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Research Article

Analytical study of second inverse sum Indeg index of special graphs

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Abstract: In this paper, we investigate the second inverse sum indeg index ISI_2 of graphs, a topological index that has significant applications in chemical graph theory. Upper and lower bounds for ISI_2 of graphs and trees with a specified number of pendent edges are established. Furthermore, ISI_2 of various bridge graphs are computed. The main contribution of this work lies in presenting precise bounds and exact expressions for particular families of graphs, offering resources for researchers and engineers in mathematical chemistry and applied graph theory.

Keywords: topological index, second inverse sum indeg index, bridge graph.

AMS Subject classification: 05C92, 92E10

1. Introduction

A topological index of the graph G is a numerical quantity reflecting structural details of G, that is expected to be of some importance in both mathematical and applied studies. These indices have become a prominent research topic, not only due to their mathematical and computational properties but also because of their extensive applications across various fields of knowledge, mainly chemistry. Within the framework

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of mathematical analysis, we will focus on the study of optimization problems related to topological indices (see [6, 8]). For an in-depth exploration of topological indices pertaining to various graph families, the reader is referred to [11, 12, 16, 26, 29, 32, 33].
Let G be a simple undirected graph, with its vertex set denoted by $V(G)$ and edge set by $E(G)$. Additionally, let $n = |V(G)|$ and $m = |E(G)|$ be the number of vertices and edges of the graph, respectively.
A plethora of topological indices have been proposed and examined in the context of graph theory, each offering specific insights into the structural properties of graphs. Among these, a significant class of indices is defined in terms of the degree of vertices. Formally, this family of indices can be defined as follows [9, 14]:

$$TI(G) = \sum_{uv \in E(G)} f\left(d_u, d_v\right),$$

where d_u is the degree of a vertex u in graph G, and f is a function satisfying the condition $f(x, y) = f(y, x)$.
Another pivotal family of topological indices is dependent on the distance between vertices. This category of indices was found to play a crucial role in the investigation of the physico-chemical properties of molecular structures [10, 16, 24, 32]. In a general framework, these indices can be formally defined as:

$$TI_2(G) = \sum_{uv \in E(G)} g(n_u, n_v).$$

where for adjacent vertices u and v of the graph G, n_u is the number of vertices lying closer to u and n_v is the number of vertices lying closer to v.
For the function g defined by $g(x, y) = x + y$, the topological index PI is given by:

$$PI_u\left(G\right) = \sum_{uv \in E(G)} \left(n_u + n_v\right).$$

This index was initially introduced in [19]. For an in-depth study and comprehensive analysis, the reader is referred to [18, 21, 22].
The vertex Szeged index was introduced by one of the present authors [20], and is defined as

$$Sz_u(G) = \sum_{uv \in E(G)} n_u n_v.$$

For details on this significant index, the reader is directed to consult [15, 25].
The second geometric-arithmetic index, with g taking the form $g(x, y) = \frac{2\sqrt{xy}}{x+y}$, is formally defined as:

$$GA_2\left(G\right) = \sum_{uv \in E(G)} \frac{2\sqrt{n_u n_v}}{n_u + n_v}.$$

This index was first introduced by Fath-Tabar et al. in 2010, as detailed in [10]. The paper provides a comprehensive analysis of both upper and lower bounds in a general framework. Upper and lower bounds pertaining to the geometric-arithmetic indices have been derived for general graphs, molecular graphs, and molecular trees, as presented in [30, 34, 35].
Another interesting family of indices in this category is defined by considering the function $g(x,y) = |x-y|$. leading to the the Mostar index,defined as

$$Mo_u(G) = \sum_{uv \in E(G)} |n_u - n_v|.$$

The studies related to this index can be found in references [1–4].
Graovac and Ghorbani, defined a new version of the atom-bond connectivity index [13], and named it second atom-bond connectivity index,

$$ABC_2\,(G) = \sum_{uv \in E(G)} \sqrt{\frac{n_u + n_v - 2}{n_u\, n_v}}.$$

Comprehensive and foundational investigations into this index have been presented in [5, 27, 28]. Examination of these references facilitates a deeper understanding of the index and its practical applications.
The second inverse sum indeg index of a graph G defined as [7]

