

Roman coalition partitions in graphs

Juan Carlos Valenzuela Tripodoro^{1,*}, Hamidreza Golmohammadi²,
Mustapha Chellali³

¹Department of Mathematics, University of Cádiz, Spain
*jcarlos.valenzuela@uca.es

²Sobolev Institute of Mathematics, Ak. Koptyug av. 4, Novosibirsk, 630090, Russia
h.golmohammadi@g.nsu.ru

³LAMDA-RO Laboratory, Dept. of Mathematics, Univ. of Blida, Blida, Algeria
m.chellali@yahoo.com

Received: 2 May 2025; Accepted: 7 February 2026

Published Online: 21 April 2026

Abstract: The Roman domination problem is a combinatorial optimization problem on a graph asking to assign a label from $\{0, 1, 2\}$ to each vertex feasibly, such that the total sum of assigned labels is minimized. Let $G = (V, E)$ be a graph, and let $U_1, U_2 \subseteq V$ be two non-empty disjoint subsets. We say that the pair $\{U_1, U_2\}$ is Roman-feasible if there exists a Roman dominating function $f : V \rightarrow \{0, 1, 2\}$ such that $V_0 = V \setminus (U_1 \cup U_2)$, $V_1 = U_i$ and $V_2 = U_{3-i}$ for some $i \in \{1, 2\}$, where V_j denotes the set of vertices assigned label j by f . The set $\{U_1, U_2, U_3\}$ is a Roman coalition if the following two conditions hold: (i) the pair $\{U_i, U_j\}$ is not Roman-feasible for any different $1 \leq i, j \leq 3$; (ii) there exists k such that the pair $\{U_i \cup U_j, U_k\}$ is Roman-feasible, where $\{i, j, k\} = \{1, 2, 3\}$. The purpose of this paper is to introduce and study the new concept of Roman coalitions in graphs, providing basic properties, lower and upper bounds, as well as exact values in specific cases.

Keywords: coalition, Roman coalition, Roman dominating set, Roman coalition partition

AMS Subject classification: 05C60

1. Introduction

In this work, only finite, simple and connected graphs $G = (V, E)$ are considered, where V is the set of vertices and E is the set of edges. We denote by $[n]$ the set of

* *Corresponding Author*

positive integers $\{1, 2, \dots, n\}$. The set of vertices adjacent to a vertex u is the *open neighborhood* of u and it is denoted by $N(u)$. A set $D \subseteq V$ in a graph G is called a *dominating set* if every vertex of G is either in D or adjacent to a vertex of D . The *domination number* of G is $\gamma(G) = \min\{|D| : D \text{ is a dominating set of } G\}$. A dominating set having cardinality γ is called a γ -set. We say that a γ -set D is a *resilient dominating set* of a graph G if there is a vertex $v \in D$ such that $D - \{v\}$ is a dominating set in $G - v$. We refer to the books [15, 16] for a detailed study of domination and its variants. Inspired by the defence strategies used to defend the Roman Empire, Cockayne and others [9] formally introduced the Roman domination problem (RDP) in 2004. A Roman dominating function (RDF) on a graph $G = (V, E)$ is a function $f : V \rightarrow \{0, 1, 2\}$ such that for every vertex $u \in V$ with $f(u) = 0$ there exists a vertex $v \in N(u)$ such that $f(v) = 2$. The weight $w(f)$ of an RDF is the sum of labels assigned to the vertices of the whole graph. The minimum weight of an RDF is known as the Roman domination number of the graph, and it is denoted by $\gamma_R(G)$. There is a bijection between the functions $f : V \rightarrow \{0, 1, 2\}$ and the three-subset collections (V_0^f, V_1^f, V_2^f) defined as $V_j^f = \{v \in V : f(v) = j\}$ for $j = 0, 1, 2$. If there is no risk of confusion, we usually write V_j instead of V_j^f . Results on Roman domination and its variants have been collected in [7, 8]. The partitioning of the vertex set of graph G into subsets with a specific attribute has been extensively studied [16, Chapter 12]. For instance, in [18] a set $\{f_1, f_2, \dots, f_d\}$ of distinct Roman dominating functions on G , with the property that $\sum_{i=1}^d f_i(v) \leq 2$ for each $v \in V(G)$, is called a *Roman dominating family* (of functions) on G . The maximum number of functions in a Roman dominating family on G is the *Roman domatic number* of G , denoted by $d_R(G)$. Recently, the concept of coalition partition in graphs has attracted much interest of late. Inspired by the idea that the union of two sets can possess a property that neither set has, Haynes et al. [10] proposed this concept as a mathematical model in graph theory based on dominating sets in graphs. Coalition in graphs has been investigated in a number of subsequent papers [4, 11–14]. In order to expand the study of coalitions, various aspects of coalitions based on different types of dominating sets have been studied [1–3, 5, 17]. In this paper, we introduce and study the analogous concept of a Roman coalition in a graph.

