

On the covering array number of 3-uniform qualitative independence hypergraphs

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Abstract: Covering arrays (CAs) are widely recognized combinatorial designs that facilitate efficient test suite generation in software testing. For a positive integer n and a set S of size g , a *covering array on a hypergraph* $H = (V, E)$ of size n and alphabet size g , denoted $CA(n, H, g)$, is an $n \times |V|$ array with entries from S , where columns correspond to the vertices in V , such that for every hyperedge $e \in E$ of size m , the sub-array indexed by the vertices of e contains all the ordered m -tuples from S^m at least once, as a row. The smallest n for which such an array exists is the *covering array number of H* , denoted $CAN(H, g)$. This parameter represents the minimal test suite size for practical applications. For a t -uniform hypergraph H and positive integers g and n , a $CA(n, H, g)$ exists if and only if there is a hypergraph homomorphism from H to t - $QI(n, g)$, a distinctive class of hypergraphs called *qualitative independence hypergraphs*. Consequently, for such a hypergraph H , $CAN(H, g) \leq CAN(t$ - $QI(n, g), g)$. In this paper, we develop a technique to determine the strong chromatic number and the size of the largest 3-cliques of 3- $QI(n, 2)$ to establish that $CAN(3$ - $QI(n, 2), 2) = n$ for some values of n and apply the results obtained to find the covering array number of certain classes of hypergraphs.

Keywords: qualitative independence, hypergraph, covering array, software testing.

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1. Introduction

In complex systems such as software applications, errors often arise due to the interaction between multiple factors; such errors are referred to as *interaction faults*. The

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surge in software development worldwide has led to the demand for effective testing tools to identify such faults. Exhaustive testing of a system with k factors, each having g possible values, requires g^k tests. This approach becomes prohibitively expensive and impractical for large k . For instance, testing just 10 factors, each with 4 possible values, would require over a million ($4^{10} = 1048576$) tests. Empirical studies by NIST and others [21], covering domains such as medicine, web browsers, software systems, and NASA applications, reveal that most software failures are caused by the interaction of only a small number of factors (typically 6 or fewer), with not all factors contributing to every failure. This insight underscores the importance of prioritizing interactions between fewer factors during testing.

Covering arrays (CAs) provide a combinatorial framework for efficient t -way interaction testing. For any t factors (where t is the maximum number of factors that interact to cause a failure), CAs ensure that all the g^t possible input combinations across each combination of t factors is tested. Since t is typically much smaller than k (the total number of factors in the system), this approach significantly reduces the number of tests compared to exhaustive testing, which requires g^k tests for k factors. Furthermore, CAs provide test suites of size $O(\log k)$ tests [16], making them far more cost-effective. As a result, they have become a vital tool in software testing [30], allowing for efficient detection of the presence of interaction faults. In addition to software testing, covering arrays find their use in various other fields. They provide a cost-effective solution for testing switching circuits [34], drug screenings [31], data compression [20], and more [32]. They also find applications in mathematical domains, including the construction of difference matrices [10], truth functions [17], and codes with small covering radius [14].

The foundational property of covering arrays is termed *t-qualitative independence*, and is also commonly referred to as *t-way coverage* or *t-way covering* in the testing literature. Formally, let n , g , and t be positive integers, then a set of vectors of length n and entries from a set S of cardinality g is said to be *t-qualitatively independent* if for any t -subset, $\{v_1, v_2, \dots, v_t\}$, of vectors and any ordered t -tuple say $(g_1, g_2, \dots, g_t) \in S^t$, there exists a j such that $v_i[j] = g_i$ for $i = 1, 2, \dots, t$. A *covering array*, denoted *t-CA*(n, k, g) or *CA*($n; t, k, g$), of strength t , alphabet size g , k factors, and size n , is an $n \times k$ array, where entries are from a set S of cardinality g (say \mathbb{Z}_g), with the property that any set of t column vectors is *t-qualitatively independent*. Therefore, CAs are also known as *qualitatively independent families* (see [22] and Remark 10.5, Chapter VI in [7]). In practical applications, this framework enables systematic testing, where each row represents a test case (trial), each column corresponds to a factor affecting the experiment, and each factor can take on g distinct values (levels). Since testing is an expensive affair [21, 26], the goal is to minimize the number of rows (tests), as fewer tests reduce costs and improve efficiency. An *optimal* covering array is one with the least possible size. The size of an optimal covering array, *t-CAN*(k, g), is known as the *covering array number*. This parameter is of primary interest to us, as the aim is to minimize the number of tests.

CAs of strength two have garnered significant attention, as studies indicate that 2-way

testing provides adequate fault detection coverage [6, 11]. In real-world systems, some factors may not interact, making it unnecessary to test combinations of their values. Such systems can be effectively represented using graphs, where vertices correspond to factors and edges indicate interactions between them. A *graph* is defined as a pair (V, E) , where V is the set of vertices and $E \subseteq V \times V$ is the set of edges. Covering arrays on graphs, introduced by Meagher and Stevens in 2005 [25], generalizes traditional CAs by requiring only the columns corresponding to adjacent vertices to be 2-qualitatively independent. When two factors are known to not interact, their corresponding vertices are not connected by an edge in the graph. This approach may result in smaller test suites, as it eliminates the need to cover all 2-tuples for non-adjacent vertices.

Pairwise (2-way interaction) testing provides a cost-effective method with reasonable fault detection, but may fail to identify up to 40% of system bugs [21]. To address higher-order interactions, hypergraphs, which generalize graphs, offer a robust framework for capturing multi-adic relationships. Covering arrays on hypergraphs build on this framework to develop more comprehensive test suites. This concept led to the investigation of *covering arrays on hypergraphs* by Akhtar and Maity [2] in 2017 and Raaphorst et al. [28] in 2018. In the hypergraph model, factors correspond to vertices, while sets of mutually interacting factors form hyperedges. A CA on a hypergraph ensures for each hyperedge of the hypergraph, that the set of columns associated with its vertices (say t many) is t -qualitatively independent. The objective is to find the minimum number of tests required to detect the presence of interaction faults from hypergraph models of real-life systems, which is known as the *covering array number* of the hypergraph. However, Seroussi and Bshouty [29] proved that determining the covering array number of a hypergraph is NP-complete, by way of reduction from the three-colouring problem in graphs, which is well-known to be NP-complete. For more details on the construction of the covering arrays on hypergraphs, refer to [1, 3, 4].

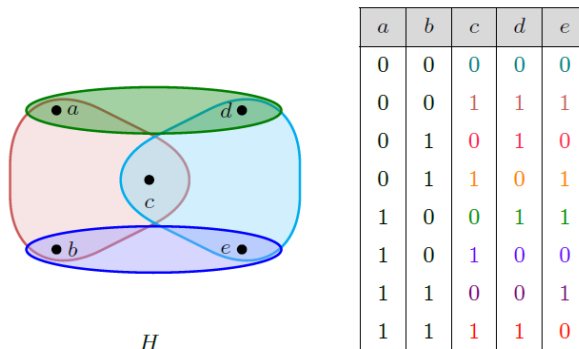
Extending the notion of qualitative independence graphs from [24], Raaphorst et al. [28] introduced a special class of hypergraphs known as *qualitative independence hypergraphs*, denoted t - $QI(n, g)$. These hypergraphs are fundamental in characterizing the covering array numbers of hypergraphs. They showed that for a t -uniform hypergraph H and positive integers g and n , a covering array $CA(n, H, g)$ exists if and only if there is a hypergraph homomorphism from H to t - $QI(n, g)$, and further established that $CAN(H, g) \leq CAN(t$ - $QI(n, g), g)$. Meagher and Stevens [25] proved that $CAN(2$ - $QI(n, 2), 2) = n$. Motivated by this result, we explore the question: What is $CAN(3$ - $QI(n, 2), 2)$? In this paper, we study the size of the largest 3-cliques and the strong chromatic number to establish that $CAN(3$ - $QI(n, 2), 2) = n$ for $n = 9, 10$, and 12. Additionally, we conjecture that this result holds for all $n \geq 8$.

The rest of the paper is organized as follows: Section 2 introduces formal definitions and key preliminaries essential for the subsequent discussions. In Section 3, we note supporting results that are instrumental in proving the main findings presented in Sections 4 and 5. In these sections, we establish that $CAN(3$ - $QI(n, 2), 2) = n$ for $n = 8, 9, 10$, and 12. Specifically, Section 4 addresses the even cases ($n = 8, 10, 12$)

by computing the size of the largest 3-cliques in $3-QI(n, 2)$. Section 5 focuses on $CAN(3-QI(9, 2), 2)$, where solely knowing the size of the largest t -cliques was insufficient, and the strong chromatic number was analyzed, to prove the result. In Section 6, we derive lower bounds on the covering array number of certain families of hypergraphs. Finally, in Section 7, we present an algorithm to verify the validity of colour classes in a strong colouring of $3-QI(n, 2)$, where $n = 8, 9, 10$ and conclude with a discussion in Section 8.

