same context. Moreover, in [23, 28], the parameters $[n, k, d]$ and $\{A_j\}_{j=1}^n$ of certain punctured versions of $\mathbb{Z}_q$-Simplex codes were thoroughly investigated.

Inspired by previous research, we have achieved the following results. Preliminaries and a few essential properties are presented in Section 2. In Section 3, we determined the parameters of unit $\mathbb{Z}_r$-Simplex code with respect to homogeneous weight metric for rank 2, where $r = \mathrm{p}^m$ and then extended the same for rank k. Further, in Section 4, its corresponding weight distribution has been evaluated for rank 2, where $r = \mathrm{p}^m$. In Section 5, the comparison of Gray images of $\mathbb{Z}_r$-Simplex code and unit $\mathbb{Z}_r$-Simplex code concerning the code rate is provided.

2. Preliminaries

Let $r = p^m$, where p is a prime number. A code C is a subset of $\mathbb{Z}_r^n$ represented using its parameters (n, r^k, d_{Hom}), where n is length, r^k is size and d_{Hom} is minimum non-zero homogeneous distance. If C is a submodule of $\mathbb{Z}_r^n$ then C is said to be a $\mathbb{Z}_r$-*linear code*. A matrix G is referred to be the generator matrix of a $\mathbb{Z}_r$-linear code C if each row in the matrix G represents a basis element of the code. An $[n, k, d_{Hom}]$ $\mathbb{Z}_r$-linear code is a $\mathbb{Z}_r$-linear code with length n, rank k and minimum homogeneous distance d_{Hom}. The weight distribution of code C can be expressed as $(1,\ A_1,\ A_2,\ A_3,\ \ldots,\ A_n)$ if A_i is the cardinality of the codewords in C with homogeneous weight i of length n. A code with s nonzero A_i in the weight distribution is said to be s-weight code.

Let C be a $\mathbb{Z}_r$-linear code and $c_1, c_2 \in C$. Since $c_1 - c_2 \in C$, the minimum homogeneous distance d_{Hom} of C is

$$\min\{d_{Hom}(c_1, c_2) \mid c_1, c_2 \in C,\ c_1 \neq c_2\} = \min\{wt_{Hom}(c) \mid c \in C,\ c \neq 0\}.$$

That is, $d_{Hom}(C)$ and the minimum homogeneous weight denoted by wt_{Hom} of the code C will be equal. Let $o(y)$ denote the additive order of y, where $y \in \mathbb{Z}_r$ and $N_r(c)$ denotes the number of coordinates of additive order r in the codeword c.

The generalised Gray map of finite chain rings under the homogeneous metric was developed in [13]. For this occasion, the homogeneous weight of $\mathbb{Z}_{\mathrm{p}^m}$ is defined in [5] as

$$wt_{Hom}(y) = \begin{cases} 0 \quad \text{if} \quad y = 0, \\ \mathrm{p}^{m-1} \quad \text{if} \quad y \in \mathrm{p}^{m-1}\mathbb{Z}_{\mathrm{p}^m} \backslash \{0\}, \\ \mathrm{p}^{m-2}(\mathrm{p}-1) \quad \text{otherwise.} \end{cases}$$

For $j = 1$, define $M_j = (1)$. For $m \geq 2$, the sequence of matrices M_j is constructed recursively as $M_j = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \cdots & \boldsymbol{p-1} \\ M_{j-1} & M_{j-1} & \cdots & M_{j-1} \end{bmatrix}$.
The Gray map $\psi : \mathbb{Z}_{p^m} \to \mathbb{F}_p^{p^{(m-1)}}$, defined in [21], is given by

$$\begin{aligned} \psi(a) &= \psi(\alpha_0 + p\alpha_1 + \cdots + p^{m-1}\alpha_{m-1}) \\ &= \begin{bmatrix} \alpha_0 & \alpha_1 & \cdots & \alpha_{m-1} \end{bmatrix} M_m, \end{aligned}$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{m-1} \in \{0, 1, 2, \ldots, p-1\} \subset \mathbb{Z}_{\mathrm{p}^{m-1}}$. The mapping ψ interprets an isometry from $(\mathbb{Z}_{p^m},\, wt_{Hom})$ to $\left(\mathbb{F}_p^{p^{(m-1)}},\, wt_{Ham}\right)$, that is, $wt_{Hom}(x-y) = wt_{Ham}(\psi(x)-\psi(y))$.

