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Research Article

The Wiener index of the associate graph of $\mathbb{Z}_n$

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Abstract: Let R be a commutative ring, the associate graph, $Ass(R)$ of ring R has the elements of ring R as vertices and two distinct vertices u and v are adjacent if u and v are associate elements of R. In this article we investigate the Wiener index of $Ass(\mathbb{Z}_n)$ and its line graph for all $n \in \mathbb{N}$. We also give some characterization results regarding degree, diameter, girth, clique number, chromatic number, domination number, and independence number of $Ass(\mathbb{Z}_n)$ and $L\left(Ass(\mathbb{Z}_n)\right)$.

Keywords: associate graph, Wiener index; line graph, commutative ring.

AMS Subject classification: 05C10, 05C25, 05C76, 05C12

1. Introduction

Topological indices are commonly classified into two classes: distance-based and degree-based. These are known as graph invariants in theoretical chemistry for predicting the chemical properties of molecules. Wiener [25] introduced topological indices while considering the boiling point of the paraffin. For some interesting results on the indices, see [1, 8–13, 15].

The idea of associating the elements of a ring with a graph and investigating the interplay between ring structure and the graph theoretical properties of the associated graph was initiated by Beck [2] in 1988. The study of associate graph of a commutative ring with unity was initiated by Subhakar [23] in 2010. Khalel and Ibrahiem [21]

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later modified the definition.
In this paper, our main focus is to find Wiener index for associate graph of commutative ring $\mathbb{Z}_n$, and for its line graph. Also, we investigate several graph-theoretical parameters of the associate graph of $\mathbb{Z}_n$.

2. Preliminaries

For terminology related to graph theory, we refer to Clark and Holton [7], follow Herstein [19], for terminology related to algebraic structures, and we follow Burton [5], for terminology related to number theory.
Khalel and Ibrahiem [21] gave the following definition.

Definition 1. For ring R, Associate graph of R is denoted by $Ass(R)$ and is a graph with $V(Ass(R)) = R$ and $E(Ass(R)) = \{uv/u = rv \ and \ v = su \ for \ some \ r \ and \ s \in R\}$.

Note that the zero element of R is adjacent to every other vertex in $Ass(R)$. Throughout this paper, we consider R as a commutative ring $\mathbb{Z}_n$.
The Wiener index, discovered by the chemist Wiener [25], plays a central role in chemistry literature for finding the distance between the molecules. The Wiener index is defined as the sum of the length of the shortest path between all pairs of vertices in the graph. In 1971, Hosaya [20] gave mathematical representation of the Wiener index as follows:

Definition 2. For a graph G, the Wiener index of G is denoted by $W(G)$ and defined as

$$W(G) = \sum_{u,v \in V(G)} d_G(u, v)$$

Definition 3 ([18]). The set $S \subseteq V(G)$ of vertices in graph G is called dominating set if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S. The minimum cardinality of a dominating set of G is called the domination number of G which is denoted by $\gamma(G)$.

Definition 4 ([18]). The set $I \subseteq V(G)$ is independent if no two vertices in I are adjacent. The independence number $\beta_0(G)$ is the maximum cardinality of an independent set in G.

Following are some notations and results used in the next section.
Notations:

- $O = \{0\}$.
- For $n \in \mathbb{N}$. The Euler's totient function of n is denoted by $\phi(n)$ and is given by the number of positive integers not exceeding n that are relatively prime to n.
- For $n \in \mathbb{N}, n \neq 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where p_i's are distinct primes and $\alpha_i \in \mathbb{N}$ for all $i = 1, 2, \dots, k$.

– Then $\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\dots\left(1-\frac{1}{p_k}\right)$.

– Let $\langle d_j \rangle$ be the sequence of proper divisors of n in increasing order. Define $\tau(n) = \{d_j : \text{ for each } j\}$ and $B_{d_j} = \{v \in \mathbb{Z}_n : gcd(v,n) = d_j\}$. Then, $|\tau(n)| = \prod_{i=1}^{k}(\alpha_i + 1) - 1$ and $\left|B_{d_j}\right| = \phi\left(\frac{n}{d_j}\right)$.

Proposition 1 ([21]). *For prime number p, $Ass(\mathbb{Z}_p) = K_p$.*

3. Associate Graph of Ring $\mathbb{Z}n$

Theorem 1. *Let $v \in V(Ass(\mathbb{Z}_n))$ and $d = gcd(v,n)$. Then*

$$deg\,(v) = \begin{cases} n-1; & \text{if } v \in O, \\ \phi\left(\frac{n}{d}\right); & \text{if } v \notin O. \end{cases}$$

Proof. The graph $Ass(\mathbb{Z}_n)$ has n vertices. By definition of associate graph, vertex 0 is adjacent to all the other vertices in $Ass(\mathbb{Z}_n)$. The vertices of $B_{d_j} \cup \{0\}$ induced a complete subgraph in $Ass(\mathbb{Z}_n)$ of order $\phi\left(\frac{n}{d_j}\right) + 1$. Hence proved. □