2. Preliminaries

In this section, we provide main definitions and present basic properties of the Roman coalition in graphs. We begin by stating the following observation.

Observation 1. Let $f = (V_0, V_1, V_2)$ be a Roman dominating function. Then the set $V_1 \cup V_2 = \{v \in V : 1 \leq f(v) \leq 2\}$ is a dominating set.

As an auxiliary tool, we define the concept of Roman-feasible as follows.

Definition 1. Given a graph $G = (V, E)$ and two non-empty disjoint subsets $W_1, W_2 \subset V$

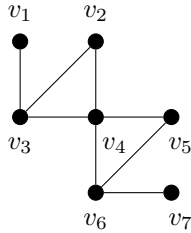


Figure 1. The sets $\{v_1, v_3\}$ and $\{v_5, v_6\}$ form a dominating set, but are not RF.

we say that the pair $\{W_1, W_2\}$ is Roman-feasible (for short, *RF*) if there exists a Roman dominating function $f : V \rightarrow \{0, 1, 2\}$ such that $V_0 = V \setminus (W_1 \cup W_2)$, $V_1 = W_i$ and $V_2 = W_{3-i}$ for some $i \in \{1, 2\}$.

Remark 1. 1. $W_1 \cup W_2$ can be a dominating set, and yet $\{W_1, W_2\}$ may not be an RF pair of subsets. This distinction highlights why the concept of "Roman-feasible" differs from that of a coalition in graphs.

For instance, consider $W_1 = \{v_1, v_3\}$ and $W_2 = \{v_5, v_6\}$ (see Figure 1). If $f = (V_0, V_1, V_2)$ would be a RDF with $\{V_1 = W_2, V_2 = W_1\}$ then vertex v_7 will not be Roman-dominated, otherwise, if $\{V_1 = W_1, V_2 = W_2\}$ then vertex v_2 will not be Roman-dominated.

2. Observe that if $\{W_1, W_2\}$ is not RF then neither W_1 nor W_2 can be a dominating set.

Lemma 1. Let $W_1, W_2 \subseteq V$ be two non-empty disjoint subsets of vertices. If there exists a vertex $v \in W_2$ having degree $d(v) = n - 2$ and $V \setminus N(v) \subseteq W_1 \cup W_2$ then $\{W_1, W_2\}$ is RF.

Proof. Let $v \in W_2$ be a vertex with degree $d(v) = n - 2$ and let us define $f(z) = 2$ for all $z \in W_2$ and $f(z') = 1$ for all $z' \in W_1$. Since the only vertex, $V \setminus N(v) = \{v'\}$, that is not Roman-dominated by v has a label of 1 if $v' \in W_1$ or a label of 2 if $v' \in W_2$, it follows that f is an RDF with $V_1 = W_1$ and $V_2 = W_2$. Hence, $\{W_1, W_2\}$ is RF. \square

We are now prepared to introduce the concepts of Roman coalition and Roman coalition partition in graphs.

Definition 2. Given a graph $G = (V, E)$ and three non-empty subsets $W_1, W_2, W_3 \subset V$ we say that $\{W_1, W_2, W_3\}$ is a Roman coalition if the following conditions hold,

1. $\{W_i, W_j\}$ is not RF for any different $1 \leq i, j \leq 3$.
2. there exists k such that $\{W_i \cup W_j, W_k\}$ is RF, where $\{i, j, k\} = \{1, 2, 3\}$.

Remark 2. 1. For the graph of Figure 1 we may check that

$$\pi_1 = \{W_1 = \{v_1\}, W_2 = \{v_4\}, W_3 = \{v_7\}\}$$

is a Roman coalition because:

- $\{\{v_1\}, \{v_4\}\}$ is not RF because it would not be possible to Roman-dominate the vertex v_7 .
- $\{\{v_1\}, \{v_7\}\}$ is not RF because it would not be possible to Roman-dominate the vertices v_2, v_4 and v_5 .
- $\{\{v_4\}, \{v_7\}\}$ is not RF because it would not be possible to Roman-dominate the vertex v_1 .
- We may define the function $f(v_1) = f(v_7) = 1, f(v_4) = 2$ and $f(v_j) = 0$ otherwise. This is an RDF such that $V_2 = W_2$ and $V_1 = W_1 \cup W_3$.

2. Analogously, we may check that

$$\pi_2 = \{\{v_1\}, \{v_2\}, \{v_6\}\} \text{ and } \pi_3 = \{\{v_3\}, \{v_5\}, \{v_7\}\}$$

are Roman coalitions in the graph depicted in Figure 1.