2. Preliminaries

In this section, we define key terms and concepts related to hypergraphs, laying the groundwork for understanding covering arrays on hypergraphs and the specialized class known as qualitative independence hypergraphs, whose covering array number is the focus of our study.



a	b	c	d	e
0	0	0	0	0
0	0	1	1	1
0	1	0	1	0
0	1	1	0	1
1	0	0	1	1
1	0	1	0	0
1	1	0	0	1
1	1	1	1	0

$CA(8, H, 2)$

Figure 1. A hypergraph H and a covering array on it, $CA(8, H, 2)$

A *hypergraph* H is a pair $H = (V, E)$, where $V = \{v_1, v_2, \dots, v_k\}$ is a set of vertices, and $E = \{e_1, e_2, \dots, e_m\}$ is a set of non-empty subsets of V , called hyperedges. Since isolated vertices are not of interest with regard to CAs, the hyperedges are taken such that $\bigcup_{i=1}^m e_i = V$. See Figure 1 for an example of a hypergraph $H = (V, E)$, where $V = \{a, b, c, d, e\}$ and $E = \{\{a, d\}, \{a, b, c\}, \{c, d, e\}, \{b, e\}\}$. A t -uniform hypergraph is one in which every hyperedge contains exactly t vertices. When $t = 2$, this is known as a *graph* (without loops).

For two hypergraphs $H = (V, E)$ and $H_1 = (V_1, E_1)$, a *hypergraph homomorphism* from H to H_1 , denoted $f : H \rightarrow H_1$, is a map $f : V \rightarrow V_1$ such that for every hyperedge $e = \{v_0, \dots, v_m\} \in E$, there exists a hyperedge $e' \in E_1$ such that the image $f(e) = \{f(v_0), \dots, f(v_m)\} \subseteq e'$ and $|e| = |f(e)|$. For example, consider $H_1 =$

(V_1, E_1) , where $V_1 = \{a', b', c', d'\}$ and $E_1 = \{\{a', b', c'\}, \{c', d'\}\}$. Then, there exists a homomorphism $f : H \rightarrow H_1$, where H is the hypergraph given in Figure 1, given by $f(a) = f(e) = a'$, $f(c) = c'$ and $f(b) = f(d) = b'$. Now, consider $H_2 = (V_2, E_2)$, where $V_2 = \{a'', b'', c'', d''\}$ and $E_2 = \{\{a'', b''\}, \{c'', d''\}, \{a'', d''\}\}$. There cannot be a homomorphism $H \rightarrow H_2$ because hyperedges of size 3 in H cannot be mapped to hyperedges in H_2 , as H_2 contains no hyperedges of size 3.

For a hypergraph $H = (V, E)$, a *strong k -colouring* (of the vertices) is a k -partition (S_1, S_2, \dots, S_k) of V such that no colour appears more than once within any hyperedge. Formally, for every hyperedge $e \in E$, $|e \cap S_i| \leq 1$ for all $i = 1, 2, \dots, k$. The *strong chromatic number* of a hypergraph H , denoted $\chi_S(H)$, is the smallest integer k for which H admits a strong k -colouring. For example, the hypergraph H in Figure 1 has $\chi_S(H) = 3$, where $S_1 = \{a, e\}$, $S_2 = \{b, d\}$, and $S_3 = \{c\}$. A *subhypergraph* $H' = (V', E')$ of a hypergraph $H = (V, E)$ is a hypergraph such that $V' \subseteq V$, $E' \subseteq E \cap \mathcal{P}(V')$, where $\mathcal{P}(V')$ is the power set of V' . If $E' = E \cap \mathcal{P}(V')$, then H' is said to be an *induced subhypergraph*. A *t -complete hypergraph of order n* , denoted K_n^t , is a t -uniform hypergraph $H = (V, E)$ such that $|V| = n$ and E consists of all t -subsets of V . Given a hypergraph H , a *t -clique* of size m in H is a subset $C \subseteq V$ of size m such that the induced subhypergraph on C is K_m^t . We write $\omega_t(H)$ to denote the size of the largest t -clique in H .

Now, we define a generalization of covering arrays, known as covering arrays on hypergraphs:

Definition 1. (Covering Array on a Hypergraph) Let n and g be positive integers and S be a set of size g . A *covering array on a hypergraph* $H = (V, E)$, of size n and alphabet size g , denoted $CA(n, H, g)$, is an $n \times |V|$ array with entries from S and columns indexed by the vertices in V such that: for any hyperedge $e \in E$ with $|e| = m$, the sub-array corresponding to the vertices in e contains every ordered m -tuple from S^m at least once as a row.

Definition 2. (Covering Array Number of a Hypergraph) The *covering array number of a hypergraph* H , $CAN(H, g)$, is the least possible size n of a $CA(n, H, g)$.

Figure 1 shows a covering array of size 8 with alphabet size 2 on the given hypergraph H . The 8 binary 3-tuples corresponding to the hyperedge $\{c, d, e\}$ are highlighted in different colours. This array is optimal since all 8 possible binary 3-tuples must be covered for hyperedges with 3 vertices, implying that $CAN(H, 2) = 8$.

The concept of *qualitative independence*, rooted in *extremal set theory*, is a key property associated with covering arrays:

Definition 3. (Qualitatively Independent Partitions) Let R be an n -element set. Consider a family of partitions of R into g classes, $\mathcal{P} = \{R_1, R_2, \dots, R_k\}$, where $R_j = (M_1^j, M_2^j, \dots, M_g^j)$ for $j = 1, 2, \dots, k$. The set \mathcal{P} is said to be *t -qualitatively independent*

if, for every t of the partitions $R_{i_1}, R_{i_2}, \dots, R_{i_t}$, and for every choice of classes $M_s^{i_j}$ (with $1 \leq j \leq t, 1 \leq s \leq g$), $\bigcap_{j=1}^t M_s^{i_j} \neq \emptyset$, for all choices of s and distinct choices of i_j .

Each of these partitions can be represented in terms of vectors, by defining a vector v corresponding to $R_j = (M_{j_1}, M_{j_2}, \dots, M_{j_g})$, where $v[i] = k - 1$ if $i \in M_{j_k}$ ($1 \leq i \leq n, 1 \leq k \leq g$). Thus, a set of partitions is t -qualitatively independent if and only if the corresponding vectors are t -qualitatively independent. For example, consider $n = 11$, $R = [11]$ ($[n]$ denotes $\{1, 2, \dots, n\}$), $g = 3$, $S = \{0, 1, 2\}$, and $t = 2$. Table 1 illustrates some 3-partitions that are 2-qualitatively independent, along with their 2-qualitatively independent vector counterparts.

Table 1. 2-qualitatively independent partitions of [11] and corresponding 2-qualitatively independent vectors

Partitions into 3 classes	Vectors
$\{\{1, 2, 7\}, \{4, 5, 6, 8\}, \{3, 9, 10, 11\}\}$	00211101222
$\{\{1, 3, 8\}, \{2, 5, 6, 9\}, \{4, 7, 10, 11\}\}$	01021120122
$\{\{1, 4, 9\}, \{2, 3, 6, 10\}, \{5, 7, 8, 11\}\}$	01102122012
$\{\{1, 5, 10\}, \{2, 3, 4, 11\}, \{6, 7, 8, 9\}\}$	01110222201
$\{\{1, 6, 11\}, \{3, 4, 5, 7\}, \{2, 8, 9, 10\}\}$	02111012220

Definition 4. (Qualitative Independence Hypergraph) Let n , g , and t be positive integers such that $t \geq 2$, $n \geq g^t$. A t -qualitative independence hypergraph, t - $QI(n, g) = (V, E)$, is a t -uniform hypergraph, where V is the set of all g -partitions of an n -set with the property that every class of the partition has size at least g^{t-1} (or equivalently, set of all vectors of length n with entries from S and with the property that every alphabet appears at least g^{t-1} times and the first appearance of every alphabet is in the lexicographic order). In this hypergraph, a set of t vertices forms a hyperedge when the corresponding partitions are t -qualitatively independent.