Remark 1. For the graph $Ass(\mathbb{Z}_n)$,

1. $diam\,(Ass(\mathbb{Z}_n)) = \begin{cases} 1; & \text{if } n \text{ is prime,} \\ 2; & \text{if } n \text{ is not a prime.} \end{cases}$
2. Since, $\chi\,(Ass\,(\mathbb{Z}_n)) = \phi(n) + 1 = cl\,(Ass(\mathbb{Z}_n))$, $Ass\,(\mathbb{Z}_n)$ is a perfect graph.
3. $\beta_0\,(Ass\,(\mathbb{Z}_n)) = |\tau(n)|$.

Theorem 2. *The Wiener index of $Ass(\mathbb{Z}_n)$ is given by*

$$W\,(Ass(\mathbb{Z}_n)) = 2\binom{n}{2} - \sum_{j=1}^{|\tau(n)|}\binom{\phi\left(\frac{n}{d_j}\right)+1}{2}, \text{ where } d_j \in \tau(n).$$

Proof. The distance between any two vertices u and v of $Ass(\mathbb{Z}_n)$ is as follow:

1. If $u \in O$ and $v \in B_{d_j}$. Then distance between u and v is 1. (There are $n-1$ pairs of such vertices.)
2. If $u, v \in B_{d_j}$. Then distance between u and v is 1. (There are $\sum_{j=1}^{|\tau(n)|}\binom{\phi\left(\frac{n}{d_j}\right)}{2}$ pairs of such vertices.)

3. If $u \in B_{d_j}$ and $v \in B_{d_k}$ for $j \neq k$. Then distance between u and v is 2. (There are $\sum_{j=1}^{|\tau(n)|-1} \left(\sum_{k=j+1}^{|\tau(n)|} \phi\left(\frac{n}{d_j}\right) \phi\left(\frac{n}{d_k}\right) \right)$ pairs of such vertices.)

Hence,

$$
\begin{aligned}
W\left(Ass(\mathbb{Z}_n)\right) &= \sum_{u,v \in V(Ass(\mathbb{Z}_n))} d(u,v) \\
&= \sum_{u \in O, v \in B_{d_j}} d(u,v) + \sum_{u,v \in B_{d_j}} d(u,v) + \sum_{u \in B_{d_j}, v \in B_{d_k}, j \neq k} d(u,v) \\
&= n - 1 + \sum_{j=1}^{|\tau(n)|} \binom{\phi\left(\frac{n}{d_j}\right)}{2} + 2 \sum_{j=1}^{|\tau(n)|-1} \left(\sum_{k=j+1}^{|\tau(n)|} \phi\left(\frac{n}{d_j}\right) \phi\left(\frac{n}{d_k}\right) \right) \\
&= 2\binom{n}{2} - \sum_{j=1}^{|\tau(n)|} \binom{\phi\left(\frac{n}{d_j}\right) + 1}{2}
\end{aligned}
$$

□

Example 1. For $n = 36 = 2^2.3^2$. The Wiener index of $Ass(\mathbb{Z}_{36})$ is 1120, when it is computed from the graph.
Now, we compute Wiener index of $Ass(\mathbb{Z}_{36})$ using Theorem 2.
The number of proper divisors of n is 8.

d_j	$\lvert B_{d_j} \rvert = \phi\left(\frac{n}{d_j}\right)$	$B_{d_j} = \{v \in \mathbb{Z}_n : gcd(v,n) = d_j\}$
1	12	1,5,7,11,13,17,19,23,25,29,31,35
2	6	2,10,14,22,26,34
3	4	3,15,21,33
4	6	4,8,16,20,28,32
6	2	6,30
9	2	9,27
12	2	12,24
18	1	18

By Theorem 2,

$$
\begin{aligned}
W\left(Ass(\mathbb{Z}_n)\right) &= 2\binom{n}{2} - \sum_{j=1}^{|\tau(n)|} \binom{\phi\left(\frac{n}{d_j}\right) + 1}{2} \\
W\left(Ass(\mathbb{Z}_{36})\right) &= 2\binom{36}{2} - \left[\binom{13}{2} + \binom{7}{2} + \binom{5}{2} + \binom{7}{2} + \binom{3}{2} \right. \\
&\qquad \left. + \binom{3}{2} + \binom{3}{2} + \binom{2}{2} \right]
\end{aligned}
$$

$$W\left(Ass(\mathbb{Z}_{36})\right) = 1120.$$

Thus, the Wiener index of $Ass(\mathbb{Z}_{36})$ computed using graph and Theorem 2 is the same.