3. Parameters of unit $\mathbb{Z}_r$-Simplex code under the homogeneous weight metric

In this section, we obtain the various parameters of unit $\mathbb{Z}_r$-Simplex code under the homogeneous weight metric.
Let

$$\mathbb{G}_2'(r) = \left[\begin{array}{c|c|cccc} 1 & 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{\phi(r)} \\ \hline 0 & 1 & 1 & 1 & \cdots & 1 \end{array}\right],$$

where $\phi(r)$ is the Euler's ϕ-function. Obviously, the rank of $\mathbb{G}_2'(r)$ is 2 and the number of columns in the matrix $\mathbb{G}_2'(r)$ is $\phi(r)+2$, which represents the length of the $\mathbb{Z}_r$-linear code. Let $x = (10\alpha_1\alpha_2\cdots\alpha_{\phi(r)})$ and $y = (01111\cdots1)$. Then, $\mathbf{C} = \{\, \lambda x + \eta y \mid \lambda, \eta \in \mathbb{Z}_r \,\}$. Now, we define inductively

$$\mathbb{G}_k'(r) = \left[\begin{array}{c|c|c|c|c|c} 1 & 00\cdots0 & \alpha_1\alpha_1\cdots\alpha_1 & \alpha_2\alpha_2\cdots\alpha_2 & \cdots & \alpha_{\phi(r)}\alpha_{\phi(r)}\cdots\alpha_{\phi(r)} \\ \hline 0 & & & & & \\ \vdots & \mathbb{G}_{k-1}'(r) & \mathbb{G}_{k-1}'(r) & \mathbb{G}_{k-1}'(r) & \cdots & \mathbb{G}_{k-1}'(r) \\ 0 & & & & & \end{array}\right]$$

for $k \geq 2$. Clearly, the matrix $\mathbb{G}_k'(r)$ generates $[n_k = \frac{\varrho^k-1}{\varrho-1}, k, d_{Hom}]$ $\mathbb{Z}_r$-linear code of rank k, where $\varrho = \phi(r)+1$. The code $S_k(r)$ generated by the matrix $\mathbb{G}_k'(r)$ is called unit $\mathbb{Z}_r$-Simplex code of rank k. Hence,

$$S_k(r) = \{\eta(1\,\mathbf{0}\,\boldsymbol{\alpha_1}\,\boldsymbol{\alpha_2}\ldots\boldsymbol{\alpha}_{\phi(\mathbf{r})}) + (0\underbrace{c\,c\ldots\,c}_{\phi(r)+1}) \mid \eta \in \mathbb{Z}_r, c \in S_{k-1}(r),$$
$$\boldsymbol{\alpha_i} = \alpha_i\alpha_i\cdots\alpha_i, \mathbf{0} = 00\cdots0 \in \mathbb{Z}_r^{n_{k-1}}\}.$$

Consider the sets $\mathbb{E}_{\mathrm{p}^j} = \{y \in \mathbb{Z}_r \mid o(y) = \mathrm{p}^j\}$ for $0 \leq j \leq m$. Now, the sets $\{\mathbb{E}_{\mathrm{p}^j} \mid 0 \leq j \leq m\}$ form a partition of $\mathbb{Z}_r$. Furthermore, observe that the set $\mathbb{E}_{\mathrm{p}} \cup \{0\}$ with the operation of addition forms a group. Additionally, when $2 \leq j \leq m$, the set $\mathbb{E}_{p^j}$ is closed under additive inverse.

Lemma 1. *Let C be an $[n, k, d_{Hom}]$ unit $\mathbb{Z}_r$-Simplex code generated by the matrix $\mathbb{G}_k'(r)$. Let $c \in C$ and $\eta \in Z_r$ such that $o(\eta) = p^j$, where $1 \leq j \leq m$. If $c' = \eta(1\,\mathbf{0}\,\boldsymbol{\alpha_1}\,\boldsymbol{\alpha_2}\,\cdots\,\boldsymbol{\alpha}_{\phi(\mathbf{r})})+$*

$(0\underbrace{c\,c\ldots\,c}_{\phi(r)+1})$, *then*

$$wt_{Hom}(c^{'}) = \begin{cases} wt_{Hom}(c)+h[1+n(p^m-p^{m-1})] & \text{if } o(\eta)=p^m,\\ wt_{Hom}(c)+h^2p\sum_{i=2}^{m}N_{p^i}(c)+p^{m-1}[(p-2)p^{m-1}N_p(c) & \\ \qquad +phN_1(c)+1] & \text{if } o(\eta)=p,\\ wt_{Hom}(c)+ph^2(n-N_{p^j}(c)) & \\ \qquad +hp^{m-j+1}\left[N_{p^j}(c)(p^{j-2}-1)+1\right]+h & \text{otherwise,} \end{cases}$$

where $h=(p-1)p^{m-2}$ *and* $N_r(c)$ *denotes the number of* r*-th order elements in* c.

Proof. We consider the following cases.