For example, the graph 2 - $QI(4, 2)$ has vertices $\{\{1,2\}\{3,4\}\}$, $\{\{1,3\}\{2,4\}\}$, $\{\{1,4\},\{2,3\}\}$ and has edges between each pair of these vertices, so it is equivalent to the complete graph on three vertices, K_3 .

This special class of hypergraphs led to the following characterization for the existence of a CA on a given t -uniform hypergraph and of a given size, derived from [28].

Theorem 1. For a t -uniform hypergraph H and positive integers g and n , a $CA(n, H, g)$ exists if and only if there is a hypergraph homomorphism $H \rightarrow t$ - $QI(n, g)$.

3. Basic Results

In this section, we present a few results that are frequently used in Sections 4 and 5. Let X be a $(t+1)$ - $CA(n, k+1, g)$. Suppose that we select a column in X , say the first column c_1 , and focus on the rows where the value in this column is a fixed symbol,

say a . Then, these rows without c_1 , form a covering array of strength t , say X_1 . This follows because for every choice of t columns $c_{i_1}, c_{i_2}, \dots, c_{i_t}$ in X_1 , the $(t+1)$ -strength property ensures that all $(t+1)$ -tuples with the first entry as a are covered in the columns $c_1, c_{i_1}, c_{i_2}, \dots, c_{i_t}$ of X . This observation establishes an upper bound for the covering array number t -CAN(k, g) based on the covering array number for strength $(t+1)$ with one additional column. The result is summarized in the following theorem:

Theorem 2 (Derivation [9]). *Let k, g, t be positive integers. Then,*

$$t\text{-CAN}(k, g) \leq \left\lfloor \frac{(t+1)\text{-CAN}(k+1, g)}{g} \right\rfloor.$$

The formula for covering array number of complete graphs with alphabet size 2, a significant breakthrough in the field, is as follows:

Theorem 3. [17, 18] *Let n, k be positive integers. Then,*

$$2\text{-CAN}(k, 2) = \min \left\{ n : \binom{n-1}{\lfloor \frac{n}{2} \rfloor - 1} \geq k \right\}.$$

This result was independently proved by Katona [17] and Kleitman and Spencer [18] using the Erdős - Ko - Rado theorem [12]. State-of-the-art reviews of known constructions of covering arrays have been surveyed in detail by Torrez-Jimenez et al., and Lu et al. in 2019 [23, 32]. Bush [5] showed that if g is a prime power and $g > t$, then $t\text{-CAN}(g+1, g) = g^t$, and specifically, when g is a power of 2, then $3\text{-CAN}(g+2, g) = g^3$. It is known that for an integer $t > 1$, $t\text{-CAN}(t+1, 2) = 2^t$, and for $g = 2^m$ (where m is a positive integer), $(g-1)\text{-CAN}(g+2, g) = g^{g-1}$ [13]. Colbourn [7] showed that when g is a prime power and $g \leq t$, $t\text{-CAN}(t+1, g) = g^t$. Johnson-Entringer [15] established that $t\text{-CAN}(t+2, 2) = \lfloor \frac{4}{3} 2^t \rfloor$. The size of an optimal strength-2 covering array with $g > 2, n > g^2$ is known only for a few parameter sets compiled in [19].

Let $f : H \rightarrow H'$ be a hypergraph homomorphism and suppose $\text{CAN}(H', g) = n$. So, there exists a $CA(n, H', g) = X$. We can use this covering array to construct a $CA(n, H, g)$ as follows. For each vertex in H , assign a column corresponding to the column of its image vertex under f in X . Since any set of vertices in H that form a hyperedge must map to a set of vertices in H' that also form a hyperedge of the same size, the corresponding columns in X will be qualitatively independent. This observation implies the following lemma:

Lemma 1. [28] *If H, H' are hypergraphs and $g \in \mathbb{N}$ such that there exists a homomorphism $H \rightarrow H'$, then $\text{CAN}(H, g) \leq \text{CAN}(H', g)$.*

This lemma, along with Theorem 1, implies that if there exists a homomorphism $H \rightarrow t\text{-}QI(n, g)$, then $CAN(H, g) \leq n$. Meagher et al. [25] established that $CAN(2\text{-}QI(n, 2), 2) = n$ for $n \geq 4$ and conjectured that $CAN(2\text{-}QI(n, g), g) = n$ for $n \geq g^2$. Motivated by their work, we ask the question: What is $CAN(3\text{-}QI(n, 2), 2)$?

Lemma 2. $CAN(3\text{-}QI(n, 2), 2) \leq n$ for $n \geq 8$.

Proof. The vector representations of the vertices of $3\text{-}QI(n, 2)$ can be arranged as columns to construct a $CA(n, 3\text{-}QI(n, 2), 2)$, since a set of vertices forms a hyperedge if and only if they are 3-qualitatively independent. Thus, $CAN(3\text{-}QI(n, 2), 2) \leq n$. \square

Corollary 1. *If there does not exist a homomorphism $3\text{-}QI(n, 2) \rightarrow 3\text{-}QI(n-1, 2)$, then $CAN(3\text{-}QI(n, 2), 2) = n$.*

Proof. From Lemma 2, $CAN(3\text{-}QI(n, 2), 2) \leq n$. Suppose $CAN(3\text{-}QI(n, 2), 2) < n$. So, there exists a $CA(n-1, 3\text{-}QI(n, 2), 2)$ (by adding a sufficient number of extra rows, if needed). By Theorem 1, this implies the existence of a homomorphism $3\text{-}QI(n, 2) \rightarrow 3\text{-}QI(n-1, 2)$, which contradicts the assumption. \square

4. $CAN(3\text{-}QI(n, 2), 2)$ for $n = 8, 10, 12$

In this section, we determine the size of the largest 3-cliques in $3\text{-}QI(n, 2)$ for $n = 8, 9, 10, 11, 12, 13$. Using these results and others from Section 3, we establish that $CAN(3\text{-}QI(n, 2), 2) = n$ for $n = 8, 10, 12$.

Theorem 4. *The covering array number of $3\text{-}QI(8, 2)$, $CAN(3\text{-}QI(8, 2), 2) = 8$.*

Proof. From Lemma 2, $CAN(3\text{-}QI(8, 2), 2) \leq 8$. For any CA on 2 symbols, if at least one set of three columns is required to be 3-qualitatively independent, it will require at least eight rows to cover all the possible $2^3 = 8$ 3-tuples from $\{0, 1\} \times \{0, 1\} \times \{0, 1\}$. Therefore, $CAN(3\text{-}QI(8, 2), 2) \geq 8$, since the hyperedges are of size 3. \square

The following theorem is derived from the observation that for any positive integer t , t -cliques are preserved under hypergraph homomorphisms.

Theorem 5. [27] *If H, H' are hypergraphs such that there is a homomorphism $H \rightarrow H'$, then for any $t \in \mathbb{N}$, $\omega_t(H) \leq \omega_t(H')$.*

Corollary 1 and Theorem 5 give us the following result, which we use to determine $CAN(3\text{-}QI(n, 2), 2)$ for $n = 10, 12$:

Corollary 2. *If $\omega_3(3\text{-}QI(n, 2)) > \omega_3(3\text{-}QI(n-1, 2))$, then $CAN(3\text{-}QI(n, 2), 2) = n$.*

Lemma 3. *The size of the largest 3-clique in $3\text{-}QI(9, 2)$, $\omega_3(3\text{-}QI(9, 2)) = 4$.*

Proof. Figure 2 shows a covering array of strength 3 with 4 factors and size 8 [7].

0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Figure 2. $3\text{-}CA(8, 4, 2)$

With the addition of another random row with entries from $\{0, 1\}$, the column vectors in this CA, when considered as vertices, form a 3-clique of size 4 in $3\text{-}QI(9, 2)$. Thus, $\omega_3(3\text{-}QI(9, 2)) \geq 4$. Now, suppose $\omega_3(3\text{-}QI(9, 2)) \geq 5$. This would imply the existence of a $3\text{-}CA(9, 5, 2)$ and by Derivation (Theorem 2), the existence of a $2\text{-}CA(4, 4, 2)$. However, by Theorem 3, for a CA of strength 2, alphabet size 2, and 4 columns to exist, at least 5 rows are required. This contradicts the assumption that such a $2\text{-}CA(4, 4, 2)$ exists. Hence, $\omega_3(3\text{-}QI(9, 2)) < 5$ and we conclude that $\omega_3(3\text{-}QI(9, 2)) = 4$. \square

The existence of a homomorphism $3\text{-}QI(8, 2) \rightarrow 3\text{-}QI(9, 2)$, where a vertex v is mapped to $(v, 0)$, along with the above result implies $\omega_3(3\text{-}QI(8, 2)) = 4$.