4. Line Graph of an Associate Graph of Ring $\mathbb{Z}_n$

The line graph associated to graph G is denoted by $L(G)$, where each vertex of $L(G)$ represents an edge of G and two distinct vertices of $L(G)$ are adjacent if the corresponding edges are incident in G. The formal definition of line graph was given by Harary and Norman [17].
For more interesting results related to line graph associated with graphs obtained from algebraic structures can be found in [3, 4, 6, 14, 22, 26]. In this section, we have discussed the line graph of $Ass(\mathbb{Z}_n)$. Following are few propositions required.

Proposition 2 ([16]). *Let $L(G)$ be the line graph of G. Then for any vertex $[u,v]$ of $L(G)$, $deg_{L(G)}[u,v] = deg_G(u) + deg_G(v) - 2$.*

Proposition 3 ([16]). *$diam(L(K_n)) = 2$, for $n \geq 4$.*

Proposition 4 ([24]). $\beta_0\left(L(K_n)\right) = \left\lfloor \dfrac{n}{2} \right\rfloor$.

Theorem 3. *Let $[u,v] \in V(L(Ass(\mathbb{Z}_n)))$. Then*

$$deg\left([u,v]\right) = \begin{cases} n + \phi\left(\dfrac{n}{d_j}\right) - 3; & \textit{if } u = 0 \textit{ and } v \in B_{d_j}, \\ 2\left(\phi\left(\dfrac{n}{d_j}\right) - 1\right); & \textit{if } u, v \in B_{d_j}. \end{cases}$$

Proof. In $Ass(\mathbb{Z}_n)$, 0 is adjacent to all other vertices. Let v be any non zero vertex of $Ass(\mathbb{Z}_n)$. Then $[0,v]$ be a vertex of $L(Ass(\mathbb{Z}_n))$.
By Proposition 2,

$$\begin{aligned} deg([o,v]) &= deg(0) + deg(v) - 2 \\ &= n - 1 + \phi\left(\frac{n}{d_j}\right) - 2 \qquad \text{(By Theorem 1)} \\ &= n + \phi\left(\frac{n}{d_j}\right) - 3. \end{aligned}$$

The vertices of $B_{d_j} \cup \{0\}$ induced a complete subgraph in $Ass(\mathbb{Z}_n)$ of order $\phi\left(\dfrac{n}{d_j}\right) + 1$.
Thus, for any $u, v \in B_{d_j}$, $[u,v]$ be a vertex of $L(Ass(\mathbb{Z}_n))$.

By Proposition 2,

$$\begin{aligned} deg([u,v]) &= deg(u) + deg(v) - 2 \\ &= \phi\left(\frac{n}{d_j}\right) + \phi\left(\frac{n}{d_j}\right) - 2 \qquad \text{(By Theorem 1)} \\ &= 2\left(\phi\left(\frac{n}{d_j}\right) - 1\right). \end{aligned}$$

□

Remark 2. For the graph $L\left(Ass(\mathbb{Z}_n)\right)$,

1. Girth of $L\left(Ass(\mathbb{Z}_n)\right)$ is 3 for $n \geqslant 3$.
2. Since, $\chi\left(L\left(Ass\left(\mathbb{Z}_n\right)\right)\right) = n - 1 = cl\left(L\left(Ass(\mathbb{Z}_n)\right)\right)$, $L\left(Ass\left(\mathbb{Z}_n\right)\right)$ is a perfect graph.

Lemma 1. *$\gamma\left(L(K_n)\right) = \left\lfloor \frac{n}{2} \right\rfloor$, for $n \geq 2$.*

Proof. Let K_n be the complete graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. If n is even, then define $S = \left\{[v_{2i-1}, v_{2i}] : 1 \leq i \leq \frac{n}{2}\right\}$ while n is odd, then let $S = \left\{[v_{2i-1}, v_{2i}] : 1 \leq i \leq \frac{n-1}{2}\right\}$. In either case, S is a minimal dominating set of $L(K_n)$ with minimum cardinality. Therefore, $\gamma(L(K_n)) = \left\lfloor \frac{n}{2} \right\rfloor$. □

Theorem 4. *For the graph $L\left(Ass(\mathbb{Z}_n)\right)$,*

$$\gamma\left(L\left(Ass(\mathbb{Z}_n)\right)\right) = \begin{cases} 1; & \text{if } n = 2, \\ \sum_{j=1}^{|\tau(n)|} \lfloor (\phi(n/d_j) + 1)/2 \rfloor; & \text{if } n \text{ is odd }, \\ \sum_{j=1}^{|\tau(n)|-1} \lfloor (\phi(n/d_j) + 1)/2 \rfloor; & \text{if } n \text{ is even and } n \neq 2. \end{cases}$$

Proof. In $L\left(Ass(\mathbb{Z}_n)\right)$, the vertices of $B_{d_j} \cup \{0\}$ induces a complete subgraph in $Ass(\mathbb{Z}_n)$ of order $\phi(n/d_j) + 1$. Thus, $L\left(Ass(\mathbb{Z}_n)\right)$ contains $L(K_m)$ as a subgraph, where $m = \phi(n/d_j) + 1$ and there are $|\tau(n)|$ such graphs.
Let $B_{d_j} \cup \{0\} = \left\{v_{1,j}, v_{2,j}, \ldots, v_{m,j} : m = \phi(\frac{n}{d_j}) + 1\right\}$, and take $S_{d_j} = \{[v_{2i-1,j}, v_{2i,j}] : 1 \leqslant i \leqslant \left\lfloor \frac{m}{2} \right\rfloor\}$. We will prove the result using following three cases.