Definition 3. Let G be a graph with $\Delta(G) \leq n - 3$. A partition of the vertex set $\pi = \{W_1, W_2, \dots, W_k\}$ is a Roman coalition partition (RCP) if no pair $\{W_i, W_j\}$ is RF and for every W_i there exists W_j, W_l with $i, j, l \in \{1, \dots, k\}$ such that $\{W_i, W_j, W_l\}$ is a Roman coalition. The maximum cardinality of an RCP is the Roman coalition number (RCN), denoted by $C_R(G)$, of a graph G .

The restriction on the maximum degree of the graph in Definition 3 is crucial. If $\Delta(G) \geq n - 2$, then, by Lemma 1, the subset containing a vertex of degree at least $n - 2$ would be RF considered together with another subset. Namely, this subset would be RF together with either the subset containing the only vertex not adjacent to it (when $\Delta = n - 2$) or with any other subset of the partition (when $\Delta = n - 1$). Thus, from now on, we only consider those graphs of order n having maximum degree at most $n - 3$. Therefore, we have to keep in mind the following fact.

Remark 3. Observe that given a graph G of order n with $\Delta \leq n - 3$, no pair of subsets, each consisting of a single vertex, can be Roman feasible.

Next, we give an example to illustrate the concept of Roman coalition partition of a graph G .

Example 1. Let G be the graph depicted in Figure 1. Then $\pi = \{W_j = \{v_j\} : j = 1, \dots, 7\}$ is a Roman coalition partition with cardinality 7. By applying Remark 3, no pair of vertices can be RF. So, it is sufficient to check that

1. $\{\{v_1\}, \{v_2\}, \{v_6\}\}$, is a Roman coalition such that $V_1 = W_1, V_2 = W_2 \cup W_6$.
2. $\{\{v_1\}, \{v_4\}, \{v_7\}\}$, is a Roman coalition such that $V_1 = W_1 \cup W_7, V_2 = W_4$.
3. $\{\{v_3\}, \{v_5\}, \{v_7\}\}$ is a Roman coalition such that $V_1 = W_7, V_2 = W_3 \cup W_5$.

Thus, π is a Roman coalition partition with maximum possible cardinality and $C_R(G) = n = 7$.

3. Bounds on the Roman coalition number

In this section, we establish lower and upper bounds on the Roman coalition number of a graph. We first recall known results needed later.

Theorem A. [6] If G is a graph without isolated vertices, then G has a minimum dominating set D such that for all $d \in D$, there exists a neighbor $f(d) \in V \setminus D$ of d such that $f(d)$ is not a neighbor of any vertex $x \in D \setminus \{d\}$.

Recall that such a vertex, $f(d)$, is called a *private neighbor* of vertex d and the set of all private neighbors of a vertex $d \in V(G)$ is denoted by $PN(d)$.

Theorem B. [9] For any graph G , $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.

Theorem C. [18] If G is a graph of order $n \geq 2$, then $\gamma_R(G) + d_R(G) \leq n + 2$.

Clearly, we have that $3 \leq C_R(G) \leq n$ for all non trivial connected graph with $\Delta \leq n - 3$. In this context, the next result gives an upper bound to the Roman coalition number of a graph in terms of its order and its domination number, which improves on the trivial bound.

Proposition 1. Let G be a graph of order n such that $\Delta \leq n - 3$. Then $C_R(G) \leq n - \gamma(G) + 3$.

Proof. Let $C_R(G) = k$, and consider a maximum Roman coalition partition $\pi = \{W_1, W_2, \dots, W_k\}$. By definition, for W_1 there exists two different sets in π , say W_2, W_3 , such that $\{W_1, W_2, W_3\}$ is a Roman coalition. This means that $W_1 \cup W_2 \cup W_3$ is a dominating set of G , and thus $\gamma(G) \leq |W_1| + |W_2| + |W_3|$. Accordingly,

$$\begin{aligned} n &= |W_1| + |W_2| + |W_3| + \dots + |W_k| \\ &\geq |W_1| + |W_2| + |W_3| + (k - 3) \\ &\geq \gamma(G) + (k - 3), \end{aligned}$$

leading us to the desired upper bound. □

As an immediate consequence of Proposition 1, we have the following result.

Corollary 1. If G is a graph of order n and maximum degree $\Delta \leq n - 3$ such that $C_R(G) = n$, then $\gamma(G) \in \{2, 3\}$.

Plugging the result of Proposition 1 into the bound given by Theorem B immediately yields the following result.

Corollary 2. *If G is a graph of order n and maximum degree $\Delta \leq n - 3$ then $C_R(G) \leq n - \left\lfloor \frac{\gamma_R(G)}{2} \right\rfloor + 3$.*

By applying Theorem C and Corollary 2, we obtain a relation between the Roman coalition number and the Roman domatic number.