Case i: $o(\eta)=p^m$. Then,

$$\begin{aligned} wt_{Hom}(c^{'}) &= p^{m-1}N_p(c)+(p-1)p^{m-2}[n-N_p(c)-N_1(c)]+p^{m-1}(p-1)N_{p^m}(c)\\ &\quad +(p-1)p^{m-2}(p^m-p^{m-1}-p)N_{p^m}(c)\\ &\quad +(p-1)p^{m-2}(p^m-p^{m-1})[n-N_{p^m}(c)]+(p-1)p^{m-2}\\ &= wt_{Hom}(c)+(p^{m-2}(p-1)N_{p^m}(c))\big[p+(p^m-p^{m-1})-p\\ &\quad -(p^m-p^{m-1})\big]+p^{m-2}(p-1)(p^m-p^{m-1})n+p^{m-2}(p-1)\\ &= wt_{Hom}(c)+h[1+n(p^m-p^{m-1})]. \end{aligned}$$

Case ii: $o(\eta)=p$. Then,

$$\begin{aligned} wt_{Hom}(c^{'}) &= p^{m-1}N_p(c)+(p-1)p^{m-2}[n-N_p(c)-N_1(c)]\\ &\quad +(p-1)p^{m-2}(p^m-p^{m-1})\left(\sum_{i\,=\,2}^{m}N_{p^i}(c)\right)+p^{m-1}(p-2)p^{m-1}N_p(c)\\ &\quad +p^{m-1}(p^m-p^{m-1})N_1(c)+p^{m-1}\\ &= wt_{Hom}(c)+h^2p\left(\sum_{i=2}^{m}N_{p^i}(c)\right)+p^{m-1}[(p-2)p^{m-1}N_p(c)+phN_1(c)+1]. \end{aligned}$$

Case iii: $o(\eta)=p^j$, where $2\le j\le m-1$. Then,

$$\begin{aligned} wt_{Hom}(c^{'}) =& p^{m-1}N_p(c)+(p-1)p^{m-2}[n-N_p(c)-N_1(c)]\\ &+(p-1)p^{m-2}(p^m-p^{m-1})\left(\sum_{i\,=\,j+1}^{m}N_{p^i}(c)\right)\\ &+(p-1)p^{m-2}\sum_{i\,=\,2}^{j-1}(p^i-p^{i-1})\left(N_{p^j}(c)\right)+p^{m-1}(p-1)\left(\frac{\phi(r)}{\phi(p^j)}\right)\\ &+(p-1)p^{m-2}(p^m-p^{m-1})\left(\sum_{i\,=\,1}^{j-1}N_{p^i}(c)\right) \end{aligned}$$

$$
\begin{aligned}
&+ (p-1)p^{m-2}(p^m - p^{m-1})N_1(c) + (p-1)p^{m-2}\\
=&wt_{Hom}(c) + ph^2(n - N_{p^j}(c)) + hp^{m-j+1}\left[N_{p^j}(c)(p^{j-2}-1)+1\right] + h.
\end{aligned}
$$

□

Theorem 1. *Let $S_2(r)$ be a unit $\mathbb{Z}_r$-Simplex code generated by the matrix $\mathbb{G}_2'(r)$. If $r = p^m$ then the parameters of the code are*

$$
\begin{cases}
\left[\phi(r)+2,\ 2\ , (p-1)(p^2-p+1))\right] & \text{when } m = 2,\\
\left[\phi(r)+2,\ 2\ , ((p-2)p^{2m-2} + 2p^{m-1})\right] & \text{when } m > 2.
\end{cases}
$$

Proof. We know that $S_2(r) = \{\lambda(1\,0\,\alpha_1\,\alpha_2 \cdots \alpha_{\phi(r)}) + \eta(1\,0\,\underbrace{1\,1\cdots 1}_{\phi(r)}) \mid \lambda, \eta \in \mathbb{Z}_r\}$.

Now, consider the following cases:
Case i: $\lambda = 0$ and $\eta = 0$. Then, $wt_{Hom}(\lambda x + \eta y) = 0$.
Case ii: $\lambda \neq 0$ and $\eta = 0$. Then, for $\lambda \notin p^{m-1}\mathbb{Z}_r$, we have

$$
\underset{\lambda\in\mathbb{Z}_r}{wt_{Hom}}(\lambda x) = \underset{\lambda\in\mathbb{Z}_r}{wt_{Hom}}(\lambda(1\,0\,\alpha_1\,\alpha_2\ \cdots\ \alpha_{\phi(r)})) = (p-1)p^{m-2}(p^m - p^{m-1} + 1).
$$

Case iii: $\lambda = 0$ and $\eta \neq 0$. Then, for $\eta \notin p^{m-1}\mathbb{Z}_r$,

$$
\underset{\eta\in\mathbb{Z}_r}{wt_{Hom}}(\eta y) = \underset{\eta\in\mathbb{Z}_r}{wt_{Hom}}(\eta(1\,0\,1\,1\ \cdots\ 1)) = (p-1)p^{m-2}(p^m - p^{m-1} + 1).
$$