Case 1. $n = 2$.
The graph $Ass(\mathbb{Z}_2)$ is same as K_2. Hence, by Lemma 1, $\gamma\left(L(Ass(\mathbb{Z}_2))\right) = 1$.

Case 2. n is odd.
Let S be a minimal dominating set of $L(Ass(\mathbb{Z}_n))$ with minimum cardinality. Then

$|S| \geq \sum_{j=1}^{|\tau(n)|} \lfloor(\phi(n/d_j)+1)/2\rfloor$. Define $S = \bigcup_{j=1}^{|\tau(n)|} S_{d_j}$, then S is a minimal dominating set of $L\left(Ass(\mathbb{Z}_n)\right)$ and $|S| = \sum_{j=1}^{|\tau(n)|} \lfloor(\phi(n/d_j)+1)/2\rfloor$. Thus, S is a minimal dominating set of $L\left(Ass(\mathbb{Z}_n)\right)$ with minimum cardinality. Hence, $\gamma\left(L\left(Ass(\mathbb{Z}_n)\right)\right) = \sum_{j=1}^{|\tau(n)|} \lfloor(\phi(n/d_j)+1)/2\rfloor$.

Case 3. n is even and $n \neq 2$.
For $d_j = \dfrac{n}{2}$, $B_{d_j} = \left\{\dfrac{n}{2}\right\}$. Thus, $L\left(Ass(\mathbb{Z}_n)\right)$ has only one vertex $\left[0, \dfrac{n}{2}\right]$ corresponding to this $B_{d_j} \cup \{0\}$. This vertex is dominated by other vertex of $L\left(Ass(\mathbb{Z}_n)\right)$ corresponding to other $B_{d_j} \cup \{0\}$ selected in case 2. Hence, $|S| \geq \sum_{j=1}^{|\tau(n)|-1} \lfloor(\phi(n/d_j)+1)/2\rfloor$. Now, define $S = \bigcup_{j=1}^{|\tau(n)|-1} S_{d_j}$, where $d_j \neq \frac{n}{2}$, then S is a minimal dominating set of $L\left(Ass(\mathbb{Z}_n)\right)$ and $|S| = \sum_{j=1}^{|\tau(n)|-1} \lfloor(\phi(n/d_j)+1)/2\rfloor$. Thus, S is a minimal dominating set of $L\left(Ass(\mathbb{Z}_n)\right)$ with minimum cardinality. Hence, $\gamma\left(L\left(Ass(\mathbb{Z}_n)\right)\right) = \sum_{j=1}^{|\tau(n)|-1} \lfloor(\phi(n/d_j)+1)/2\rfloor$. □

Theorem 5. *For the graph $L\left(Ass(\mathbb{Z}_n)\right)$,*

$$\beta_0\left(L\left(Ass(\mathbb{Z}_n)\right)\right) = \begin{cases} \lfloor n/2 \rfloor; & \text{if } n \text{ is prime,} \\ \displaystyle\sum_{j=1}^{|\tau(n)|} \frac{\phi(n/d_j)}{2}; & \text{if } n \text{ is odd and not a prime,} \\ \displaystyle\sum_{j=1}^{|\tau(n)|-1} \frac{\phi(n/d_j)}{2} + 1; & \text{if } n \text{ is even and } n \neq 2. \end{cases}$$

Proof. We consider following three cases.

Case 1. n is a prime number.
By Proposition 1, $Ass(\mathbb{Z}_n) = K_n$, and by Proposition 4, $\beta_0\left(L\left(Ass(\mathbb{Z}_n)\right)\right) = \left\lfloor \frac{n}{2} \right\rfloor$.