$$ISI_2(G) = \sum_{uv \in E(G)} \frac{n_u\, n_v}{n_u + n_v}.$$

It is evident that this index results from the choice $g(x,y) = \frac{xy}{x+y}$.
In our previous paper [17], we systematically investigated ISI_2, and reported upper and lower bounds for general graphs and trees.
Also, since the second inverse sum indeg index is a distance-based topological index, one may start studying the second inverse sum indeg index of eccentrical graphs of trees as such graphs are connected with diameter at most 3 (see [31]).
In this paper, we conduct a novel study and analysis of the ISI_2 index. This investigation encompasses the derivation of upper and lower bounds for specific graphs and trees characterized by a predetermined number of pendent vertices. Furthermore, we explore bridge graphs and compute the ISI_2 index for a selection of such graphs.
The structure of this paper is organized as follows. Section 2 introduces the preliminary definitions and notations for graphs. In Section 3, upper and lower bounds for ISI_2 are derived for specific graphs and trees, characterized by a predetermined number of pendent vertices. Section 4 is dedicated to the examination of a variety of bridge graphs, including the computation of their ISI_2 index. Lastly, Section 5 presents concluding remarks, summarizes the key findings, and outlines potential directions for future research.

2. Preliminaries

Let K_n, C_n, S_n, and P_n denote the complete graph, cycle graph, star graph, and path graph on n vertices, respectively. Furthermore, let $K_{m,n}$ represent the complete bipartite graph with partition sets of sizes m and n.
For $m, n \geq 2$, the tree formed by connecting the central vertices of S_m and S_n is denoted by $S_{m,n}$.
A tree with exactly one vertex of degree larger than two is called a star-like tree and this vertex is said to be its root. We denote a star-like tree $SL_n(2k, l)$ which has a center vertex v of degree $k + l$ and

$$SL_n(2k,l) - v \cong \underbrace{P_2 \cup \ldots \cup P_2}_{k} \cup \underbrace{P_1 \cup \ldots \cup P_1}_{l}.$$

The star-like tree has $2k + l + 1 = n$ vertices and $k + l$ branches. As an example, Figure 1 illustrates a star-like tree $SL_{10}(6, 3)$.

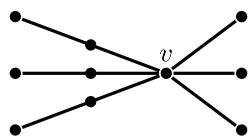

Figure 1. **The star-like tree $SL_{10}(6, 3)$.**

Let us briefly recall the definition of bridge graphs. Let P_m be a path with m vertices. Then the bridge graph $B[G, m]$ is obtained by connecting a copy of the graph G to all vertices of P_m. see Figure 2.

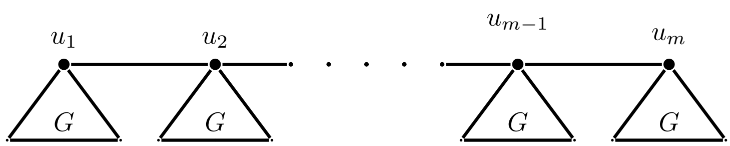

Figure 2. **The bridge graph.**

3. Bounds for ISI_2 of graphs and trees with p pendent vertices

In this section, we investigate the second inverse sum indeg index for some particular families of graphs and trees.

Theorem 1. *Let G be a connected graph of order n with m edges and p pendent vertices. Then*

$$ISI_2(G) \leq \frac{mn^2}{8} + p\left(1 - \frac{1}{n} - \frac{n^2}{8}\right),$$

with equality if and only if graph $G \cong K_{1,n-1}$.

Proof. Suppose $n \geq 2$. For each pendent edge in the graph G, the values of n_u and n_v are given by $n_u = 1$ and $n_v = n-1$ respectively. Also, for a non-pendent edge uv, we can write

$$\begin{cases} n_u n_v \leq \frac{n^2}{4}, \\ \frac{1}{n_u+n_v} \leq \frac{1}{2}, \end{cases} \quad \Rightarrow \quad \frac{n_u\, n_v}{n_u + n_v} \leq \frac{n^2}{8},$$

therefore,

$$\begin{aligned} ISI_2\,(G) &= \sum_{uv\in E} \frac{n_u\, n_v}{n_u + n_v} = \sum_{uv\in E, d_u=1} \frac{n_u\, n_v}{n_u + n_v} + \sum_{uv\in E, d_u, d_v \neq 1} \frac{n_u\, n_v}{n_u + n_v} \\ &\leq p\left(\frac{n-1}{n}\right) + (m-p)\,\frac{n^2}{8} = \frac{mn^2}{8} + p\left(1 - \frac{1}{n} - \frac{n^2}{8}\right). \end{aligned} \tag{3.1}$$

The above inequality holds as an equality if and only if $m = p$. Therefore, $G \cong K_{1,n-1}$. □

Theorem 2. *Let G be a connected graph of order n with m edges and p pendent vertices. Then*

$$ISI_2\,(G) \geq \frac{m}{n} + p\left(1 - \frac{2}{n}\right),$$

with equality if and only if $G \cong K_{1,n-1}$.

