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Short Note

A note on independent domination in almost-regular graphs

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In honour of Odile Favaron's 88th birthday

Abstract: A classic result in domination theory is that a regular graph has independent domination number at most half the order. We strengthen this result to "almost-regular" graphs by showing that if a graph has minimum degree $\delta > 0$ and maximum degree at most $\delta + 3$, and the subgraph induced by the vertices of degree $\delta + 3$ (if any) is bipartite, then the independent domination number is at most half the order. We also discuss related questions.

Keywords: Independent domination; Almost-regular graph; one-half bound.

AMS Subject classification: 05C69

1. Introduction

Recall that a set S of vertices in a graph G is a *dominating set* if every vertex not in S is adjacent to a vertex in S. If, in addition, S is an independent set, then S is an *independent dominating set*, abbreviated ID-set. The *independent domination number*, denoted $i(G)$, of G is the minimum cardinality among all ID-set in G. Equivalently, an independent dominating set is a maximal independent set of vertices in G. For more on independent domination and other graph theory terminology not defined herein, the reader is referred to [6, 10].
In 1988, Odile Favaron [5] determined the maximum value of $i(G)$ of a graph G in terms of its order n. Namely, $i(G) \le n + 2 - 2\sqrt{n}$. The graphs that attain this bound

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have minimum degree 1 and maximum degree $\sqrt{n}$. So a natural question is what happens if one bounds the maximum degree. A first step was provided by Akbari et al. [1] for $\Delta(G) \leq 3$, and by Cho, Choi, and Park [2] for $\Delta(G) \leq 4$. The general question was resolved by Cho, Kim, Kim, and Oum [3]. In the other direction, Sun and Wang [13] determined the maximum value of $i(G)$ for fixed minimum degree. At the same time it is straight-forward to show that a regular graph has independent domination number at most $n/2$. There has been considerable work on independent domination in regular graphs, both for small degree and large degree; see for example [2, 8, 9, 11].

In light of the above, a natural question is what happens in general if one bounds both the minimum and maximum degree. And in particular, for what choices of $\delta(G)$ and $\Delta(G)$ does it follow that $i(G) \leq n/2$? (One cannot hope for better than this, as discussed in Section 3.) Akbari et al. [1] showed this for $\delta(G) \geq 1$ and $\Delta(G) \leq 3$. Since the "4-special graphs" given by Choi et al. [3] all have end-vertices, it follows from their main theorem that a graph with $\Delta(G) \leq 4$ and $\delta(G) \geq 2$ has $i(G) \leq n/2$. We conjecture that this result is also true for $\Delta(G) = 5$.

In this note we first show that if G is an isolate-free graph of order n with $\Delta(G) \leq \delta(G) + 2$, then $i(G) \leq n/2$. Furthermore, we extend this to graphs with $\Delta(G) = \delta(G) + 3$, provided the subgraph induced by the vertices of degree $\Delta(G)$ is bipartite. We believe the bipartiteness requirement is unnecessary for $\delta(G) \geq 2$. (As shown by Choi et al. [3], the maximum value of $i(G)$ for graphs with $\delta(G) = 1$ and $\Delta(G) = 4$ is $(n+1)/2$.) Maybe even more is true. For example, perhaps $\Delta(G) \leq 2\delta(G)$ implies an $n/2$ upper bound.

2. Main Results

Theorem 1. *Let G be a graph of order n with minimum degree $\delta > 0$ and maximum degree Δ. If $\Delta \leq \delta + 2$, then $i(G) \leq n/2$.*

Proof. We use a coloring approach. Consider a partial proper 2-coloring of the vertices using colors red and blue. Say a vertex is ***forlorn*** if none of its neighbors is colored. Out of all partial 2-colorings, choose one such that:

C1: The number of colored vertices is as large as possible.

Let X denote the set of uncolored vertices. Let F denote the set of forlorn vertices. By Condition C1, all forlorn vertices are colored.

We claim that every vertex of X has both a red and a blue neighbor not in F. For suppose vertex x is uncolored and all its red neighbors are forlorn. Then we can recolor its red neighbors blue and then color x red, a contradiction.

Now, we construct two ID-sets for G. We create J_B by starting with all the blue vertices and then adding all vertices without a blue neighbor (that is, the red forlorn vertices), and create J_R by starting with all red vertices and then adding all vertices without a red neighbor. The vertices of X are in neither set. The sets overlap in exactly F.

Consider the subgraph H induced by all edges of G with one end in F and one end in X. By the claim, every vertex of X has at most $\Delta - 2 \le \delta$ neighbors in F. On the other hand, every edge incident with a forlorn vertex joins it to X. If H has h edges, it follows that $\delta|F| \le h \le \delta|X|$, whence $|F| \le |X|$. Hence $|J_B|+|J_R| = n+|F|-|X| \le n$. By averaging it follows that $i(G) \le n/2$. □

We can improve the result slightly.

Theorem 2. *Let G be a graph of order n with minimum degree $\delta > 0$ and maximum degree Δ. If $\Delta \le \delta + 3$ and the vertices of degree Δ induce a bipartite subgraph, then $i(G) \le n/2$.*

Proof. We use the same approach as Theorem 1. Namely, we show that there exists a partial 2-coloring where every vertex has at most $\Delta - 3 \le \delta$ forlorn neighbors, whence $|F| \le |X|$, and the result then follows by constructing J_B and J_R as before. Out of all partial 2-colorings, choose one such that:

- C1: The number of colored vertices is as large as possible.
- C2: Subject to this, the number of forlorn vertices is as small as possible.