Case iv: $\lambda \neq 0$ and $\eta \neq 0$. Then, for $\lambda \in p^{m-1}\ \mathbb{Z}_r$ and $\eta \in p^{m-1}\ \mathbb{Z}_r$,

$$
\underset{\lambda,\eta\in\mathbb{Z}_r}{wt_{Hom}}(\lambda x + \eta y) = \underset{\lambda,\eta\in\mathbb{Z}_r}{wt_{Hom}}(\lambda(1\,0\,\alpha_1\,\alpha_2\ \cdots\ \alpha_{\phi(r)}) + \eta(1\,0\,1\,1\ \cdots\ 1)) = (p-2)p^{2m-2} + 2p^{m-1}.
$$

By the above four cases, if $r = p^2$ then the minimum homogeneous distance for rank $k = 2$ is

$$
\begin{aligned}
d_{Hom}(S_2(r)) &= \min\{(p-1)p^{m-2}(p^m - p^{m-1} + 1), p^{m-1}[(p-2)p^{m-1} + 2]\}\\
&= (p-1)p^{m-2}(p^m - p^{m-1} + 1),
\end{aligned}
$$

and if $r = p^m$, where $m > 2$, then the minimum homogeneous distance for rank $k = 2$ is

$$
\begin{aligned}
d_{Hom}(S_2(r)) &= \min\{(p-1)p^{m-2}(p^m - p^{m-1} + 1), p^{m-1}[(p-2)p^{m-1} + 2]\}\\
&= (p-2)p^{2m-2} + 2p^{m-1}.
\end{aligned}
$$

□

Example 1. By Theorem 1, the parameters $[n, k, d_{Hom}]$ of the code generated by $\mathbb{G}_2'(r)$ when p $= 2, 3$ and 5 and $m = 2, 3, 4$ and 5 are given in the Tables 1-3.

Table 1. **Parameters of $S_2(p^m)$ for $m=2$**

$S_2(r)$	$[\phi(r)+2\ ,\ \ 2\ ,\ (p-1)p^{m-2}(p^m-p^{m-1}+1)]$
$p=2$	[4, 2, 3]
$p=3$	[8, 2, 14]
$p=5$	[22, 2, 84]

Table 2. **Parameters of $S_2(p^m)$ for $m=3$**

$S_2(r)$	$[\phi(r)+2\ ,\ \ 2\ ,\ (p-2)p^{2m-2}+2p^{m-1}]$
$p=2$	[6, 2, 8]
$p=3$	[20, 2, 99]
$p=5$	[102, 2, 1925]

Table 3. **Parameters of $S_2(p^m)$ for $m=4$**

$S_2(r)$	$[\phi(r)+2\ ,\ \ 2\ ,\ (p-2)p^{2m-2}+2p^{m-1}]$
$p=2$	[10, 2, 16]
$p=3$	[56, 2, 783]
$p=5$	[502, 2, 47125]

Theorem 2. *Let $r=p^m$, where p is a prime number. Then the parameters of $S_k(r)$ are*

$$\begin{cases} \left[\frac{\varrho^k-1}{\varrho-1},\ k\ ,\varrho^{k-1}(p-1)\right] & \text{if } \ m=2, \\ \left[\frac{\varrho^k-1}{\varrho-1},\ k\ ,\varrho^{k-2}((p-2)p^{2m-2}+2p^{m-1})\right] & \text{if } \ m>2, \end{cases}$$

where $\varrho=\phi(r)+1$.

Proof. Let us prove this theorem using induction on k. By Theorem 1, the assertion holds for $S_2(k)$. Assume that the assertion is true for all j less than k. Now, we show that the result is true for k. Based on the assumption, $S_j(r)$ is an $\left[\dfrac{\varrho^j-1}{\varrho-1}, j, \varrho^{j-1}[(p-1)]\right]$ $\mathbb{Z}_r$-linear code for $m=2$ and an $\left[\dfrac{\varrho^j-1}{\varrho-1}, j, \varrho^{j-2}[(p-2)p^{2m-2}+2p^{m-1}]\right]$ $\mathbb{Z}_r$-linear code for $m>2$. Now, let $S_{k-1}(r)$ be an $[n_{k-1}, k-1, d_{Hom}]$ linear code and

$$S_k(r)=\{\eta(1\,\mathbf{0}\,\boldsymbol{\alpha_1}\,\boldsymbol{\alpha_2}\ \cdots\ \boldsymbol{\alpha}_{\phi(\mathbf{r})})+(0\,\underbrace{c\,c\,\cdots\,c}_{\phi(r)+1})\mid \eta\in\mathbb{Z}_r, c\in S_{k-1}(r)\}.$$

Then,

$$d_{Hom}(S_k(r)) = \min\{wt_{Hom}(\eta(1\mathbf{0}\boldsymbol{\alpha}_\mathbf{1}\,\boldsymbol{\alpha}_\mathbf{2}\,\cdots\,\boldsymbol{\alpha}_{\phi(\mathbf{r})})+(0\underbrace{c\,c\,\cdots\,c}_{\phi(r)+1}) \mid \eta \in \mathbb{Z}_r, c \in S_{k-1}(r)\}.$$