Proof. For any pendent edge uv in the graph under the aforementioned conditions, we have $n_u = 1$ and $n_v = n-1$ clearly, for the remaining edges, we have $n_u, n_v \geq 1$. Now, the second inverse sum indeg index for this family of graphs is as follows:

$$\begin{aligned} ISI_2\,(G) &= \sum_{uv\in E} \frac{n_u\, n_v}{n_u + n_v} = \sum_{uv\in E, d_u=1} \frac{n_u\, n_v}{n_u + n_v} + \sum_{uv\in E, d_u, d_v \neq 1} \frac{n_u\, n_v}{n_u + n_v} \\ &\geq p\left(\frac{n-1}{n}\right) + (m-p)\,\frac{1}{n} = \frac{m}{n} + p\left(1 - \frac{2}{n}\right). \end{aligned} \tag{3.2}$$

It is straightforward to verify that if $m = p$, then equality holds. This occurs only if $G \cong K_{1,n-1}$. □

Remark 1. For the star graph $K_{1,n-1}$, the upper bound in Theorem 1 and the lower bound in Theorem 2 both evaluate to $\frac{(n-1)^2}{n}$. Hence, the star is the unique connected graph of order n for which these bounds coincide, thereby characterizing it as the unique extremal graph that simultaneously minimizes and maximizes $ISI_2(G)$ under the given constraints.

Theorem 3. *Let T be a tree of order of $n > 2$ with p pendent vertices. Then*

$$ISI_2\left(T\right) \leq \frac{1}{n}\left[(n-1)\left(p+\frac{n^2}{4}\right)-\frac{n^2 p}{4}\right],$$

with equality if and only if $T \cong S_n$ or $T \cong S_{\frac{n}{2},\frac{n}{2}}$ for n even.

Proof. Let T be an arbitrary tree with n vertices. For each edge of the tree T, we have $n_u + n_v = n$. Therefore, the second inverse sum indeg index for trees can be expressed as:

$$ISI_2\left(T\right) = \sum_{uv\in E(T)} \frac{n_u\, n_v}{n_u + n_v} = \frac{1}{n}\sum_{uv\in E(T)} n_u\, n_v. \tag{3.3}$$

Moreover, if T has p pendent vertices, then for the pendent edges, $n_u = 1$ and $n_v = n-1$, and for the other edges, $2 \leq n_u, n_v \leq n-2$. Now, the second inverse sum indeg index for this class of trees is as follows:

$$\begin{aligned} ISI_2\left(T\right) &= \sum_{uv\in E(T)} \frac{n_u\, n_v}{n_u + n_v} = \frac{1}{n}\sum_{uv\in E(T)} n_u\, n_v \\ &= \frac{1}{n}\left[\sum_{uv\in E(T), d_u=1} n_u\, n_v + \sum_{uv\in E(T), d_u, d_v\neq 1} n_u n_v\right] \\ &\leq \frac{1}{n}\left[p\left(n-1\right)+\frac{n^2}{4}\left(n-p-1\right)\right] \\ &= \frac{1}{n}\left[(n-1)\left(p+\frac{n^2}{4}\right)-\frac{n^2 p}{4}\right]. \end{aligned} \tag{3.4}$$

To find the trees for which the above equality holds, we consider the following cases.

(i) $p = n-1$, in this case for all edges uv we have $n_u = 1$ and $n_v = n-1$, in other words, all edges of the tree T are pendent, a condition that is satisfied by a star tree S_n.

(ii) $p < n-1$, it is evident that any tree T has at least two pendent vertices. Therefore, in this case, for the equality to hold, it is necessary that $n_u = n_v = \frac{n}{2}$ for all non-pendent edges. It is worth noting that the integer n must be even. Consequently, given the manner in which the equality holds in relation (3.4), the tree T must be a $S_{\frac{n}{2},\frac{n}{2}}$.