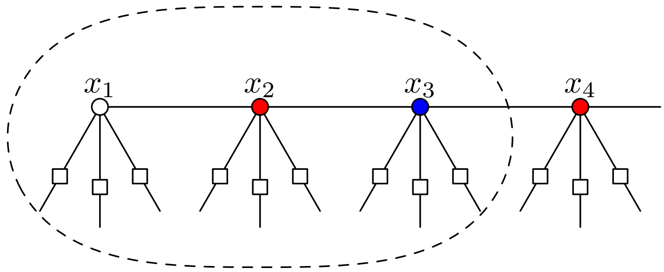
Suppose vertex x_1 has at least $\Delta - 2$ forlorn neighbors. As before, Condition C1 implies that x_1 has both a red and a blue neighbor that are not forlorn. It follows that x_1 has degree Δ, and the non-forlorn red and blue neighbors are unique. Let S_1 denote the set of $\Delta - 2$ forlorn neighbors of x_1. Say the non-forlorn red neighbor of x_1 is x_2. Call the original coloring $\mathcal{A}_1$.

Define coloring $\mathcal{A}_2$ from $\mathcal{A}_1$ by un-coloring x_2, coloring x_1 red, and re-coloring S_1 (if necessary) to be all blue. This coloring has the same number of colored vertices. Thus by Condition C1 vertex x_2 must still have a blue neighbor x_3 that is not forlorn. Further, since the vertices in S_1 are no longer forlorn, by Condition C2 there must be a set S_2 of $\Delta - 2$ neighbors of x_2 that are now forlorn. In particular, vertex x_2 has degree Δ in G, and x_3 is its unique blue non-forlorn neighbor under $\mathcal{A}_1$. Note that S_2 is an independent set; also since each vertex is colored, there is no edge between S_1 and S_2.

Define coloring $\mathcal{A}_3$ from $\mathcal{A}_1$ by un-coloring x_3, coloring x_1 red, and re-coloring x_2 blue, S_1 blue, and S_2 red. Again it follows that under $\mathcal{A}_3$ there is a set S_3 of $\Delta - 2$ neighbors of x_3 that are forlorn, while vertex x_3 has a non-forlorn neighbor of each color. In particular vertex x_3 has degree Δ. The figure below illustrates the situation for the case $\Delta = 5$; the vertices inside the dashed line are re-colored to yield coloring $\mathcal{A}_3$.

If the red non-forlorn neighbor of x_3 under $\mathcal{A}_3$ is not x_1, then call it x_4. By a similar argument, there is a set S_4 of $\Delta - 2$ vertices whose only colored neighbor under $\mathcal{A}_1$ is x_4, while x_4 has a blue neighbor other than x_3 that has a red neighbor.

Thus we obtain a sequence P of distinct vertices $x_2, x_3, x_4, \ldots$ alternating colors, each having degree Δ, and inducing a path. Because the graph G is finite, this process



must eventually terminate. The only way it can terminate is that we reach a k, with k odd, where under $\mathcal{A}_k$ the red non-forlorn neighbor of x_k is x_1. But that yields an odd cycle all of whose vertices have degree Δ, a contradiction. $\square$

3. Extremal Graphs and General Bounds

In general, for positive integers δ, Δ one can ask for the minimum constant $c_{\delta,\Delta}$ such that $i(G) \leq c_{\delta,\Delta} n$ for all graphs G of order n, minimum degree (at least) δ, and maximum degree (at most) Δ.

As commented earlier, $c_{\delta,\Delta} \geq \frac{1}{2}$. Indeed, there exist graphs with independent domination number $n/2$ for all values of $0 < \delta < \Delta$. One construction is to take two disjoint copies of $K_{\delta,\Delta-\delta}$ and add all possible edges between the two partite sets of size δ. Another construction is to take disjoint copies of $K_{\delta,\delta}$ and $K_{\Delta-\delta,\Delta-\delta}$ and add all possible edges between one partite set of the one graph and one partite set of the other.

We can also provide a general upper bound:

Theorem 3. *If G is a graph of order n with minimum degree δ and maximum degree Δ, then*

$$i(G) \leq \left(\frac{\Delta}{\delta + \Delta} \right) n.$$

Proof. Indeed the bound holds for the independence number. Let J be a maximum independent set, and let j denote the number of edges incident with J. Then $|J|\delta \leq j \leq \Delta(n - |J|)$, which rearranges to the bound. $\square$

It seems unlikely this bound is best possible. For example, we believe that the bipartite requirement in Theorem 2 can be relaxed, and pose the following conjecture.

Conjecture 1. If G is a connected graph of order n with minimum degree δ and maximum degree Δ, where $\delta \geq 2$ and $\Delta \leq \delta + 3$, then $i(G) \leq \frac{1}{2}n$.

4. Two Disjoint Independent Dominating Sets

A stronger condition than $i(G) \leq n/2$ is having two disjoint ID-sets. It follows from Favaron's bound that a graph is not guaranteed to have two disjoint ID-sets. Indeed, even for regular graphs, where the upper bound of $n/2$ is immediate, Payan [12] showed that a regular graph need not have two disjoint ID-sets. In [4] it is stated that Berge showed that cubic graphs have two disjoint ID-sets. We [7] extended this to a graph with minimum degree at least 2 and maximum degree at most 3. It is not hard to show that this property does not extend to graphs with $\delta = 1$ and $\Delta = 3$. Nor does it extend to graphs with $\delta = 2$ and $\Delta = 5$: consider for example the graph obtained from K_6 by taking a perfect matching and subdividing each edge of the perfect matching once. But it is unclear what happens in graphs with $\delta = 2$ and $\Delta = 4$.

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Data Availability. Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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