Corollary 3. *If G is a graph of order $n \geq 2$ and maximum degree $\Delta \leq n - 3$ then $C_R(G) + d_R(G) \leq 2n - \frac{3\gamma_R(G)}{2} + 5$.*

Restricted to graphs G with $\gamma(G) \geq 4$, we provide a lower bound for the Roman coalition partition number in terms of the order and the domination number, which, by Proposition 1, results in a very good framing of the Roman coalition number.

Theorem 2. *Let G be a connected graph of order n and $\Delta \leq n - 3$ such that $\gamma(G) \geq 4$. Then $C_R(G) \geq n - \gamma(G) + 2$, and this bound is sharp.*

Proof. Let $V(G) = \{w_1, w_2, \dots, w_n\}$. Let D be a minimum dominating set of G satisfying the property of Theorem A, and set $|D| = k$. Relabeling vertices, if necessary, we may assume $D = \{w_1, w_2, \dots, w_k\}$. Consider the partition $\pi = \{A, B, W_{k+1}, \dots, W_n\}$, where $A = \{w_1, \dots, w_{k-2}\}$, $B = \{w_{k-1}, w_k\}$ and $W_i = \{w_i\}$ for every $i \in \{k+1, \dots, n\}$. Note that $|A| \geq 2$, since $\gamma(G) \geq 4$. We first show that any pair of π is not Roman-feasible. For this purpose, assume there is a pair $\{X, Y\}$ which is Roman-feasible, and let $f = (V_0, V_1, V_2)$ be the resulting RDF. It follows that $X \cup Y$ is a dominating set of G , where without loss of generality, $V_1 = X$ and $V_2 = Y$. By Remark 3, we know that $\{X, Y\} \cap \{A, B\} \neq \emptyset$. Clearly, by our choice of D , X and Y cannot both belong to $\{A, B\}$, since each vertex in these sets has at least one private neighbor. Moreover, if $X \notin \{A, B\}$ and $Y \notin \{A, B\}$, then each of X and Y is a singleton set, and thus the vertex in Y is adjacent to all $V \setminus (X \cup Y)$, implying that $X \cup Y$ is a dominating set of G of cardinality 2, a contradiction. Hence either $X \notin \{A, B\}$ and $Y \in \{A, B\}$ or $X \in \{A, B\}$ and $Y \notin \{A, B\}$. In the former situation, assume that $Y = A$ (the case $Y = B$ is similar) and thus X is a singleton set. Moreover, all vertices of A are assigned the label 2 under f . Now, since $B \subseteq V_0$, vertices of B as well as all their private neighbors, except possibly the vertex in X , must have neighbors in A . But then $A \cup X$ would be a dominating set smaller than $A \cup B$, a contradiction. Hence we consider the latter situation, that is $X \in \{A, B\}$ and $Y \notin \{A, B\}$. Hence Y is a singleton set. Without loss of generality, let $V_2 = \{w_n\}$, and $X = A$ (the case $X = B$ is similar). Then w_n must be adjacent to all vertices, except possibly to those in A . But then $A \cup \{w_n\}$ would be a dominating set smaller than D , a contradiction too. Consequently, no pair of π is Roman-feasible.

Now, since $A \cup B$ is a dominating set of G , for every W_j we have $\{A, B, W_j\}$ is a Roman coalition as long as $W_j \in \pi - \{A, B\}$. Therefore $C_R(G) \geq |\pi| = n - |A \cup B| + 2 = n - \gamma(G) + 2$.

To see the sharpness of the bound in Theorem 2, let G be a graph obtained from a connected graph H of order $p \geq 4$, by adding for each vertex v of H a new vertex v' and the edge vv' . Denote by v_i the i th vertex of H and v'_i the corresponding vertex that has been added. Clearly, $n(G) = 2p$, $\gamma(G) = p$ and thus by Theorem 2, $C_R(G) \geq n(G) - \gamma(G) + 2 = p + 2$. Now, if $C_R(G) > p + 2$, then by Proposition 1, we have $C_R(G) = p + 3 = n(G) - \gamma(G) + 3$. In this case, let $\pi^* = \{X_1, X_2, \dots, X_{p+3}\}$ be a maximum RCP of G . Clearly, $|X_i \cup X_j| < p$ for any different $i, j \in [p + 3]$, and thus $X_i \cup X_j$ is not a dominating set of G . Also, since π^* must contain $p + 1$ singleton sets, and two non-singleton sets, let without loss of generality, X_1 and X_2 be the non-singleton sets of π^* . Thus, any singleton set of π^* forms a Roman coalition with X_1 and X_2 . Let v'_r be a vertex not dominated by $X_1 \cup X_2$, and let v_s be a vertex of H not in $X_1 \cup X_2$. If $r \neq s$, then clearly $X_1 \cup X_2 \cup \{v_r\}$ cannot dominate v'_r . Thus we can assume that $r = s$. Then for any singleton set X_t whose vertex is dominated by $X_1 \cup X_2$, again $X_1 \cup X_2 \cup X_t$ does not dominate v'_r . Therefore $C_R(G) = p + 2 = n(G) - \gamma(G) + 2$. \square

4. Exact values

In this section, our aim is to obtain the exact value of $C_R(G)$ for some particular graphs. We first present necessary and sufficient conditions for a graph G to satisfy $C_R(G) = n$.