□

Theorem 4. *Let T be a tree of order of $n > 2$ with p pendent vertices. Then*

$$ISI_2(T) \geq \frac{1}{n}(n-1)(2n-p-2),$$

with equality if and only if $T \cong S_n$ or $T \cong SL_n(2k,l)$.

Proof. If a tree has p pendent vertices, then for each pendent edge uv, we can write $n_u = 1$ and $n_v = n-1$ and for the other edges we can write $2 \leq n_u, n_v \leq n-2$. Then, using Eq. (3.3) from the proof of Theorem 3, we have

$$\begin{aligned} ISI_2(T) &= \sum_{uv\in E(T)} \frac{n_u n_v}{n_u+n_v} = \frac{1}{n}\sum_{uv\in E(T)} n_u n_v \\ &= \frac{1}{n}\left[\sum_{uv\in E(T),d_u=1} n_u n_v + \sum_{uv\in E(T),d_u,d_v\neq 1} n_u n_v\right] \\ &\geq \frac{1}{n}\left[p(n-1)+2(n-2)(n-p-1)\right] \\ &= \frac{1}{n}\left[(n-3)(2n-p)+4\right]. \end{aligned} \tag{3.5}$$

In order to find the trees for which the above equality holds, we consider the following cases.

(i) $p = n-1$. in this case, for all edges uv of T we have $n_u = 1$ and $n_v = n-1$. Therefore, all edges of the tree T are pendent and $T \cong S_n$.

(ii) $p < n-1$. For the equality to hold, it is necessary that $n_u = 2$ and $n_v = n-2$ for all non-pendent edges. In the present case, these conditions imply that T is the star-like tree $SL_n(2n-2p-2, 2p-n+1)$.

□

Lemma 1. *Let H_n, $n > 3$ be a helm graph. Then the number of vertices in H_n is $2n+1$ and the number of edges is $3n$.*

Proof. See Figure 3. For detailed proofs, consult Reference [23]. □

Theorem 5. *Let H_n, $n \geq 3$ be a Helm graph. Then,*

$$ISI_2(H_n) = \begin{cases} 5 + \frac{18}{7}, & n = 3, \\ \frac{2n^2}{2n+1} + \frac{4n^2-10n}{2n-3} + 2n, & n \geq 4. \end{cases}$$

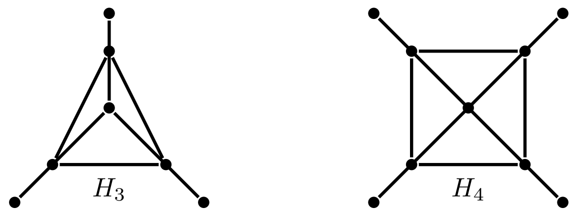

Figure 3. **The Helm graphs H_3 and H_4.**

Proof. For $n = 3$, the computations can be carried out easily. Now, assume that $n \geq 4$. In this case, according to Figure 3, the set of edges the Helm graph H_n can be classified into three categories.
The first set consists of pendent edges. The number of these edges is equal to n, and for them we have, $n_u = 1$ and $n_v = 2n$. The second set of edges, consists of those emanating from the central vertex. The number of these edges is also equal to n, and for them we have $n_u = 2$ and $n_v = 2n - 5$. The remaining edges of the Helm graph H_n, whose number is n, fall into the third category. For these latter edges, $n_u = n_v = 4$. Therefore, for $n \geq 4$

$$ISI_2(H_n) = n\left(\frac{2n}{2n+1}\right) + n\left(\frac{2(2n-5)}{2n-3}\right) + n\left(\frac{16}{8}\right)$$

$$= \frac{2n^2}{2n+1} + \frac{4n^2 - 10n}{2n-3} + 2n.$$

□

4. Computation of $\mathbf{ISI_2}$ for bridge graphs

In this section, we consider several types of bridge graphs and compute their second inverse sum indeg index. In order to calculate ISI_2, we must initially classify the edges of bridge graphs. Such a classification facilitates the computational process.