Case 2. n is odd and is not prime number.
In $L\left(Ass(\mathbb{Z}_n)\right)$, the vertices of $B_{d_j} \cup \{0\}$ induces a complete subgraph in $Ass(\mathbb{Z}_n)$ of order $\phi(n/d_j)+1$. Thus, $L\left(Ass(\mathbb{Z}_n)\right)$ contains $L(K_m)$ as a subgraph, where $m = \phi\left(\dfrac{n}{d_j}\right)+1$ and there are $|\tau(n)|$ such graphs. Then by Proposition 4,

$$\beta_0\left(L\left(Ass(\mathbb{Z}_n)\right)\right) \leq \sum_{j=1}^{|\tau(n)|} \lfloor(\phi(n/d_j)+1)/2\rfloor = \sum_{j=1}^{|\tau(n)|} \phi(n/d_j)/2.$$

Let $B_{d_j} = \{v_{1,j}, v_{2,j}, \ldots, v_{m,j} : m = \phi(n/d_j)\}$, and take $I_{d_j} = \{[v_{2i-1,j}, v_{2i,j}] : 1 \leqslant i \leqslant \dfrac{m}{2}\}$. Define $I = \displaystyle\bigcup_{j=1}^{|\tau(n)|} I_{d_j}$, then I is the maximal independent set of $L\left(Ass(\mathbb{Z}_n)\right)$. Hence, $\beta_0\left(L\left(Ass\left(\mathbb{Z}_n\right)\right)\right) = \displaystyle\sum_{j=1}^{|\tau(n)|} \phi(n/d_j)/2$.

Case 3. n is even and $n \neq 2$.
For $d_j = \frac{n}{2}$, $B_{d_j} = \left\{\frac{n}{2}\right\}$. Thus, $L\left(Ass(\mathbb{Z}_n)\right)$ has only one vertex $\left[0, \frac{n}{2}\right]$ corresponding to this $B_{d_j} \cup \{0\}$. This vertex is not a neighbour of any other vertex of $L\left(Ass(\mathbb{Z}_n)\right)$ corresponding to other B_{d_j} chosen as per case 2. Hence, $\beta_0\left(L\left(Ass(\mathbb{Z}_n)\right)\right) \leq \sum_{j=1}^{|\tau(n)|-1} \phi(n/d_j)/2+1$. Now, define $I = \bigcup_{j=1}^{|\tau(n)|-1} I_{d_j}$, where $d_j \neq \frac{n}{2}$, and add $\left[0, \frac{n}{2}\right]$ vertex of $L\left(Ass(\mathbb{Z}_n)\right)$ in I. Then I is a maximal independent set of $L\left(Ass(\mathbb{Z}_n)\right)$. Hence, $\beta_0\left(L\left(Ass(\mathbb{Z}_n)\right)\right) = \sum_{j=1}^{|\tau(n)|-1} \phi(n/d_j)/2+1$. □

Theorem 6. *$diam\left(L\left(Ass(\mathbb{Z}_n)\right)\right) = 3$, for $n \geq 6$ and n is not a prime number.*

Proof. Let $[u_1, v_1]$ and $[u_2, v_2]$ be the vertices of $L(Ass(\mathbb{Z}_n))$, where $u_1, v_1, u_2, v_2 \in \mathbb{Z}_n$. Then distance between $[u_1, v_1]$ and $[u_2, v_2]$ can be calculated by following cases.
Case 1. ($u_1 = 0$ or $v_1 = 0$) and ($u_2 = 0$ or $v_2 = 0$).
Then $d\left([u_1, v_1], [u_2, v_2]\right)$ is 1. (There are $\binom{n-1}{2}$ pairs of such vertices.)
Case 2. $u = [u_1, v_1]$ and $v = [u_2, v_2]$, where $u_1 = 0$ and $v_1, u_2, v_2 \in B_{d_j}$ for some d_j. Then distance between u and v is either 1 or 2. Moreover, there are

$$\sum_{j=1}^{|\tau(n)|} \phi(n/d_j)\left(\phi(n/d_j)-1\right)$$

pairs of such vertices for distance 1 and there are

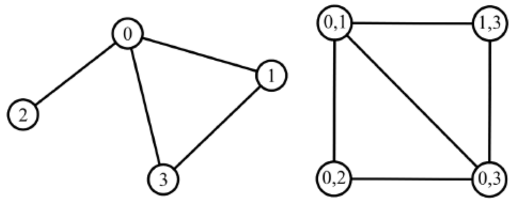
$$\sum_{j=1}^{|\tau(n)|} \frac{1}{2}\phi\left(n/d_j\right)\left(\phi(n/d_j)-1\right)\left(\phi(n/d_j)-2\right)$$

pairs of such vertices for distance 2.
Case 3. $u = [u_1, v_1]$ and $v = [u_2, v_2]$, where $u_1, v_1, u_2, v_2 \in B_{d_j}$ for some d_j. Then distance between u and v is either 1 or 2. Moreover, There are

$$\sum_{j=1}^{|\tau(n)|} \frac{1}{2}\phi(n/d_j)\left(\phi(n/d_j)-1\right)\left(\phi(n/d_j)-2\right)$$

pairs of such vertices for distance 1 and there are

$$\sum_{j=1}^{|\tau(n)|} \frac{1}{8}\phi(n/d_j)\left(\phi(n/d_j)-1\right)\left(\phi(n/d_j)-2\right)\left(\phi(n/d_j)-3\right)$$



(a) $Ass(\mathbb{Z}_4)$ (b) $L\left(Ass(\mathbb{Z}_4)\right)$

Figure 1. **Associate graph of $\mathbb{Z}_4$ and its line graph.**

pairs of such vertices for distance 2.