Now, consider the following cases:
Case i: $\eta = 0$. Then,

$$\min\{wt_{Hom}(0\,c\,c\,\cdots\,c) \mid c \in C, c \neq \mathbf{0}\} = \begin{cases} \varrho^{k-1}(p-1) & \text{if} \quad m = 2, \\ \varrho^{k-2}((p-2)p^{2m-2} + 2p^{m-1}) & \text{if} \quad m > 2. \end{cases}$$

Case ii: $c = 0$.
Subcase i: If $\eta \in \mathbb{E}_{p^j}$, where $2 \leq j \leq m$, then

$$\begin{aligned} wt_{Hom}(\eta(1\,\mathbf{0}\,\boldsymbol{\alpha}_\mathbf{1}\boldsymbol{\alpha}_\mathbf{2}\cdots\boldsymbol{\alpha}_{\phi(\mathbf{r})})) &= \left[\left(\frac{\varrho^k - 1}{\varrho - 1}\right) - \left(\frac{\varrho^{k-1} - 1}{\varrho - 1}\right)\right]((p-1)p^{m-2}) \\ &= \varrho^{k-1}((p-1)p^{m-2}). \end{aligned}$$

Subcase ii: If $\eta \in \mathbb{E}_p$ then

$$wt_{Hom}(\eta(1\,\mathbf{0}\,\boldsymbol{\alpha}_\mathbf{1}\boldsymbol{\alpha}_\mathbf{2}\cdots\boldsymbol{\alpha}_{\phi(\mathbf{r})})) = \left[\left(\frac{\varrho^k - 1}{\varrho - 1}\right) - \left(\frac{\varrho^{k-1} - 1}{\varrho - 1}\right)\right]p^{m-1} = \varrho^{k-1}p^{m-1}.$$

Case iii $\eta \neq 0$, and $c \neq \mathbf{0}$. Set $c' = \eta(1\,\mathbf{0}\,\boldsymbol{\alpha}_\mathbf{1}\,\boldsymbol{\alpha}_\mathbf{2}\,\cdots\,\boldsymbol{\alpha}_{\phi(\mathbf{r})}) + (0\,c\,c\,\cdots\,c)$. By Lemma 1, we have
Subcase i: If $\eta \in \mathbb{E}_{\mathrm{p}^m}$ then

$$\begin{aligned} wt_{Hom}(c') &= \min\{wt_{Hom}(c) + h[1 + n_{k-1}(p^m - p^{m-1}) - pN_{p^m}(c)]\} \\ &= p^{m-2}(p-1)\left[\varrho^{k-2} + \frac{\varrho^{k-1} - 1}{\varrho - 1}(p^m - p^{m-1} - p) + 1\right]. \end{aligned}$$

Subcase ii: If $\eta \in \mathbb{E}_{p^j}$, where $2 \leq j \leq m$, then

$$\begin{aligned} wt_{Hom}(c') &= \min\{wt_{Hom}(c) + ph^2(n_{k-1} - N_{p^j}(c)) \\ &\quad + hp^{m-j+1}\left[N_{p^j}(c)(p^{j-2} - 1) + 1\right] + h\} \\ &= p^{m-2}(p-1)\Bigg[\varrho^{k-2} + p^{m-1}(p-1)\left(\frac{\varrho^{k-2} - 1}{\varrho - 1}\right) \\ &\quad + p^{m-j+1}(\varrho^{k-2}(p^{j-2} - 1) + 1) + 1\Bigg]. \end{aligned}$$

Subcase iii: If $\eta \in \mathbb{E}_p$ then

$$\begin{aligned} wt_{Hom}(c^{'}) &= \min\{wt_{Hom}(c) + h^2 p\left(\sum_{i=2}^{m} N_{p^i}(c)\right) \\ &\quad + p^{m-1}[(p-2)p^{m-1}N_p(c) + phN_1(c) + 1]\} \\ &= p^{m-1}\left[\varrho^{k-2} + (p-2)p^{m-1}\varrho^{k-2} + (p^m - p^{m-1})\left(\frac{\varrho^{k-2}-1}{\varrho-1} + 1\right)\right]. \end{aligned}$$

Thus,

$$d_{Hom}(S_k(r)) = \min\left\{\varrho^{k-2}((p-2)p^{2m-2} + 2p^{m-1}),\ \varrho^{k-1}((p-1)p^{m-2}), \varrho^{k-1}p^{m-1}\right\}.$$

$$d_{Hom}(S_k(r)) = \begin{cases} \varrho^{k-1}(p-1) & \text{if} \quad r = p^2, \\ \varrho^{k-2}((p-2)p^{2m-2} + 2p^{m-1}) & \text{otherwise.} \end{cases}$$