Lemma 2. *Let $B[P_n, m]$ be a bridge graph over the path P_n, depicted in Figure 4. Then, the number of vertices of $B[P_n, m]$ is nm whereas the number of edges is $nm - 1$.*

Proof. See Figure 4. For detailed proofs, consult Reference [23]. □

Theorem 6. *Let $B[P_n, m]$, $n > 2$, be the bridge graph over path depicted in Figure 4. Then*

$$ISI_2(B[P_n, m]) = \frac{n-1}{6}(3mn - 2n + 1) + \frac{n(m^2-1)}{6}.$$

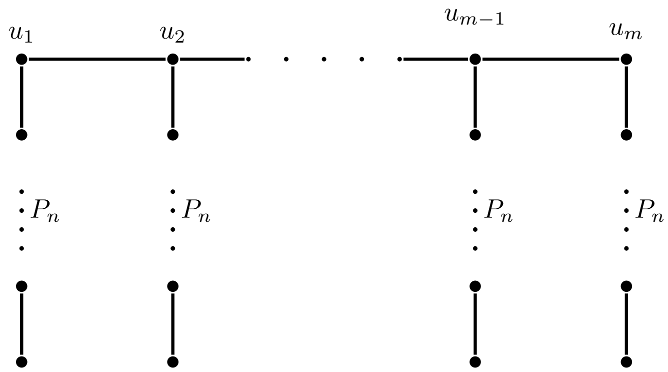

Figure 4. **Bridge graph over path P_n.**

Proof. According to Lemma 2, the number of vertices and edges of $B\left[P_n, m\right]$ are equal to nm and $nm-1$. It is evident that this bridge graph is a tree. Therefore, for each edge, we have $n_u + n_u = mn$ and

$$ISI_2\left(B\left[P_n, m\right]\right) = \sum_{uv\in E} \frac{n_u\, n_v}{n_u + n_v} = \frac{1}{mn} \sum_{uv\in E} n_u\, n_v.$$

Let us at first consider the edges of $B\left[P_n, m\right]$ in the following manner.
The edges of all paths in $B\left[P_n, m\right]$ are divided into n distinct categories. This classification is as follows for category j:

$$n_u\, n_v = j(mn-j), \qquad forj = 1, 2, \ldots, n-1.$$

Alternatively, for the n-th category of edge grouping includes $m-1$ edges $u_i u_{i+1}$ for $i = 1, 2, \ldots m-1$, we have $n_{u_i} = in$ and $n_{u_{i+1}} = mn - in$. Therefore $n_{u_i} n_{u_{i+1}} = in(mn - in)$.

$$\begin{aligned} ISI_2\left(B\left[P_n, m\right]\right) &= \sum_{uv\in E} \frac{n_u\, n_v}{n_u + n_v} = \frac{1}{mn} \sum_{uv\in E} n_u\, n_v \\ &= \frac{1}{mn}\left[\sum_{i=1}^{n-1} im\left(mn - i\right) + \sum_{i=1}^{m-1} in\left(mn - in\right)\right] \qquad (4.1) \\ &= \frac{n-1}{6}\left(3mn - 2n + 1\right) + \frac{n\left(m^2-1\right)}{6}. \end{aligned}$$

Thus, the calculations are complete, and the proof of the theorem is finalized. □

Lemma 3. *Let $B[C_n, m]$, $n > 3$ be a bridge graph over cycle C_n. Then, the number of vertices of $B[C_n, m]$ is mn and the number of edges $mn + m - 1$.*

Proof. See Figure 5. For the proof details, consult Reference [23]. □

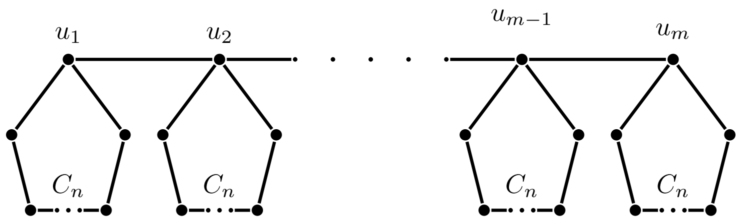

Figure 5. **Bridge graph over cycle C_n.**

Theorem 7. *Let $B[C_n, m]$, $n > 3$, be the bridge graph over cycle C_n. Then*

$$ISI_2(B[C_n, m]) = \begin{cases} \frac{m}{4}(1-3n) + \frac{mn}{6}(m+3n) + \frac{n}{12}(7-3n) - \frac{3}{4} + \frac{1}{4n}, & \text{if } n \text{ is odd}, \\ \frac{n}{12}\left(6mn - 3n + 2m^2 - 2\right), & \text{if } n \text{ is even}. \end{cases}$$

Proof. In order to begin the calculation, we first consider two cases for n.