Case 4. $u_1 = 0, v_1 \in B_{d_j},\ u_2, v_2 \in B_{d_k}$ and $j \neq k$.
Then $d\left([u_1, v_1], [u_2, v_2]\right)$ is 2. Moreover, there are

$$\sum_{j=1}^{|\tau(n)|}\left[\frac{1}{2}\phi\left(\frac{n}{d_j}\right)\left(\sum_{k=1}^{|\tau(n)|}\left(\left(\phi\left(\frac{n}{d_k}\right)\right)^2-\phi\left(\frac{n}{d_k}\right)\right)-\left(\left(\phi\left(\frac{n}{d_j}\right)\right)^2-\phi\left(\frac{n}{d_j}\right)\right)\right)\right]$$

pairs of such vertices.

Case 5. $u_1, v_1 \in B_{d_j},\ u_2, v_2 \in B_{d_k},\ j \neq k$.
Then $d\left([u_1, v_1], [u_2, v_2]\right)$ is 3. Also there are

$$\sum_{j=1}^{|\tau(n)|-1}\left[\sum_{k=j+1}^{|\tau(n)|}\left(\frac{1}{2}\phi\left(\frac{n}{d_j}\right)\left(\phi\left(\frac{n}{d_j}\right)-1\right)\right)\left(\frac{1}{2}\phi\left(\frac{n}{d_k}\right)\left(\phi\left(\frac{n}{d_k}\right)-1\right)\right)\right]$$

pairs of such vertices.
Thus, the maximum distance between any two vertices of $L\left(Ass(\mathbb{Z}_n)\right)$ is 3.
Hence, $diam\left(L\left(Ass(\mathbb{Z}_n)\right)\right) = 3$. □

Theorem 7. *$diam\left(L\left(Ass(\mathbb{Z}_p)\right)\right) = 2$, for p is prime and $p \geq 5$.*

Proof. By Proposition 1, the graph $Ass(\mathbb{Z}_p)$ is same as K_p. By Proposition 3, $diam(L(K_p)) = 2$. Thus, $diam\left(L\left(Ass(\mathbb{Z}_p)\right)\right) = 2$, for p is prime and $p \geq 5$. □

Remark 3. $diam\left(L\left(Ass(\mathbb{Z}_3)\right)\right) = 1$. (As $L\left(Ass(\mathbb{Z}_3)\right) = K_3$ and diameter of K_3 is 1.)

Remark 4. $diam\left(L\left(Ass(\mathbb{Z}_4)\right)\right) = 2$ as shown in Figures 1.

Theorem 8. *The Wiener index of $L\left(Ass(\mathbb{Z}_n)\right)$ is given by*

$$\begin{aligned}
W\left(L\left(Ass(\mathbb{Z}_n)\right)\right) &= \binom{n-1}{2} + \frac{1}{4}\sum_{j=1}^{|\tau(n)|}\left[\left(\phi\left(\frac{n}{d_j}\right)\right)^4 - 3\cdot\left(\phi\left(\frac{n}{d_j}\right)\right)^2 + 2\cdot\phi\left(\frac{n}{d_j}\right)\right] \\
&+ \sum_{j=1}^{|\tau(n)|}\left[\phi\left(\frac{n}{d_j}\right)\left(\sum_{k=1}^{|\tau(n)|}\left(\left(\phi\left(\frac{n}{d_k}\right)\right)^2 - \phi\left(\frac{n}{d_k}\right)\right)\right.\right. \\
&\qquad\left.\left. - \left(\left(\phi\left(\frac{n}{d_j}\right)\right)^2 - \phi\left(\frac{n}{d_j}\right)\right)\right)\right] \\
&+ \frac{3}{4}\sum_{j=1}^{|\tau(n)|-1}\left[\sum_{k=j+1}^{|\tau(n)|}\left(\left(\phi\left(\frac{n}{d_j}\right)\right)^2 - \phi\left(\frac{n}{d_j}\right)\right)\right. \\
&\qquad\left. \times\left(\left(\phi\left(\frac{n}{d_k}\right)\right)^2 - \phi\left(\frac{n}{d_k}\right)\right)\right].
\end{aligned}$$