Theorem 3. *Let G be a connected graph of order n and $\Delta \leq n - 3$. Then $C_R(G) = n$ if and only if one of the following conditions hold*

1. $\gamma(G) = 2$.
2. $\gamma(G) = 3$ and every vertex u belongs to a resilient dominating set.

Proof. Let $V(G) = \{v_1, \dots, v_n\}$ be the set of vertices of a connected graph G . If $C_R(G) = n$ then, by applying Corollary 1 we have that $\gamma(G) \in \{2, 3\}$. If $\gamma(G) = 3$ then $\pi = \{W_i\}$, where $W_i = \{v_i\}$, is a Roman coalition partition of G . Thus, for all W_{i_1} there must exist W_{i_2}, W_{i_3} such that $\{W_{i_1}, W_{i_2}, W_{i_3}\}$ is a RC. So, there is a RDF $f = (V_0, V_1, V_2)$ with $V_1 = W_{i_j}$ and $V_2 = W_{i_k} \cup W_{i_l}$ where $\{i_j, i_k, i_l\} = \{i_1, i_2, i_3\}$. Let $D = V_1 \cup V_2$ be a dominating set of G . Clearly, $v_{i_1} \in D$ and $N[D \setminus \{v_{i_j}\}] = V(G) - v_{i_j}$ and the result holds.

Let us prove the converse. Thus, suppose that $\gamma(G) = 2$. Assume that the pair $\{W_i, W_j\}$ is RF for some different $i, j \in [n]$, and let f be the resulting RDF. Then, the set $\{v_i, v_j\}$ dominates all vertices of G , and one of these two vertices, say v_i , is assigned 2 under f . Hence $f(v_j) = 1$, and thus v_i is adjacent to all $V - \{v_i, v_j\}$. But then $\Delta \geq n - 2$, a contradiction. Therefore, no pair $\{W_i, W_j\}$ is RF for any different $i, j \in [n]$. Now, without loss of generality, we may assume that $\{v_1, v_2\}$ is a minimum dominating set of G because $\gamma(G) = 2$. For every W_j we have that $\{W_1, W_2, W_j\}$ is a Roman coalition as long as $j \in [n] - \{1, 2\}$. It is sufficient to define the following function $f = (V \setminus (W_1 \cup W_2 \cup W_j), W_j, W_1 \cup W_2)$.

If $\gamma(G) = 3$, and the condition described in the statement holds, then we consider the partition $\pi = \{W_i\}$, where $W_i = \{v_i\}$. No pair of vertex subsets $\{W_i, W_j\}$ is RF because $\gamma(G) = 3$. Moreover, if $W_{i_1} = \{v_{i_1}\}$ is an element of π then we know that there exists a resilient dominating set D such that $v_{i_1} \in D$. Thus, there must exist a vertex $v'_{i_1} \in D$ (eventually, it could be $v'_{i_1} = v_{i_1}$) with $|N[D - \{v'_{i_1}\}]| = n - 1$. Let us consider the function $f = (V \setminus D, \{v'_{i_1}\}, D - \{v'_{i_1}\})$. Since $\gamma(G) = 3$ then $|D - \{v'_{i_1}\}| = 2$ and we deduce that $\{W_{i'_1}, W_{i'_2}, W_{i'_3}\}$ is an RC where $D = \{v_{i'_1}, v_{i'_2}, v_{i'_3}\}$ and there exists an integer j such that $v_{i_1} = v_{i'_j}$. \square

Since any complete bipartite graph, different from a star, has a domination number equal to 2, then the following corollary is an immediate consequence of Theorem 3.

Corollary 4. *Let $3 \leq p \leq q$ be two positive integers. Then $C_R(K_{p,q}) = n$.*

Next, we characterize those graphs G satisfying $C_R(G) = n - 1$.