Lemma 4. *The size of the largest 3-clique in $3\text{-}QI(10, 2)$, $\omega_3(3\text{-}QI(10, 2)) = 5$.*

Proof. Since there exists a $3\text{-}CA(10, 5, 2)$ [8], $\omega_3(3\text{-}QI(10, 2)) \geq 5$. Now, suppose $\omega_3(3\text{-}QI(10, 2)) \geq 6$. Then its vertices, when considered as column vectors, would form a $3\text{-}CA(10, 6, 2)$. By Derivation (Theorem 2), there exists a $2\text{-}CA(5, 5, 2)$, which is a contradiction to Theorem 3, which says that for a CA of strength 2 with alphabet size 2 and 5 columns to exist, at least 6 rows are required. Therefore, $\omega_3(3\text{-}QI(10, 2)) \leq 5$ and hence $\omega_3(3\text{-}QI(10, 2)) = 5$. \square

Now, we use Lemmas 3 and 4 to establish the following result.

Theorem 6. *The covering array number of $3\text{-}QI(10, 2)$, $CAN(3\text{-}QI(10, 2), 2) = 10$.*

Proof. Since $\omega_3(3\text{-}QI(10, 2)) = 5 > 4 = \omega_3(3\text{-}QI(9, 2))$, it follows from Corollary 2 that $CAN(3\text{-}QI(10, 2), 2) = 10$. \square

Lemma 5. *The size of the largest 3-clique in $3\text{-}QI(11, 2)$, $\omega_3(3\text{-}QI(11, 2)) = 5$.*

Proof. From the proof of Lemma 4, there exists a 3-CA(10, 5, 2). Addition of an arbitrary row to this, gives a 3-CA(11, 5, 2). If the column vectors are now considered as vertices of 3-QI(11, 2), we have a 3-clique of size 5, so $\omega_3(3\text{-QI}(11, 2)) \geq 5$. However, if $\omega_3(3\text{-QI}(11, 2)) \geq 6$, then there exists 3-CA(11, 6, 2). This means by Derivation (Theorem 2), there exists a 2-CA(5, 5, 2), a contradiction to Theorem 3, which says that for a CA of alphabet size 2 and 5 columns to exist, at least 6 rows are required. Therefore, $\omega_3(3\text{-QI}(10, 2)) \leq 5$ and hence $\omega_3(3\text{-QI}(10, 2)) = 5$. \square

A lower bound on the size of the largest 3-clique in 3-QI(12, 2) is provided in [8], showing that $\omega_3(3\text{-QI}(12, 2)) \geq 11$. This leads us to the following result.

Theorem 7. *The covering array number of 3-QI(12, 2), $CAN(3\text{-QI}(12, 2), 2) = 12$.*

Proof. Since $\omega_3(3\text{-QI}(12, 2)) \geq 11 > 5 = \omega_3(3\text{-QI}(11, 2))$, it follows from Corollary 2 that $CAN(3\text{-QI}(12, 2), 2) = 12$. \square

Lemma 6. *The size of the largest 3-clique in 3-QI(13, 2), $\omega_3(3\text{-QI}(13, 2)) = 11$.*

Proof. Since $\omega_3(3\text{-QI}(12, 2)) \geq 11$, the existence of homomorphism $3\text{-QI}(12, 2) \rightarrow 3\text{-QI}(13, 2)$ implies $\omega_3(3\text{-QI}(13, 2)) \geq 11$. Now, suppose $\omega_3(3\text{-QI}(13, 2)) \geq 12$. This would imply the existence of 3-CA(13, 12, 2) and by Derivation (Theorem 2), the existence of a 2-CA(6, 11, 2). This contradicts Theorem 3, which says that for a CA of alphabet size 2 and 11 columns to exist, at least 7 rows are required. Therefore, $\omega_3(3\text{-QI}(13, 2)) \leq 11$ and hence $\omega_3(3\text{-QI}(13, 2)) = 11$. \square

The above result further implies that $\omega_3(3\text{-QI}(12, 2)) = \omega_3(3\text{-QI}(13, 2)) = 11$. Notably, since $\omega_3(3\text{-QI}(8, 2)) = \omega_3(3\text{-QI}(9, 2))$, $CAN(3\text{-QI}(9, 2), 2)$ could not be determined using this technique. Computing the size of the largest cliques did not provide sufficient insight to establish the covering array number for cases $n = 9, 11$. Consequently, we proceeded to explore whether the strong chromatic number could be utilized to resolve such cases, and the results are presented in the next section.

5. $CAN(3\text{-QI}(9, 2), 2)$

We now compute the strong chromatic number of 3-QI(8, 2) and establish bounds for $\chi_s(3\text{-QI}(9, 2))$ that paved the way to establish that $CAN(3\text{-QI}(9, 2), 2) = 9$. To proceed, we first prove the following theorem, which is essential for determining the covering array number of 3-QI(n , 2) via its strong chromatic number.

Theorem 8. *If H, H' are hypergraphs such that there is a homomorphism $H \rightarrow H'$, then $\chi_s(H) \leq \chi_s(H')$.*

Proof. Let $f : H \rightarrow H'$ be a homomorphism and $\chi_S(H') = k$. Consider a strong k -colouring of H' . Let $e = \{v_1, v_2, \dots, v_m\}$ be an edge in H . Colour each v_i with the colour assigned to $f(v_i)$. If two vertices in e are given the same colour, then their images also have the same colour, which is a contradiction to the fact that in a strong k -colouring, no two vertices in a hyperedge are coloured the same. Hence, we have a k -colouring of H . So, $\chi_S(H) \leq k$. \square

Using Corollary 1 and Theorem 8, we derive the following result, which will be instrumental in determining $CAN(3-QI(n, 2), 2)$ for $n = 9, 11$.

Corollary 3. *If $\chi_S(3-QI(n, 2)) > \chi_S(3-QI(n-1, 2))$, then $CAN(3-QI(n, 2), 2) = n$.*

To determine the structure of the colour classes in $3-QI(n, 2)$, we establish the necessary representation for its vertices. Henceforth, each vertex will be represented by the smaller part in the associated 2-part partition of $[n]$. In cases where both parts are of equal size, either part can be chosen for representation. For instance, the vertex $\{A, \bar{A}\}$, where $A = \{1, 2, 3, 4\}$ (\bar{A} denotes the complement of A) in $3-QI(8, 2)$ will be represented by A . Furthermore, when explicitly writing out the elements of the part A , we drop the curly braces and use a string format. Thus, the vertex A , defined above, will be written as 1234. The following condition serves as a key criterion for determining the membership of vertices within hyperedges of $3-QI(n, 2)$.

Condition 1. A pair of distinct vertices A, B , cannot belong to a hyperedge in $3-QI(n, 2)$ if at least one of the intersections $A \cap B, A \cap \bar{B}, \bar{A} \cap B, \bar{A} \cap \bar{B}$ has cardinality at most one.

Proof. Let X be one of the sets $A \cap B, A \cap \bar{B}, \bar{A} \cap B, \bar{A} \cap \bar{B}$ with cardinality less than two. Now, suppose there is a hyperedge in $3-QI(n, 2)$ with vertices A, B , and C . Then, by definition of qualitative independence, each of $A \cap B \cap C, A \cap B \cap \bar{C}, A \cap \bar{B} \cap C, \bar{A} \cap \bar{B} \cap \bar{C}, \bar{A} \cap B \cap C, \bar{A} \cap \bar{B} \cap C, \bar{A} \cap B \cap \bar{C}, A \cap \bar{B} \cap \bar{C}$ must be non-empty. But here, either $X \cap C = \emptyset$ or $X \cap \bar{C} = \emptyset$, which contradicts the fact that vertices in a hyperedge in $3-QI(n, 2)$ are 3-qualitatively independent. \square

Lemma 7. *Let F be a set with cardinality n , where n is even, and let X be an $\frac{n}{2}$ -subset of F . Let \mathcal{F}_1 be the family of $\frac{n}{2}$ -subsets of F , intersecting X at exactly one element, and \mathcal{F}_2 be the family of $\frac{n}{2}$ -subsets of F , intersecting X at $(\frac{n}{2} - 1)$ elements. Then $|\mathcal{F}_1| = |\mathcal{F}_2|$.*

Proof. Let $A \in \mathcal{F}_1$. Since $|X \cap A| = 1$, all the remaining $(\frac{n}{2} - 1)$ elements of X are in \bar{A} . Hence, $|X \cap \bar{A}| = (\frac{n}{2} - 1)$, and the same holds vice versa as well. We show that the function $f : \mathcal{F}_1 \rightarrow \mathcal{F}_2$, where $f(A) = \bar{A}$ is a bijection. If $\bar{A} = \bar{B}$, then $\bar{\bar{A}} = \bar{\bar{B}}$, implying $A = B$ and hence, f is injective. For any element $B \in \mathcal{F}_2$, its complement is also an $\frac{n}{2}$ -set, since n is even, such that $f(\bar{B}) = B$, making f surjective. Therefore, f is a bijection. \square

Table 2. A sample colour class partitioning of the vertices of $3\text{-}QI(8, 2)$.