Proof. Distance between any two vertices of $L\left(Ass(\mathbb{Z}_n)\right)$ is given by Theorem 6. Hence,

$$\begin{aligned}
W\left(L\left(Ass(\mathbb{Z}_n)\right)\right) &= \sum_{[u_1,v_1],[u_2,v_2]\in V(L(Ass(\mathbb{Z}_n)))} d(u,v) \\
&= \sum_{(u_1=0 \text{ or } v_1=0) \text{ or } (u_2=0 \text{ or } v_2=0)} d(u,v) \\
&+ \sum_{u=[u_1,v_1],v=[u_2,v_2],u_1=0,v_1,u_2,v_2\in B_{d_j}} d(u,v) \\
&+ \sum_{u=[u_1,v_1],v=[u_2,v_2],u_1,v_1,u_2,v_2\in B_{d_j}} d(u,v) \\
&+ \sum_{u_1=0,v_1\in B_{d_j},u_2,v_2\in B_{d_k},j\neq k} d(u,v) + \sum_{u_1,v_1\in B_{d_j},u_2,v_2\in B_{d_k},j\neq k} d(u,v) \\
&= \binom{n-1}{2} + \sum_{j=1}^{|\tau(n)|}\phi\left(\frac{n}{d_j}\right)\left(\phi\left(\frac{n}{d_j}\right)-1\right) \\
&+ 2\sum_{j=1}^{|\tau(n)|}\left[\frac{1}{2}\phi\left(\frac{n}{d_j}\right)\left(\phi\left(\frac{n}{d_j}\right)-1\right)\left(\phi\left(\frac{n}{d_j}\right)-2\right)\right] \\
&+ \sum_{j=1}^{|\tau(n)|}\left[\frac{1}{2}\phi\left(\frac{n}{d_j}\right)\left(\phi\left(\frac{n}{d_j}\right)-1\right)\left(\phi\left(\frac{n}{d_j}\right)-2\right)\right] \\
&+ 2\sum_{j=1}^{|\tau(n)|}\left[\frac{1}{8}\phi\left(\frac{n}{d_j}\right)\left(\phi\left(\frac{n}{d_j}\right)-1\right)\right. \\
&\qquad\left. \times\left(\phi\left(\frac{n}{d_j}\right)-2\right)\left(\phi\left(\frac{n}{d_j}\right)-3\right)\right]
\end{aligned}$$

$$+2\sum_{j=1}^{|\tau(n)|}\left[\frac{1}{2}\phi\left(\frac{n}{d_j}\right)\left(\sum_{k=1}^{|\tau(n)|}\left(\left(\phi\left(\frac{n}{d_k}\right)\right)^2-\phi\left(\frac{n}{d_k}\right)\right)\right.\right.$$
$$\left.\left.-\left(\left(\phi\left(\frac{n}{d_j}\right)\right)^2-\phi\left(\frac{n}{d_j}\right)\right)\right)\right]$$
$$+3\sum_{j=1}^{|\tau(n)|-1}\left[\sum_{k=j+1}^{|\tau(n)|}\left(\frac{1}{2}\phi\left(\frac{n}{d_j}\right)\left(\phi\left(\frac{n}{d_j}\right)-1\right)\right)\right.$$
$$\left.\times\left(\frac{1}{2}\phi\left(\frac{n}{d_k}\right)\left(\phi\left(\frac{n}{d_k}\right)-1\right)\right)\right]$$
$$=\binom{n-1}{2}+\frac{1}{4}\sum_{j=1}^{|\tau(n)|}\left[\left(\phi\left(\frac{n}{d_j}\right)\right)^4-3\cdot\left(\phi\left(\frac{n}{d_j}\right)\right)^2+2\cdot\phi\left(\frac{n}{d_j}\right)\right]$$

$$+\sum_{j=1}^{|\tau(n)|}\left[\phi\left(\frac{n}{d_j}\right)\left(\sum_{k=1}^{|\tau(n)|}\left(\left(\phi\left(\frac{n}{d_k}\right)\right)^2-\phi\left(\frac{n}{d_k}\right)\right)\right.\right.$$
$$\left.\left.-\left(\left(\phi\left(\frac{n}{d_j}\right)\right)^2-\phi\left(\frac{n}{d_j}\right)\right)\right)\right]$$
$$+\frac{3}{4}\sum_{j=1}^{|\tau(n)|-1}\left[\sum_{k=j+1}^{|\tau(n)|}\left(\left(\phi\left(\frac{n}{d_j}\right)\right)^2-\phi\left(\frac{n}{d_j}\right)\right)\right.$$
$$\left.\times\left(\left(\phi\left(\frac{n}{d_k}\right)\right)^2-\phi\left(\frac{n}{d_k}\right)\right)\right]$$

□

5. Conclusion

The associate graph of the ring of integers modulo n is a union of complete graphs with the vertex 0 dominates all the other vertices. In this article, we study properties like clique number, chromatic number, domination number and independence number of $Ass(\mathbb{Z}_n)$. These properties are useful in the study of social networks. We also compute the Wiener index of this graph, and these results can play an important role in visualizing the topology of the graph under consideration. Similar properties have been discussed for the $L\left(Ass\left(\mathbb{Z}_n\right)\right)$ which help in understanding the behavior of the network during expansion.