Theorem 4. *Let G be a connected graph of order n with $\Delta \leq n - 3$. Then $C_R(G) = n - 1$ if and only if $\gamma(G) = 3$ and there exists a vertex u that does not belong to any resilient dominating set.*

Proof. If $\gamma(G) = 3$ and there is a vertex that does not belong to a resilient dominating set, then by applying Theorem 3, we know that $C_R(G) \leq n - 1$. Let v_1 be such a vertex and let $D = \{v_1, v_2, v_3\}$ be a non-resilient dominating set of G . We may readily verify that the partition $\pi = \{\{v_1, v_2\}, \{v_3\}, \{v_4\}, \dots, \{v_n\}\}$ is an RCP because D is a non-resilient dominating set, implying that: (i) no pair of subsets can be RF; and (ii) the collection $\{\{v_1, v_2\}, \{v_3\}, \{v_j\}\}$ is an RC for all $j \geq 4$. Thus, $C_R(G) = n - 1$. On the other hand, let us suppose $C_R(G) = n - 1$, which implies that $\gamma > 2$. Then, we consider any RCP $\pi = \{\{v_1\}, \{v_2\}, \dots, \{v_{n-2}\}, \{v_{n-1}, v_n\}\}$. Since G is connected we know that $N(\{v_{n-1}, v_n\}) \setminus \{v_{n-1}, v_n\} \neq \emptyset$. Without loss of generality, assume that $v_1 \in N(v_n)$. As π is an RCP there must exist two subsets that form an RC together with $\{v_1\}$. If $\{\{v_1\}, \{v_i\}, \{v_j\}\}$ is an RC, then $\{v_1, v_i, v_j\}$ is a dominating set. If $\{\{v_1\}, \{v_i\}, \{v_{n-1}, v_n\}\}$ is an RC, then the set $\{v_i, v_{n-1}, v_n\}$ is a dominating set, because $v_1 \in N(v_n)$. In any case $\gamma = 3$ and, by applying Theorem 3, the result holds. \square

The Roman coalition number of a path P_n and cycle C_n on $n \geq 5$ vertices is given by the following closed formulas.

Theorem 5. *Let $n \geq 5$ be a positive integer. Then*

$$C_R(P_n) = \begin{cases} \left\lfloor \frac{2n+6}{3} \right\rfloor, & \text{if } n \neq 7, \\ \left\lfloor \frac{2n+6}{3} \right\rfloor = 7, & \text{if } n = 7. \end{cases}$$

Proof. Let $n \geq 5$ be a positive integer, and let $P_n = v_1 v_2 \dots v_n$ denote the path consisting of n consecutive vertices.

First of all, recall that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$. If $n \in \{5, 6\}$, then $\gamma(P_n) = 2$ and thus, by Theorem 3, we have that $C_R(P_n) = n = \lfloor \frac{2n+6}{3} \rfloor$. Assume that $n = 7$ and consider the partition $\pi = \{W_i : 1 \leq i \leq 7\}$ where $W_i = \{v_i\}$. Since $|W_i \cup W_j| = 2$ then $W_i \cup W_j$ is not a dominating set and, therefore, $\{W_i, W_j\}$ is not RF. Besides, observe that $\{W_1, W_3, W_6\}, \{W_2, W_5, W_7\}, \{W_2, W_4, W_6\}$ are RC's and, thus, we can deduce that $C_R(P_7) = 7$, that proves the result for $n = 7$.

For $n \in \{8, 9\}$, observe that the partition $\pi = \{W_i : 1 \leq i \leq n\}$ is not an RCP. Otherwise, $\{W_i, W_j, W_l\}$ would be an RC, for some different i, j, l , but $|N[W_i \cup W_j]| \leq 6$ and $|W_l| = 1$ which implies that $|W_l \cup N[W_i \cup W_j]| = |V_1 \cup N[V_2]| \leq 7$ and, consequently, $V_1 \cup V_2$ would not be a dominating set, a contradiction. On the other hand, let us consider the partition $\pi = \{\{2, 5\}, \{i\} : 1 \leq i \leq n, i \neq 2, 5\}$. No pair of two subsets is RF because $|W_i \cup N[W_j]| \leq 7$, for all $W_i, W_j \subseteq \pi$. Moreover, $\{\{i\}, \{2, 5\}, \{8\}\}$ is an RC, for every $i \in [n] \setminus \{2, 5, 8\}$, because $f = ([n] \setminus \{i, 2, 5, 8\}, \{i\}, \{2, 5, 8\})$ is a Roman dominating function on P_n . Since $|\pi| = n - 1$ we deduce that $C_R(P_n) = n - 1$, and thus the result is valid for $n \in \{8, 9\}$.