Colour Class	Vertices					
Class 1	1234 5678	2345 1678	2346 1578	2347 1568	2348 1567	
Class 2	1235 4678	1356 2478	1357 2468	1358 2467	1345 2678	
Class 3	1236 4578	1246 3578	1256 3478	1267 3458	1268 3457	
Class 4	1247 3568	1347 2568	1457 2368	1467 2358	1478 2356	
Class 5	1257 3468	2357 1468	2457 1368	2567 1348	2578 1346	
Class 6	1367 2458	2367 1458	3467 1258	3567 1248	3678 1245	
Class 7	1237 4568	1238 4567	1278 3456	1378 2456	2378 1456	

Lemma 8. Any pair of distinct $(\lfloor \frac{n}{2} \rfloor - 1)$ -subsets of a $\lfloor \frac{n}{2} \rfloor$ -set has exactly $(\lfloor \frac{n}{2} \rfloor - 2)$ elements in common, for $n \geq 4$.

Proof. To have distinct subsets, at least one element must be different. Since we have only a total of $\lfloor \frac{n}{2} \rfloor$ elements, and the subset size is $(\lfloor \frac{n}{2} \rfloor - 1)$, by Pigeonhole principle any two distinct $(\lfloor \frac{n}{2} \rfloor - 1)$ -subsets differ exactly by one element and hence have exactly $(\lfloor \frac{n}{2} \rfloor - 2)$ elements in common. \square

Latin squares are combinatorial designs that are widely used in the design of experiments. We use their existence to prove the results in this section. Let n be a positive integer. A *Latin square of order n* , based on the set S of cardinality n , is an $n \times n$ array, each of whose entries is an element of S such that each of the n elements of S occurs once (and hence exactly once) in each row and once in each column.

Theorem 9. [33] For any positive integer n , a Latin square of order n always exists.

Theorem 10. The strong chromatic number of $3\text{-}QI(8, 2)$, $\chi_S(3\text{-}QI(8, 2)) = 7$.

Proof. In $3\text{-}QI(8, 2)$, the vertices correspond to 2-partitions of $[8]$ where each class of the partition has size at least $2^{3-1} = 4$. Thus, each vertex is a 2-partition $\{A, \bar{A}\}$, where $|A| = |\bar{A}| = 4$. These partitions can be represented by either A or \bar{A} . Consequently, the set of vertices can be viewed as a subset of all 4-subsets of $[8]$. Each 4-subset of $[8]$ represents a valid vertex, and each vertex $\{A, \bar{A}\}$ appears exactly twice in the set of 4-subsets of $[8]$: once as A and once as \bar{A} . Therefore, $3\text{-}QI(8, 2)$ has $\frac{1}{2} \binom{8}{4} = 35$ vertices. These 4-subsets can be partitioned into 7 groups (one such grouping is given in Table 2 where subsets are represented as strings) such that any two vertices in a group cannot belong to the same hyperedge, as dictated by Condition 1. Each of these groups contains five vertices, marked in bold and their complements (when considered as set) in normal font, and forms a colour class of five elements each (verified using a code based on Algorithm 1 given in Section 7). This shows that the minimum number of colour classes, $\chi_S(3\text{-}QI(8, 2)) \leq \frac{35}{5} = 7$. Now if $\chi_S(3\text{-}QI(8, 2)) < 7$, then Pigeonhole principle would imply that at least one colour

class contains more than five vertices. The rest of this proof deals with establishing that no colour class can have more than five vertices, thereby confirming that $\chi_S(3\text{-}QI(8, 2)) = 7$.

Consider a 4-subset, $A = abcd$, where a, b, c, d are distinct elements in $[8]$. For another 4-subset B to belong to the same colour class as A , B must intersect with A at either zero (in which case $B = A$), one, or three elements, as required by Condition 1. We count the number of such subsets B . By Lemma 7, the 4-subsets that intersect A at exactly one element are in one-to-one correspondence with those that intersect A at exactly three elements, under the complement function. Effectively, each vertex is mapped to itself under this map. So, without loss of generality, it is sufficient to consider only those subsets that intersect A at three elements, as the subsets intersecting at one element are automatically complements of these and represent the same set of vertices. There are $\binom{4}{3} = 4$ ways to choose which of the three elements of A are intersecting with B and $\binom{4}{1} = 4$ ways to choose the 4th element in B which is not in A . Hence, there are at most $4 \times 4 = 16$ subsets B that intersect A at exactly three elements. These subsets are listed in Figure 3, where $\bar{A} = efgh$.

abce	acde	abde	bcd e
abcf	acdf	abdf	bcd f
abcg	acd g	abd g	bcd g
abch	acd h	abd h	bcd h

Figure 3. Vertices in $3\text{-}QI(8, 2)$ that cannot share a hyperedge with $A = abcd$

We now claim that the entries in each column of the array in Figure 3 can be rearranged in such a way that any two vertices in the same row cannot belong to the same colour class. If these 16 subsets can be organized into 4 rows such that any two subsets in the same row intersect at exactly two elements, they would form a hyperedge and, consequently, cannot belong to the same colour class. This arrangement ensures that, at most one subset from each row can belong to the same colour class as A , resulting in a maximum colour class size of 5. By Lemma 8, the first 3 digits in any two subsets in distinct columns in the array in Figure 3 intersect at exactly two elements. So, to establish the claim, the fourth digit in each entry must be distinct in any row. If we label each subset by its 4th digit, we are looking for a Latin square of order 4 with symbols e, f, g, h , whose existence is guaranteed by Theorem 9. For instance, see Figure 4 for the Latin square and the corresponding rearrangement. Hence, a colour class can have at most five vertices and $\chi_S(3\text{-}QI(8, 2)) = 7$. \square

e	f	g	h
f	g	h	e
g	h	e	f
h	e	f	g

abce	acdf	abd g	bcd h
abcf	acde	abd h	bcd g
abcg	acd h	abde	bcd f
abch	acd g	abdf	bcd e

Figure 4. A Latin square with symbols e, f, g, h (on the left) and a rearrangement of the array in Figure 3 such that any two vertices in a row intersect at exactly two digits, ensuring they form a hyperedge and thus cannot belong to the same colour class

In the following results, we analyze the maximum number of vertices of specific types that can belong to the same colour class as a given vertex A in a strong colouring of the vertices in $3\text{-}QI(9, 2)$. Specifically, we categorize the vertices based on their intersection properties with A and determine the limits imposed by the structural properties of $3\text{-}QI(9, 2)$. Moving forward, we represent the vertices of $3\text{-}QI(9, 2)$ using a 4-subset of $[9]$ in string format. For example, the vertex A is denoted as $abcd$, where a, b, c, d are distinct elements in $[9]$.

Lemma 9. *Let $A = abcd$ be a vertex of $3\text{-}QI(9, 2) = (V, E)$. Consider a family $\mathcal{F} = \{B \in V : |A \cap B| = 0, |B| = 4\}$. Then, $|\mathcal{F}| = 5$ and \mathcal{F} along with A can form a colour class in $3\text{-}QI(9, 2)$.*

Proof. For any $B \in \mathcal{F}$, $B \subseteq \bar{A}$. Thus, there are $\binom{5}{4} = 5$ ways to form a 4-subset B by choosing 4 digits from $\bar{A} = efghi$ (say), namely

$$efgh, efgi, eghi, efhi, fghi.$$

By the Pigeonhole principle, any two of these subsets must intersect at exactly three digits since they are all 4-subsets of the 5-set $efghi$. According to Condition 1, subsets that intersect in this manner cannot be in the same hyperedge. Therefore, they can all belong to the same colour class along with A . \square

Lemma 10. *Let $A = abcd$ be a vertex of $3\text{-}QI(9, 2) = (V, E)$. Consider a family $\mathcal{F} = \{B \in V : |A \cap B| = 1, |B| = 4\}$. Then, in any strong colouring of $3\text{-}QI(9, 2)$, a colour class containing A can have at most six from \mathcal{F} .*

Proof. For each $x \in A$, let $\mathcal{F}_x = \{B \in \mathcal{F} : A \cap B = \{x\}\}$ implying $|\mathcal{F}_x| = \binom{5}{3} = 10$. Let $B_1, B_2 \in \mathcal{F}_x$. To be in the same colour class, $|B_1 \cap B_2| \neq 2$; that is, $|B_1 \cap B_2| = 0, 1$, or 3. However, intersection at zero or one digit is not possible as $x \in B_1 \cap B_2$ and rest of the elements of B_1 and B_2 must be selected from the complement set, $efghi$, which has only 5 elements. By Pigeonhole Principle, any two 3-subsets of the complement will intersect at least at one digit, so $|B_1 \cap B_2| \geq 2$. Hence, to belong to the same colour class we must have $|B_1 \cap B_2| = 3$. This means $B_1 \setminus \{x\}$ must intersect with B_2 at exactly two elements from $efghi$ (\bar{A}).