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References

[1] T. Asir and V. Rabikka, *The Wiener index of the zero-divisor graph of* $\mathbb{Z}_n$, Discrete Appl. Math. **319** (2022), no. 38, 461–471. https://doi.org/10.1016/j.dam.2021.02.035.

[2] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226. https://doi.org/10.1016/0021–8693(88)90202-5.

[3] L. Boro, M.M. Singh, and J. Goswami, *Line graph associated to the intersection graph of ideals of rings*, J. Math. Comput. Sci. **11** (2021), no. 3, 2736–2745. https://doi.org/10.28919/jmcs/5567.

[4] ———, *On the line graphs associated to the unit graphs of rings*, Palestine J. Math. **11** (2022), no. 4, 139–145.

[5] D.M. Burton, *Elementary Number Theory*, McGraw-Hill Companies, 2011.

[6] H.J. Chiang-Hsieh, P.F. Lee, and H.J. Wang, *The embedding of line graphs associated to the zero-divisor graphs of commutative rings*, Israel J. Math. **180** (2010), no. 1, 193–222. https://doi.org/10.1007/s11856-010-0101-2.

[7] J. Clark and D.A. Holton, *A First Look at Graph Theory*, Allied Publishers Ltd., 1995.

[8] M. Dehmer and F. Emmert-Streib, *Quantitative Graph Theory: Mathematical Foundations and Applications*, CRC Press, 2014.

[9] J. Devillers and A. Balaban, *Topological indices and related descriptors in QSAR and QSPR*, Gordon and Breach Science Publishers, 1999.

[10] A. Dobrynin, R. Entringer, and I. Gutman, *Wiener index of trees: Theory and applications*, Acta Appl. Math. **66** (2001), no. 3, 211–249. https://doi.org/10.1023/A:1010767517079.

[11] A. Dobrynin and A. Iranmanesh, *Wiener index of edge thorny graphs of catacondensed benzenoids*, Mathematics **8** (2020), no. 4, 467–479. https://doi.org/10.3390/math8040467.

[12] K. Elahi, A. Ahmad, and R. Hasni, *Construction algorithm for zero divisor graphs of finite commutative rings and their vertex-based eccentric topological indices*, Mathematics **6** (2018), no. 12, 301–309. https://doi.org/10.3390/math6120301.

[13] N. Gohain, T. Ali, and A. Akhtar, *Reducing redundancy of codons through total graph*, Current Research in Bioinformatics **4** (2015), no. 1, 1–6. https://doi.org/10.3844/ajbsp.2015.1.6.

[14] J. Goswami and H.K. Saikia, *On the line graph associated to the line graph of a*

module, Matematika **31** (2015), no. 1, 7–13.
https://doi.org/10.11113/matematika.v31.n1.676.

[15] I. Gutman and O.E. Polansky, *Mathematical Concept in Organic Chemistry*, Springer-Verlag, 1986.

[16] F. Harary, *Graph Theory*, Addison-Wesley Publishing Company, 1969.

[17] F. Harary and R.Z. Norman, *Some properties of line digraphs*, Rendiconti del Circolo Matematico di Palermo **9** (1960), no. 2, 161–169.
https://doi.org/10.1007/BF02854581.

[18] T. Haynes, S.T. Hedetniemi, and P.J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker, New York, 1988.

[19] I.N. Herstein, *Topics in Algebra*, John Wiley & Sons, 1975.

[20] H. Hosaya, *Topological index. a newly proposed quantity characterizing the topological nature of structural isomers of saturated hydrocarbons*, Bull. Chem. Soc. Japan **44** (1971), no. 9, 2332–2339.
https://doi.org/10.1246/bcsj.44.2332.

[21] N.J. Khalel and N.E. Arif, *Associate graph of a commutative ring*, J. Discrete Math. Sci. Cryp. **26** (2023), no. 7, 1883–1887.
https://doi.org/10.47974/JDMSC-1679.

[22] M. Sarmah and K. Patra, *Line graph associated to total graph of idealization*, Afrika Matematika **27** (2015), no. 3, 485–490.
https://doi.org/10.1007/s13370-015-0355-2.

[23] M.J. Subhakar, *Associate ring graphs*, Mapana J. Sciences **9** (2010), no. 1, 31–40.
https://doi.org/10.12723/mjs.16.5.

[24] C. Susanth and S.J. Kalayathankal, *The sum and product of independence numbers of graphs and their line graphs*, J. Inf. Math. Sci. **6** (2014), no. 2, 77–85.

[25] H. Wiener, *Structural determination of paraffin boiling points*, J. American Chem. Soc. **69** (1947), no. 1, 17–20.
https://doi.org/10.1021/ja01193a005.

[26] Z.P. Zoran and S.P. Zoran, *The line graph associated to the total graph of commutative rings*, Ars Combin. **127** (2016), 185–195.