Hence, $S_k(r)$ is a $\left[n_k = \frac{\varrho^k - 1}{\varrho - 1}, k, \varrho^{k-1}(p-1)\right]$ $\mathbb{Z}_r$-linear code if $k > 2$ and $m = 2$, and a $\left[n_k = \frac{\varrho^k - 1}{\varrho - 1}, k, \varrho^{k-2}((p-2)p^{2m-2} + 2p^{m-1})\right]$ $\mathbb{Z}_r$-linear code if $k > 2$ and $m > 2$. □

Example 2. Let $r = 25$. Then the generator matrix of $S_2(25)$ is

$$\mathbb{G}_2'(25) = \left[\begin{array}{c|c|cccc} 1 & 0 & 1 & 2 & \cdots & 24 \\ \hline 0 & 1 & 1 & 1 & \cdots & 1 \end{array}\right].$$

Moreover,

$$\mathbb{G}_k'(25) = \left[\begin{array}{c|c|c|c|c|c} 1 & 0 \quad 0\cdots 0 & 1 \quad 1\cdots 1 & 2 \quad 2\cdots 2 & \cdots & 24 \quad 24\cdots 24 \\ \cline{2-6} 0 & & & & & \\ \vdots & \mathbb{G}_{k-1}'(25) & \mathbb{G}_{k-1}'(25) & \mathbb{G}_{k-1}'(25) & \cdots & \mathbb{G}_{k-1}'(25) \\ 0 & & & & & \end{array}\right]$$

for $k > 2$. By Theorem 2, the parameters of the codes generated by the matrices $\mathbb{G}_2'(25)$, $\mathbb{G}_3'(25)$, $\cdots$, $\mathbb{G}_9'(25)$ are given in Table 4.

4. Weight Distribution of $S_2(r)$ under the homogeneous metric where $r = \mathrm{p}^m$

In general, weight distribution plays an essential role in error-detection and error-correction of codes. In [28], the weight distributions of unit $\mathbb{Z}_q$-Simplex codes are determined. In this section, we find the weight distribution of $S_2(r)$ under the homogeneous weight metric.

Table 4. **Parameters of $S_k(r)$**

$S_k(r)$	$\left[\frac{\varrho^k-1}{\varrho-1},\ k,\ \varrho^{k-1}[p^{m-2}(\mathrm{p}-1)]\right]$
$S_2(25)$	$[22, 2, 84]$
$S_3(25)$	$[463, 3, 1764]$
$S_4(25)$	$[9724, 4, 37044]$
$S_5(25)$	$[204205, 5, 777924]$
$S_6(25)$	$[4288306, 6, 16336404]$
$S_7(25)$	$[90054427, 7, 343064484]$
$S_8(25)$	$[1891142968, 8, 7204354164]$
$S_9(25)$	$[39714002329, 9, 151291437444]$

Theorem 3. *Let $r = \mathrm{p}^m$, where p is a prime. Then, $S_2(r)$ has the weight distribution*

$$\begin{array}{rcl} A_0 & = & 1, \\ A_{(p-1)p^{m-2}(p^m-p^{m-1}+1)} & = & 2(p^m-p), \\ A_{p^{m-1}(p^m-p^{m-1}+1)} & = & 2(p-1), \\ A_{(p-1)[p^{m-1}+(p^m-p^{m-1}+2-p)p^{m-2}]} & = & (p-1)^2 \sum\limits_{i=2}^{m} p^{2(i-1)}, \\ A_{p^{m-1}\left[p^m-2p^{m-1}+2\right]} & = & (p-1)^2, \\ A_{(p-1)p^{m-2}\left[p^m-p^{m-1}+1\right]+p^{m-1}} & = & 2(p^m-p)(p-1). \end{array}$$

The other A_i's are equal to zero.

Proof. Consider the generator matrix $\mathbb{G}_2'(r)$. Then, $S_2(r) = \{\alpha_1 x + \alpha_2 y \mid \alpha_1, \alpha_2 \in \mathbb{Z}_r\}$. Now, consider the following cases:
Case 1: $\alpha_1, \alpha_2 \in \mathbb{E}_1$. Then, $wt_{Hom}(\alpha_1 x + \alpha_2 y) = 0$. Therefore, $A_0 = 1$.
Case 2: $\alpha_1 \in \mathbb{E}_1$ and $\alpha_2 \in \mathbb{E}_{p^j}$. Then,
Subcase i: If $\alpha_2 \in \mathbb{E}_{\mathrm{p}^j}$, where $2 \leq j \leq m$, then

$$wt_{Hom}(\alpha_2 y) = wt_{Hom}(\alpha_2(0\,1\,1\cdots 1)) = (0\ \underbrace{\alpha_2\,\alpha_2\cdots\alpha_2}_{p^m\ -\ p^{m-1}\ +1}) = (p-1)p^{m-2}(p^m-p^{m-1}+1).$$

Subcase ii: If $\alpha_2 \in \mathbb{E}_p$ then

$$wt_{Hom}(\alpha_2 y) = wt_{Hom}(\alpha_2(0\,1\,1\cdots 1)) = (0\ \underbrace{\alpha_2\,\alpha_2\cdots\alpha_2}_{p^m\ -\ p^{m-1}\ +1}) = p^{m-1}(p^m-p^{m-1}+1).$$

So, there are $p^m - p$ codewords of homogeneous weight $(p-1)p^{m-2}(p^m-p^{m-1}+1)$ and $p-1$ codewords of homogeneous weight $p^{m-1}(p^m-p^{m-1}+1)$. That is, $A_{(p-1)p^{m-2}(p^m-p^{m-1}+1)} = p^m - p$ and $A_{p^{m-1}(p^m-p^{m-1}+1)} = p-1$.