Now, we investigate how many subsets from \mathcal{F} can belong to the same colour class as A . Let B be a subset such that $|A \cap B| = 1$, say $B = aefg$ without loss of generality. Consider $\mathcal{G}_x = \{C : |B \cap C| \neq 2, x \in C\} \subseteq \mathcal{F}_x$ for $x = a, b, c, d$. Within \mathcal{G}_x , for $x \neq a$, any subset must have exactly one or three digits from efg . Thus, $|\mathcal{G}_x| = 4$, for $x = b, c, d$ as subsets intersecting with efg at exactly two digits ($\binom{3}{2} \cdot \binom{2}{1} = 6$) are excluded. For example, $\mathcal{G}_b = \{befg, behi, bfhi, bghi\}$. Among these, at most three can coexist in a colour class. If we choose $B_1 = befg$, no additional subsets can be chosen from \mathcal{G}_b as they intersect at two digits with B_1 . Furthermore, no subset from \mathcal{G}_a can be chosen, as any vertex from \mathcal{G}_a intersecting B at three digits, would intersect B_1 at two digits. Similarly, choosing $B_2 = cefg$ invalidates all other subsets from \mathcal{G}_c .

Now selecting $B_3 = defg$ excludes the remaining subsets from \mathcal{G}_d . This led to a total of 4 options from \mathcal{F} , namely $ae fg$, $be fg$, $ce fg$ and $de fg$. Alternatively, if B_3 is one of the other three options, say $de hi$, we have 6 possibilities in total from \mathcal{F} : $ae fg$, $be fg$, $ce fg$, $de hi$, $df hi$, and $dgh i$. See Figure 5 for an exhaustive tree representation of all the 8 possibilities. The option we just mentioned is represented by the paths marked 1 and 2 in the figure.

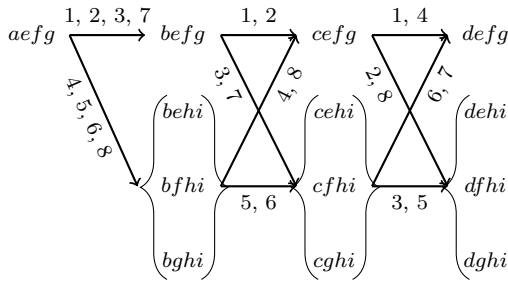


Figure 5. Ways to choose vertices in a colour class containing $A = abcd$ and $B = ae fg$

Next, consider the 3×3 sub-array, avoiding the first row in Figure 5. It can be rearranged in such a way that any two subsets in a row intersect at two digits (we are looking for a Latin square with symbols ehi , fhi , and ghi since these intersect pairwise at precisely two digits, which we know exists by Theorem 9. See Figure 6). Thus, a colour class can have at most three elements from this sub-array.

$be hi$	$ce hi$	$dgh i$
$bf hi$	$cg hi$	$de hi$
$bg hi$	$ch fi$	$df hi$

Figure 6. Latin square with symbols ehi , fhi and ghi

Moreover, choosing a subset $be fg$, $ce fg$, or $de fg$ from the first row in Figure 5, invalidates every other subset in the same column and leaves only $ae hi$, $af hi$, and $ag hi$ as options from \mathcal{G}_a and none of these can coexist with $ae fg$ in the same colour class. This gives at most 5 elements (say, $ae fg$, $be fg$, and 3 from the sub-array) in the colour class. If instead a subset is selected from the other three rows, say $be hi$ (without loss of generality), it leaves just two options ($af gh$, $af gi$) from \mathcal{G}_a apart from $B = ae fg$, again leading to at most 6 elements ($ae fg$, $be hi$, $af gh$, $af gi$, and two more from the sub-array). On the other hand, in any vertex chosen from \mathcal{G}_a to be in the same colour class as that of $B = ae fg$, note that exactly two elements from efg have to be present. From the six such options $ae fh$, $ae fi$, $ae gh$, $ae gi$, $af gh$, $af gi$ at most three can have pairwise intersection at three digits. Thus, if none from \mathcal{G}_x , $x = b, c, d$ are chosen, we only have a total of four options. In conclusion, we see that in all cases, a maximum of six subsets from \mathcal{F} can belong to the colour class of A . \square

Lemma 11. *Let $A = abcd$ be a vertex of $3\text{-}QI(9, 2) = (V, E)$. Consider a family $\mathcal{F} = \{B \in V : |A \cap B| = 3, |B| = 4\}$. Then, at most five from \mathcal{F} along with A can form a colour class in a strong colouring of $3\text{-}QI(9, 2)$.*

Proof. Since for any $B \in \mathcal{F}$, $|A \cap B| = 3$, there are $\binom{4}{3} \cdot \binom{5}{1} = 20$ choices for B . We arrange them in a 4×5 array as shown in Figure 7. Each row corresponds to a 3-subset of A and each column corresponds to an element from \bar{A} that could possibly be in B . We claim that the entries in each column of this array can be rearranged such

$abce$	$abcf$	$abcg$	$abch$	$abci$
$abde$	$abdf$	$abdg$	$abdh$	$abdi$
$acde$	$acdf$	$acdg$	$acdh$	$acdi$
$bcde$	$bcdf$	$bcdg$	$bcdh$	$bcdi$

$abce$	$abdf$	$acdg$	$bcdh$	$abci$
$abde$	$acdf$	$bcdg$	$abch$	$abdi$
$acde$	$bcdf$	$abcg$	$abdh$	$acdi$
$bcde$	$abcf$	$abdg$	$acdh$	$bcdi$

Figure 7. A family $\mathcal{F} = \{B \in V : |A \cap B| = 3, |B| = 4\}$ (on the left) and a rearrangement of it as described in Lemma 11

that, in any row of a 4×4 sub-array, any pair of subsets belong to a hyperedge. For this to hold, the subsets in each row of the rearranged array must intersect at exactly two digits. By Condition 1, subsets from distinct rows in the original array intersect pairwise at least two digits. Therefore, in the rearranged array, we must ensure that the element from \bar{A} is distinct for each entry in a row. This requirement corresponds to constructing a Latin square of order 4, with entries abc, abd, acd , and bcd . By Theorem 9, such a Latin square exists, proving our claim. One such rearrangement is shown in Figure 7 (on the right) in which entries within a row in the first four columns intersect pairwise at two digits. Thus, the first four columns can contribute at most 4 vertices, one from each row, to a colour class. Furthermore, selecting an entry from this 4×4 sub-array of the rearranged array eliminates all the options in the remaining column (fifth column) of the array except the one that shares three digits with it (since the other vertices intersect with it at two digits). Consequently, a colour class of A can have at most 5 vertices from 4×5 array representing \mathcal{F} . \square

Theorem 11. *For the 3-qualitative independence hypergraph $3\text{-}QI(9, 2)$, $11 \leq \chi_S(3\text{-}QI(9, 2)) \leq 19$.*

Proof. The vertices of $3\text{-}QI(9, 2)$ are 2-partitions of $[9]$, where one part has four elements and the other part contains five elements; so we represent them by the 4-subsets of $[9]$. There are therefore, $\binom{9}{4} = 126$ vertices in this hypergraph. These 4-subsets can be partitioned into groups, with one such partitioning presented in Table 3.

Each of these groups forms a colour class, as no two subsets within the same group can form a hyperedge. According to Condition 1, this requires that any two subsets in a group must intersect at either zero, one, or three digits. This property has

Table 3. A sample set of colour classes of $3\text{-}QI(9, 2)$.