Let $n \geq 10$ be an integer. Since $\gamma(P_n) \geq 4$, then, by applying Proposition 1 and Theorem 2, we have that $n - \gamma(P_n) + 2 \leq C_R(P_n) \leq n - \gamma(P_n) + 3$. If $C_R(P_n) = n - \gamma(P_n) + 3$, then there exists an RCP, $\pi = \{W_1, \dots, W_{n-\gamma+3}\}$. Without loss of generality, we can assume that $|W_i| \leq |W_{i+1}|$, for all $1 \leq i \leq n - \gamma + 2$. Let x be the number of subsets W_j of cardinality 1 and let y be the number of subsets W_i containing at least two vertices. Then, $x + y = n - \gamma + 3$ and $n \geq x + 2y$ which implies that $y \leq \gamma - 3$ and $x \geq n - 2\gamma + 6$. As $n \geq 10$ and it is well-known that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$, then we may derive that $x \geq 8$.

If $y \geq 3$ then $|W_i \cup W_j| \leq \gamma - 2$ for i, j such that $|W_i|, |W_j| \geq 2$. Otherwise, if $|W_i|, |W_j|, |W_l| \geq 2$ and $|W_i \cup W_j| > \gamma - 2$, then

$$n = \sum_{k=1}^{n-\gamma+3} |W_k| = \sum_{\substack{k=1 \\ k \neq i, j, l}}^{n-\gamma+3} |W_k| + |W_i| + |W_j| + |W_l| > n - \gamma + 2 + \gamma - 2 = n,$$

a contradiction.

Hence, since $x \geq 8$ we know that $|W_1| = 1$ and we conclude that $\{W_1, W_i, W_j\}$ could not be an RC because otherwise there exists an RDF f such that

$$|V_1 \cup V_2| \leq |W_1| + |N[W_i] \cup N[W_j]| \leq 1 + 3(\gamma - 2) = 1 + 3\left(\left\lceil \frac{n}{3} \right\rceil - 2\right) \leq n - 2.$$

But this implies that $V_1 \cup V_2$ is not a dominating set, a contradiction. Therefore, it must be $1 \leq y \leq 2$.

If $y = 2$, then $x = n - \gamma + 1$ and $|W_{n-\gamma+2} \cup W_{n-\gamma+3}| = \gamma - 1$. On the other hand, if $y = 1$ then $x = n - \gamma + 2$ and $|W_{n-\gamma+3}| = \gamma - 2$. Clearly, three singleton sets $\{W_i, W_j, W_l\}$ cannot form an RC. We have to study two different cases.

Case 1. If $N[W_{n-\gamma+2}] \cap N[W_{n-\gamma+3}] \neq \emptyset$, then $|N[W_i] \cup N[W_j]| \leq |N[W_{n-\gamma+2}] \cup N[W_{n-\gamma+3}]| \leq 3(\gamma - 1) - 1 \leq n - 2$. Therefore, $\{W_1, W_i, W_j\}$ is not a RC for all i, j because if not, there exists a RDF f with $|V_1| + |V_2| \leq 1 + |N[W_i] \cup N[W_j]| \leq n - 1$, a contradiction.

Case 2. If $N[W_{n-\gamma+2}] \cap N[W_{n-\gamma+3}] = \emptyset$ then $n - 1 \geq |N[W_{n-\gamma+2}] \cup N[W_{n-\gamma+3}]| \geq 3(\gamma - 2) + 4 \geq n - 3$. Since $n \geq 10$, then $x \geq n - \gamma + 1 \geq 7$ and thus there is at least a vertex v_j such that $W_j = \{v_j\} \subseteq N(W_{n-\gamma+2}) \cup N(W_{n-\gamma+3})$. Therefore, $\{W_j, W_{n-\gamma+2}, W_{n-\gamma+3}\}$ is not an RC.

So, it is deduced that $C_R(P_n) = n - \gamma + 2 = n - \lceil \frac{n}{3} \rceil + 2 = \lfloor \frac{2n+6}{3} \rfloor$, for all $n \geq 10$, which concludes the proof. \square

Theorem 6. *Let $n \geq 5$ be a positive integer. Then*

$$C_R(C_n) = \begin{cases} \lfloor \frac{2n+6}{3} \rfloor, & \text{if } n \neq 7, \\ \lceil \frac{2n+6}{3} \rceil, & \text{if } n = 7. \end{cases}$$

Proof. Let us point out that the argument used to obtain the Roman coalition number of a path P_n leads us to the Roman coalition number of a cycle C_n . Hence, the proof is omitted. \square

5. Conclusion

In this paper, we initiated the study of Roman coalition in graphs. We obtained upper and lower bounds for Roman coalition number. We provided necessary and sufficient conditions for a graph G to satisfy $C_R(G) = n$ and $C_R(G) = n - 1$. We close the paper with the following problems we have yet to settle.

1. Establish upper and lower bounds for the Roman coalition number of a graph in terms of its maximum degree $\Delta(G)$ and minimum degree $\delta(G)$.
2. Investigate possible relationships between $C_R(G)$ and other coalition-related parameters.
3. Analyze Roman coalitions in particular graphs.
4. Explore total Roman coalitions in graphs, as derived from total Roman domination, as well as other domination variants.