Case 3: $\alpha_1 \in \mathbb{E}_{p^j}$ and $\alpha_2 \in \mathbb{E}_1$. Then,
Subcase i: If $\alpha_1 \in \mathbb{E}_{p^j}$, where $2 \leq j \leq m$, then

$$wt_{Hom}(\alpha_1 x) \;=\; wt_{Hom}(\alpha_1(1\,0\,\underbrace{\alpha_1\,\alpha_2\,\cdots\,\alpha_{\phi(r)}}_{p^m\,-\,p^{m-1}})) = (p-1)p^{m-2}(p^m - p^{m-1} + 1).$$

Subcase ii: If $\alpha_1 \in \mathbb{E}_p$ then

$$wt_{Hom}(\alpha_1 x) \;=\; wt_{Hom}(\alpha_1(1\,0\,\underbrace{\alpha_1\,\alpha_2\,\cdots\,\alpha_{\phi(r)}}_{p^m\,-\,p^{m-1}})) = p^{m-1}(p^m - p^{m-1} + 1).$$

So, there are $p^m - p$ codewords of homogeneous weight $(p-1)p^{m-2}(p^m - p^{m-1} + 1)$ and $p-1$ codewords of homogeneous weight $p^{m-1}(p^m - p^{m-1} + 1)$. That is, $A_{(p-1)p^{m-2}(p^m - p^{m-1}+1)} = p^m - p$ and, $A_{p^{m-1}(p^m - p^{m-1}+1)} = p - 1$.

Case 4: $\alpha_1, \alpha_2 \notin \mathbb{E}_1$.
Subcase i: If $o(\alpha_1) = \mathrm{p}^j = o(\alpha_2)$, where $2 \leq j \leq m$, then

$$\begin{aligned} wt_{Hom}(\alpha_1 x + \alpha_2 y) &= wt_{Hom}(\alpha_1(1\,0\,\alpha_1\,\alpha_2\,\cdots\,\alpha_{\phi(r)}) + \alpha_2(0\,1\,1\cdots 1)) \\ &= (p-1)p^{m-1} + (p^m - p^{m-1} + 2 - p)(p-1)p^{m-2} \\ &= (p-1)[p^{m-1} + (p^m - p^{m-1} + 2 - p)p^{m-2}]. \end{aligned}$$

Subcase ii: If $o(\alpha_1) = \mathrm{p} = o(\alpha_2)$ then

$$\begin{aligned} wt_{Hom}(\alpha_1 x + \alpha_2 y) &= wt_{Hom}(\alpha_1(1\,0\,\alpha_1\,\alpha_2\,\cdots\,\alpha_{\phi(r)}) + \alpha_2(0\,1\,1\cdots 1)) \\ &= p^{m-1}\left[p^m - p^{m-1} + 2 - p^{m-1}\right] = p^{m-1}\left[p^m - 2p^{m-1} + 2\right]. \end{aligned}$$

Subcase iii: If $o(\alpha_1) = \mathrm{p}$ and $o(\alpha_2) = \mathrm{p}^j$, where $2 \leq j \leq m$, then

$$\begin{aligned} wt_{Hom}(\alpha_1 x + \alpha_2 y) &= wt_{Hom}(\alpha_1(1\,0\,\alpha_1\,\alpha_2\,\cdots\,\alpha_{\phi(r)}) + \alpha_2(0\,1\,1\cdots 1)) \\ &= (p-1)p^{m-2}\left[p^m - p^{m-1} + 1\right] + p^{m-1}. \end{aligned}$$

Subcase iv: If $o(\alpha_2) = \mathrm{p}$ and $o(\alpha_1) = \mathrm{p}^j$, where $2 \leq j \leq m$, then

$$\begin{aligned} wt_{Hom}(\alpha_1 x + \alpha_2 y) &= wt_{Hom}(\alpha_1(1\,0\,\alpha_1\,\alpha_2\,\cdots\,\alpha_{\phi(r)}) + \alpha_2(0\,1\,1\cdots 1)) \\ &= (p-1)p^{m-2}\left[p^m - p^{m-1} + 1\right] + p^{m-1}. \end{aligned}$$