Colour Class	Vertices
Class 1	1234, 1235, 1236, 1237, 1238, 1239, 5678, 5679, 5689, 5789, 6789
Class 2	1256, 1356, 1456, 1567, 1568, 1569, 3478, 3479, 3489, 3789, 4789
Class 3	1278, 1378, 1478, 1578, 1678, 1789, 3456, 3459, 3569, 3469, 4569
Class 4	1357, 2357, 3457, 3567, 3578, 3579, 2468, 2469, 2689, 2489, 4689
Class 5	1368, 2368, 3468, 3568, 3678, 3689, 2457, 2459, 2479, 2579, 4579
Class 6	1467, 2467, 3467, 4567, 4678, 4679, 1358, 1359, 1389, 1589, 3589
Class 7	1249, 1349, 1459, 1469, 1479, 1489, 2356, 2567, 2367
Class 8	1245, 1246, 1247, 1248, 3679
Class 9	1257, 1258, 1259, 1346, 2346
Class 10	1267, 1268, 1269, 1345, 2345
Class 11	2458, 3458, 4568, 4578, 4589
Class 12	1347, 1348, 2589, 2568, 2569
Class 13	1367, 1369, 1379, 1458, 1679
Class 14	1457, 1579, 2349, 2369, 2389
Class 15	1468, 1689, 2358, 2359
Class 16	2347, 2348, 2478, 2378
Class 17	1279, 1289, 2456
Class 18	2379, 2679, 2789
Class 19	2578, 2678

been verified using a code based on Algorithm 1 in Section 7. Therefore, the strong chromatic number of $3\text{-}QI(9, 2)$ satisfies:

$$\chi_S(3\text{-}QI(9, 2)) = \text{minimum number of colour classes} \leq 19.$$

Next, we show that $\chi_S(3\text{-}QI(9, 2)) \geq 11$. To establish this, we prove that a colour class can contain at most 12 vertices. Consequently, the strong chromatic number satisfies:

$$\chi_S(3\text{-}QI(9, 2)) \geq \frac{\text{Number of vertices}}{\text{Maximum size of a colour class}} = \left\lceil \frac{126}{12} \right\rceil = 11.$$

We count the number of subsets that can coexist in the same colour class as a given subset $A = abcd$. If another 4-subset B belongs to the same colour class as A , then by Condition 1, B must intersect A at either zero, one, or three digits. Each of these cases are discussed separately in Lemmas 9, 10, and 11, respectively.

Case 1. When $|A \cap B| = 0$, by Lemma 9, there are at most five possible subsets B .

Case 2. When $|A \cap B| = 1$, Lemma 10 establishes that there are at most six possible subsets B .

Case 3. When $|A \cap B| = 3$, by Lemma 11, at most five such subsets B can belong to the same colour class.

Thus, any colour class containing A can have at most $5 + 6 + 5 + 1$ (itself) = 17 vertices. Now, observe that in Case 1, there are exactly two subsets corresponding to each 3-subset in \overline{A} (since the 4th digit has two possible choices). When selecting any subset from Case 2 (say $ae fg$), which contains a 3-subset from \overline{A} , only two subsets remain available from Case 1. For a specific 3-subset, there are four vertices in Case 2 (here $ae fg$, $be fg$, $ce fg$, $de fg$). If another vertex from Case 2 is chosen, which does not share the 3-subset efg (say $ae fh$), at least one more subset from Case 1 is eliminated along with $be fg$, $ce fg$ and $de fg$. Selecting a third subset from Case 2 with yet another distinct 3-subset removes all remaining options from Case 1. Therefore, the combined contribution of Case 1 and Case 2 can be at most six vertices, reducing the maximum size of the colour class to at most $6 + 5 + 1 = 12$. Hence, $\chi_S(3\text{-}QI(9, 2)) \geq \lceil \frac{126}{12} \rceil = 11$. \square

Theorem 12. *The covering array number of the 3-qualitative independence hypergraph $3\text{-}QI(9, 2)$, $CAN(3\text{-}QI(9, 2), 2) = 9$.*

Proof. Since $\chi_S(3\text{-}QI(8, 2)) = 7 < 11 \leq \chi_S(3\text{-}QI(9, 2))$, by Theorem 8, a homomorphism from $3\text{-}QI(9, 2)$ to $3\text{-}QI(8, 2)$ cannot exist. Thus, by Corollary 3, $CAN(3\text{-}QI(9, 2), 2) = 9$. \square

Theorem 13. *For the 3-qualitative independence hypergraph $3\text{-}QI(10, 2)$, $\chi_S(3\text{-}QI(10, 2)) \leq 30$.*

Proof. The vertices in $3\text{-}QI(10, 2)$ are 2-partitions of $[10]$ and fall into two categories:

1. Partitions where one subset contains 4 elements and the other contains 6 elements. These are represented by their corresponding 4-subsets.
2. Partitions where both subsets contain 5 elements, represented by either of the two 5-subsets.

We represent these subsets in string format, with the digit 10 denoted as 0. The 4-subsets and 5-subsets of $[10]$ which contain all the vertices of $3\text{-}QI(10, 2)$ can be partitioned into the colour classes shown in Table 4.

Condition 1 implies that two 4-subsets can belong to a hyperedge if they intersect exactly at two digits, and thus can belong to the same colour class if they intersect at zero or one or three digits. Similarly, two 5-subsets belong to a hyperedge if they intersect at exactly two or three digits, so they can belong to the same colour class if they intersect at zero or one or four digits. Additionally, a 4-subset and a 5-subset can belong to a hyperedge if they intersect at exactly two digits, so they can belong to the same colour class if they intersect at zero, one, three, or four digits. We use Algorithm 1 given in Section 7 to verify these conditions for the classes shown in Table 4. Therefore, $\chi_S(3\text{-}QI(10, 2)) \leq 30$. \square

Table 4. A sample set of colour classes of $3-QI(10, 2)$.

Colour Class	Vertices
Class 1	1234, 12345 67890, 6789, 1235, 12346 57890, 6780, 1236, 12347 56890, 6890, 1237, 12348 56790, 7890, 1238, 12349 56780, 6790, 1239, 12340 56789, 1230
Class 2	1356, 12356 47890, 3560, 2356, 23456 17890, 4789, 3456, 23567 14890, 1478, 3567, 23568 14790, 1479, 3568, 23569 14780, 1489, 3569, 23560 14789, 1789
Class 3	1257, 12357 46890, 2570, 2357, 12457 36890, 4689, 2457, 12567 34890, 3689, 2567, 12578 34690, 3469, 2578, 12579 34680, 3468, 2579, 12570 34689, 3489
Class 4	1358, 12358 46790, 1580, 1258, 13458 26790, 2467, 1458, 13568 24790, 2469, 1568, 13578 24690, 2479, 1578, 13589 24670, 4679, 1589, 13580 24679, 2679
Class 5	1367, 12367 45890, 1670, 1267, 13467 25890, 2458, 1467, 13567 24890, 2459, 1567, 13678 24590, 2489, 1678, 13679 24580, 2589, 1679, 13670 24589, 4589
Class 6	1268, 12368 45790, 2680, 2368, 12468 35790, 4579, 2468, 12568 34790, 4570, 2568, 12678 34590, 4590, 2678, 12689 34570, 4790, 2689, 12680 34579, 5790
Class 7	1378, 12378 45690, 3780, 2378, 23478 15690, 1456, 3478, 23578 14690, 1450, 3578, 23678 14590, 1460, 3678, 23789 14560, 1560, 3789, 23780 14569, 4560
Class 8	1259, 12359 46780, 3670, 1359, 12459 36780, 4670, 1459, 12569 34780, 3470, 1569, 12589 34670, 3460, 1579, 12590 34678, 3467, 1590
Class 9	2340, 12350 46789, 4567, 2350, 12360 45789, 4568, 2360, 12370 45689, 4578, 2370, 12380 45679, 4678, 2380, 12390 45678, 5678, 2390
Class 10	1269, 12369 45780, 2478, 1369, 13469 25780, 2470, 1469, 13569 24780, 2480, 1689, 13689 24570, 2780, 1690, 13690 24578, 4780
Class 11	1249, 12379 45680, 3458, 1279, 12479 35680, 3450, 1289, 12679 34580, 3580, 1290, 12789 34560, 4580, 12790 34568, 3480
Class 12	1245, 12456 37890, 3790, 2345, 24567 13890, 1790, 2456, 24568 13790, 1390, 2450, 24569 13780, 1370, 24560 13789, 1379
Class 13	1480, 12458 36790, 2367, 1468, 14568 23790, 2369, 1348, 14578 23690, 2379, 1248, 14589 23670, 3679, 14580 23679
Class 14	1389, 12389 45670, 1457, 2389, 23489 15670, 1470, 3589, 23589 14670, 1570, 3890, 23689 14570, 23890 14567
Class 15	1240, 12450 36789, 3579, 1250, 12460 35789, 5789, 1260, 12470 35689, 12480 35679, 12490 35678
Class 16	1340, 13450 26789, 2789, 1490, 13460 25789, 3490, 13470 25689, 13480 25679, 13490 25678
Class 17	1278, 12478 35690, 2569, 1780, 13478 25690, 5690, 2690, 14678 23590, 2560, 2590
Class 18	3680, 13680 24579, 2359, 4680, 14680 23579, 5680, 16780 23459, 16890 23457 2670, 12780 34569, 1346, 2790, 26780 13459, 1345, 27890 13456, 1349
Class 19	13579 24680, 2346, 14579 23680, 2460, 15679 23480, 15789 23460
Class 20	1380, 12580 34679, 2347, 1680, 15680 23479, 1890, 15780 23469
Class 21	2490, 12489 35670, 1350, 2349, 24689 13570, 1360, 24789 13560
Class 22	1246, 12467 35890, 3590, 1247, 23467 15890, 5890, 24678 13590
Class 23	4569, 12469 35780, 3570, 4690, 14679 23580, 5780, 14689 23570
Class 24	1270, 12560 34789, 3479, 1280, 12670 34589, 12690 34578
Class 25	2890, 12890 34567, 1347, 4890, 26890 13457, 1357, 3457
Class 26	2348, 15678 23490, 5689, 16789 23450, 15689 23470
Class 27	1368, 13468 25790, 5679, 23468 15790, 5670
Class 28	13479 25680, 3459, 13489 25670, 1256
Class 29	2580, 16790 23458, 3690, 2358
Class 30	