6. Acknowledgments

The authors gratefully acknowledge the helpful comments from the referees. The work of Hamidreza Golmohammadi was supported by the state contract of the Sobolev

Institute of Mathematics (project number FWNF-2026-0011). J. C. Valenzuela-Tripodoro was partially funded by the Spanish Ministry of Science, Innovation and Universities through project PID2022-139543OB-C41 and also by the European Commission's Horizon Europe Research and Innovation programme through the Marie Skłodowska-Curie Actions Staff Exchanges (MSCA-SE) under Grant Agreement no. 101182819 (COVER: (C)ombinatorial (O)ptimization for (V)ersatile Applications to (E)merging u(R)ban Problems).

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

- [1] S. Alikhani, D. Bakhshesh, and H. Golmohammadi, *Total coalitions in graphs*, Quaest. Math. **47** (2024), no. 11, 2283–2294.
<https://doi.org/10.2989/16073606.2024.2365365>.
- [2] S. Alikhani, D. Bakhshesh, H. Golmohammadi, and S. Klavžar, *On independent coalition in graphs and independent coalition graphs*, Discuss. Math. Graph Theory **45** (2025), no. 2, 533–544.
<https://doi.org/10.7151/dmgt.2543>.
- [3] S. Alikhani, D. Bakhshesh, H. Golmohammadi, and E.V. Konstantinova, *Connected coalitions in graphs*, Discuss. Math. Graph Theory **44**, 1551–1566.
<https://doi.org/10.7151/dmgt.2509>.
- [4] D. Bakhshesh, M.A. Henning, and D. Pradhan, *On the coalition number of trees*, Bull. Malays. Math. Sci. Soc. **46** (2023), no. 3, Article number: 95
<https://doi.org/10.1007/s40840-023-01492-4>.
- [5] J. Barát and Z.L. Blázsik, *General sharp upper bounds on the total coalition number*, Discuss. Math. Graph Theory **44** (2024), no. 1567-1584.
<https://doi.org/10.7151/dmgt.2511>.
- [6] B. Bollobás and E.J. Cockayne, *Graph-theoretic parameters concerning domination, independence, and irredundance*, J. Graph Theory **3** (1979), no. 3, 241–249.
<https://doi.org/10.1002/jgt.3190030306>.
- [7] M. Chellali, N. Jafari Rad, S.M. Sheikholeslami, and L. Volkmann, *Roman domination in graphs*, Topics in Domination in Graphs, vol. 64, Springer, 2020, pp. 365–409.
<https://doi.org/10.1007/978-3-030-51117-3-11>.
- [8] ———, *Varieties of Roman domination*, Structures of domination in graphs, vol. 66, Springer, 2021, pp. 273–307.
<https://doi.org/10.1007/978-3-030-58892-2-10>.
- [9] E.J. Cockayne, P.A. Dreyer Jr, S.M. Hedetniemi, and S.T. Hedetniemi, *Roman domination in graphs*, Discrete Math. **278** (2004), no. 1-3, 11–22.

- <https://doi.org/10.1016/j.disc.2003.06.004>.
- [10] T.W. Haynes, J.T. Hedetniemi, S.T. Hedetniemi, A.A. McRae, and R. Mohan, *Introduction to coalitions in graphs*, AKCE Int. J. Graphs Combin. **17** (2020), no. 2, 653–659.
<https://doi.org/10.1080/09728600.2020.1832874>.
- [11] ———, *Upper bounds on the coalition number*, Australas. J. Combin. **80** (2021), no. 3, 442–453.
- [12] ———, *Coalition graphs*, Commun. Comb. Optim. **8** (2023), no. 2, 423–430.
<https://doi.org/10.22049/cco.2022.27916.1394>.
- [13] ———, *Coalition graphs of paths, cycles, and trees*, Discuss. Math. Graph Theory **43** (2023), no. 4, 931–946.
<https://doi.org/10.7151/dmgt.2416>.
- [14] ———, *Self-coalition graphs*, Opuscula Math. **43** (2023), no. 2, 173–183.
<https://doi.org/10.7494/opmath.2023.43.2.173>.
- [15] T.W. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of domination in graphs*, CRC press, 1998.
- [16] T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, *Domination in graphs: Core concepts*, Springer, 2023.
- [17] M.A. Henning and S.N. Jogan, *A characterization of graphs with given total coalition numbers*, Discrete Appl. Math. **358** (2024), 395–403.
<https://doi.org/10.1016/j.dam.2024.07.031>.
- [18] S.M. Sheikholeslami and L. Volkmann, *The Roman domatic number of a graph*, Appl. Math. Lett. **23** (2010), no. 10, 1295–1300.
<https://doi.org/10.1016/j.aml.2010.06.016>.