So, there are $\sum\limits_{i=2}^{m} (p^i - p^{i-1})^2$ codewords of homogeneous weight $(p-1)[p^{m-1} + (p^m - p^{m-1} + 2 - p)p^{m-2}]$, $(p-1)^2$ codewords of homogeneous weight $p^{m-1}\left[p^m - 2p^{m-1} + 2\right]$ and $2(p^m - p)(p-1)$ codewords of homogeneous weight $(p-1)p^{m-2}\left[p^m - p^{m-1} + 1\right] + p^{m-1}$. That is, $A_{(p-1)[p^{m-1} + (p^m - p^{m-1} + 2 - p)p^{m-2}]} = \sum\limits_{i=2}^{m} (p^i - p^{i-1})^2 = (p-1)^2 \sum\limits_{i=2}^{m} p^{2(i-1)}$, $A_{p^{m-1}[p^m - 2p^{m-1}+2]} = (p-1)^2$ and $A_{(p-1)p^{m-2}[p^m - p^{m-1}+1] + p^{m-1}} = 2(p^m - p)(p-1)$. ☐

Example 3. Let

$$\mathbb{G}_2'(4) = \left[\begin{array}{c|c|cc} 1 & 0 & 1 & 3 \\ \hline 0 & 1 & 1 & 1 \end{array}\right]$$

$$\mathbb{G}_2'(9) = \left[\begin{array}{c|c|cccccc} 1 & 0 & 1 & 2 & 4 & 5 & 7 & 8 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\mathbb{G}_2'(25) = \left[\begin{array}{c|c|cccccccccccccccccccc} 1 & 0 & 1 & 2 & 3 & 4 & 6 & 7 & 8 & 9 & 11 & 12 & 13 & 14 & 16 & 17 & 18 & 19 & 21 & 22 & 23 & 24 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

By Theorem 3 and using MATLAB, we have listed the weight distribution of the codes generated by the matrices $\mathbb{G}_2'(4)$, $\mathbb{G}_2'(9)$ and $\mathbb{G}_2'(25)$:

$S_2(4) : A_0 = 1,\ A_3 = 4,\ \cdots\ ,A_4 = 5,\ A_5 = 4,\ A_6 = 2.$
$S_2(9) : A_0 = 1,\ A_{14} = 12,\ A_{15} = 4,\ A_{16} = 36,\ A_{17} = 24,\ A_{21} = 4.$
$S_2(25) : A_0 = 1,\ A_{84} = 40,\ A_{85} = 16,\ A_{88} = 400,\ A_{89} = 160,\ A_{105} = 8.$

5. Comparison of Gray images of $\mathbb{Z}_r$-Simplex code and unit $\mathbb{Z}_r$-Simplex code

The key aspect of coding theory is to develop codes that maintain a specified error correction capability while minimizing redundancy. A higher code rate indicates a more efficient utilization of redundancy compared to a lower code rate. Consequently, these codes are generally more desirable. From another perspective, we must also take into account the error-correcting abilities of the code. There exists a fundamental trade-off between the code rate and the minimum distance; as the code rate increases, the minimum distance decreases, and vice versa.

Based on the perspective described, Table 5 compares the code rate between $\mathbb{Z}_r$-Simplex code and the unit $\mathbb{Z}_r$-Simplex code for a specific value of r. The symbols CR and $CR^{'}$ denote the code rates for the gray images of $\mathbb{Z}_r$-Simplex code and the unit $\mathbb{Z}_r$-Simplex code, respectively. We observe from Table 5 that the code rate decreases and the minimum distance increases.

Table 5. **Code rates of CR and $CR^{'}$**

CR		$CR^{'}$	
$\left(2(\frac{4^k-1}{3}),\ 2^{2k},\ (4^{k-1})\right)$	Code rate $\frac{2k}{n}$	$\left(2(\frac{3^k-1}{2}), 2^{2k}, 3^{k-1}(2-1)\right)$	Code rate $\frac{2k'}{n'}$
$(10,\ 2^4,\ 4)$	0.4	$(8,\ 2^4,\ 3)$	0.5
$(42,\ 2^6,\ 16)$	0.143	$(26,\ 2^6,\ 9)$	0.231
$(170,\ 2^8,\ 64)$	0.0471	$(80,\ 2^8,\ 27)$	0.1
$(682,\ 2^{10},\ 256)$	0.0147	$(242,\ 2^{10}, 81)$	0.0413

6. Concluding Remarks

In this article, we obtained the various parameters of the unit $\mathbb{Z}_r$-Simplex code under the homogeneous weight metric for some particular cases. Generally, it is difficult to determine the weight distribution of higher rank codes. Here, we examined the weight distribution of unit $\mathbb{Z}_r$-Simplex code under the homogeneous weight metric for rank 2 where $r = \mathrm{p}^m$. In the last decade, the construction of nonlinear codes has been increased because of the existence of “Better than Linear Codes”(BTL) and “Better than Known Linear Code”(BTKL) [20]. Based on the puncturing technique in [10, 28], one can obtain new families of nonlinear code with better parameters.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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