6. Lower Bounds on the Covering Array Number of Certain Families of 3-uniform Hypergraphs

In this section, we highlight some consequences derived from the results obtained so far.

Theorem 14. *For any 3-uniform hypergraph H , if $\chi_S(H) > 7$, then $CAN(H, 2) > 8$.*

Proof. By Theorems 8 and 10, a homomorphism $H \rightarrow 3-QI(8, 2)$ cannot exist. So, by Theorem 1, there does not exist a $CA(8, H, 2)$. \square

Corollary 4. *For any 3-uniform hypergraphs H , if there exists a homomorphism from H to $3-QI(9, 2)$ and $\chi_S(H) > 7$, then $CAN(H, 2) = 9$.*

Proof. By Theorem 14, $CAN(H, 2) > 8$ and by Lemma 1, $CAN(H, 2) \leq 9$. Hence, $CAN(H, 2) = 9$. \square

Theorem 15. *For any 3-uniform hypergraph H , if $\omega_3(H) > 4$ or $\chi_S(H) > 19$, then $CAN(H, 2) > 9$. Furthermore, if there exists a homomorphism from H to $3-QI(10, 2)$, then $CAN(H, 2) = 10$.*

Proof. If $\omega_3(H) > 4$, by Theorem 5 and Lemma 3, a homomorphism $H \rightarrow 3-QI(9, 2)$ cannot exist. Similarly, if $\chi_S(H) > 19$, then by Theorems 8 and 11, there cannot be a homomorphism $H \rightarrow 3-QI(9, 2)$. Thus, in either case, by Theorem 1, there does not exist a $CA(9, H, 2)$. If in addition, there exists a homomorphism from H to $3-QI(10, 2)$, then by Lemma 1, $CAN(H, 2) = 10$. \square

Theorem 16. *For any 3-uniform hypergraph H , if $\chi_S(H) > 30$, then $CAN(H, 2) > 10$.*

Proof. By Theorems 8 and 13, a homomorphism $H \rightarrow 3-QI(10, 2)$ cannot exist. Therefore, by Theorem 1, there does not exist a $CA(10, H, 2)$. \square

Corollary 5. *If $\chi_S(3-QI(11, 2)) > 30$, then $CAN(3-QI(11, 2), 2) = 11$.*

Proof. If $\chi_S(3-QI(11, 2)) > 30$, then by Theorems 8 and 13, a homomorphism $3-QI(11, 2) \rightarrow 3-QI(10, 2)$ cannot exist. So, by Corollary 1, $CAN(3-QI(11, 2), 2) = 11$. \square

7. Algorithm to Verify Colour Classes

Given below is an algorithm to check if a given set of subsets written in string format is a valid colour class of the corresponding $QI(n, 2)$ for $n = 8, 9, 10$:

To represent any vertex of these hypergraphs, strings of either 4 or 5 digits are sufficient. By Condition 1, the following properties must hold for two distinct vertices in the same colour class:

1. If both are of 4 digits, they must intersect exactly at 0 or 1 or 3 digits.
2. If both are of 5 digits, they must intersect exactly at 0 or 1 or 4 digits.
3. If one is of 4 and the other is of 5 digits, they must intersect exactly at 0 or 1 or 3 or 4 digits.

We begin the algorithm by accepting a set of strings representing vertices as input. Then, for every pair of strings, we compare each digit of the first with each digit of the second. Whenever two same digits are encountered, we increment a counter variable *count*, to keep track of the number of matching digits. After each comparison, we validate the intersection count against the above-specified conditions. If a violation is detected, we display an error message and update a marker variable *Marker* to indicate invalidity. We continue this process until all pairs are checked. If no violations are detected and the marker remains unchanged, that is, $Marker = 0$, then we conclude the algorithm by confirming that the given set of strings forms a valid colour class.

8. Conclusion

Minimizing the number of tests in a test suite has been a relevant combinatorial optimization problem, particularly in software applications. In 2005, Meagher et al. [25] proved that $CAN(2-QI(n, 2), 2) = n$, for $n \geq 4$ and conjectured that $CAN(2-QI(n, g), g) = n$, for $n \geq g^2$. The covering array number of any hypergraph that admits a homomorphism to $3-QI(n, g)$ is bounded above by $CAN(3-QI(n, g), g)$, where g represents the alphabet size. In this paper, we have established that $CAN(3-QI(n, 2), 2) = n$ for $n = 8, 9, 10$, and 12 by studying the sizes of the largest cliques and the strong chromatic numbers of $3-QI(n, 2)$. Moving forward, we plan to investigate the strong chromatic number of $3-QI(11, 2)$. Proving that $\chi_S(3-QI(11, 2)) > 30$ would imply $CAN(3-QI(11, 2), 2) = 11$. Additionally, we propose the following conjecture:

Conjecture 1. $CAN(3-QI(n, 2), 2) = n$ for all $n \geq 8$.

However, computing the largest clique sizes and strong chromatic numbers for larger n becomes increasingly difficult, making our current approach less effective for higher

Algorithm 1 An algorithm to verify if a given colour class is valid

Require: The number of vertices in the colour class n and the vertices of $3-QI(n, 2)$ entered one by one as strings corresponding to a part in the corresponding partition (10 is represented by 0).

Ensure: Prints “Valid colour class” if the given set is valid, else prints “Error”.

```

for  $1 \leq i \leq n$  do
  Extract the  $i$ th vertex to  $num$ .
  for  $i < j \leq n$  do
    Extract the  $j$ th vertex to  $num2$ .
    while  $num > 0$  do
      while  $num2 > 0$  do
        if  $num \% 10 == num2 \% 10$  then
           $count = count + 1$ 
        end if
         $num2 = \text{integer part of } (num2/10)$ 
      end while
       $num = \text{integer part of } (num/10)$ 
      Reinitialize  $num2$  to be the  $j$ th vertex.
    end while
    if  $i$ th and  $j$ th vertex both have 4 digits then

      if  $count! = 1$  and  $count! = 3$  and  $count! = 0$  then
        Print “Error”
         $Marker = 1$  return
      end if
    end if
    if exactly one out of the  $i$ th and  $j$ th vertex have 4 digits then

      if  $count! = 3$  and  $count! = 1$  and  $count! = 4$  and  $count! = 0$  then
        Print “Error”
         $Marker = 2$  return
      end if
    end if
    if  $i$ th and  $j$ th vertex both have 5 digits then

      if  $count! = 1$  and  $count! = 4$  and  $count! = 0$  then
        Print “Error”
         $Marker = 3$  return
      end if
    end if
    Reinitialize  $count = 0$ .
    Reinitialize  $num$  to be the  $i$ th vertex.
  end for
  Reinitialize  $count = 0$ .
end for
if  $Marker == 0$  then
  Print “Valid colour class”.
end if

```

values of n . Furthermore, to the best of our knowledge, no substantial progress has been made in exploring the above conjectures for $g > 2$ or $t > 2$, which remains an open and challenging problem.

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