Case I. n is odd.

In this case, the edges are classified as follows.

In the first class, there are m edges for which the following relation holds $n_u = n_v = \frac{n-1}{2}$.

The second class consists of $m(n-1)$ edges. For each edge in this class, $n_u = \frac{n-1}{2}$ and $n_v = mn - \frac{n-1}{2}$.

The edges of the third class are denoted by $u_i u_{i+1}$. For these edges, the values of n_{u_i} and $n_{u_{i+1}}$ are determined as follows, $n_{u_i} = in$ and $n_{u_{i+1}} = mn - in$. The number of edges in this class is $m - 1$. This implies

$$\begin{aligned} ISI_2(B[C_n, m]) &= \sum_{uv \in E} \frac{n_u n_v}{n_u + n_v} \\ &= \frac{m(n-1)}{4} + \frac{(n-1)^2}{2n}\left(mn - \frac{n-1}{2}\right) + \frac{1}{mn}\left[\sum_{i=1}^{m-1} in(mn - in)\right] \\ &= \frac{m}{4}(1-3n) + \frac{mn}{6}(m+3n) + \frac{n}{12}(7-3n) - \frac{3}{4} + \frac{1}{4n}. \end{aligned} \tag{4.2}$$

Case II: n is even.

In this case, the edges are classified as follows.

In the first class, there are mn edges for which $n_u = \frac{n}{2}$ and $n_v = mn - \frac{n}{2}$.
The edges of the second class are denoted by u_iu_{i+1}. For these edges, the values of n_{u_i} and $n_{u_{i+1}}$ are determined as follows, $n_{u_i} = in$ and $n_{u_{i+1}} = mn - in$. The number of edges in this class is $m - 1$. This implies

$$\begin{aligned} ISI_2\left(B\left[C_n, m\right]\right) &= \sum_{uv \in E} \frac{n_u\, n_v}{n_u + n_v} \\ &= \frac{n}{2}\left(mn - \frac{n}{2}\right) + \frac{1}{mn}\left[\sum_{i=1}^{m-1} in\left(mn - in\right)\right] \qquad (4.3) \\ &= \frac{n}{12}\left(6mn - 3n + 2m^2 - 2\right). \end{aligned}$$

□

Lemma 4. *Let $B\left[S_n, m\right]$, $n > 3$ be the bridge graph over the star S_n. Then the number of vertices of $B\left[S_n, m\right]$ is mn whereas the number of edges is $mn - 1$.*

Proof. The proof is straightforward, see Figure 6. □

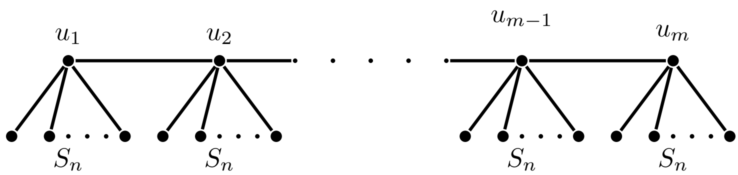

Figure 6. **Bridge graph over star S_n.**

Theorem 8. *Let $B\left[S_n, m\right]$, $n > 3$, be the bridge graph over the star S_n. Then*

$$ISI_2\left(B\left[S_n, m\right]\right) = \frac{1}{n} + \frac{n}{6}\left(m^2 - 1\right) + mn - m - 1.$$

Proof. Since $B\left[S_n, m\right]$ is a tree, for every edge we have $n_u + n_v = mn$. In order to continue calculating the second inverse sum indeg index of this bridge graph, we first divide the set of edges into two categories.
The first category of edge grouping includes $m(n-1)$ edges uv for which $n_u = 1$ and $n_v = mn - 1$.

The second category of edge grouping includes $m-1$ edges u_iu_{i+1} for $i = 1, 2, \ldots m-1$, we have $n_{u_i} = in$ and $n_{u_{i+1}} = mn - in$. This yields

$$\begin{aligned} ISI_2\left(B\left[S_n, m\right]\right) &= \sum_{uv \in E} \frac{n_u\, n_v}{n_u + n_v} = \frac{1}{mn} \sum_{uv \in E} n_u\, n_v \\ &= \frac{1}{n}\left(n-1\right)\left(mn-1\right) + \frac{1}{mn}\left[\sum_{i=1}^{m-1} in\left(mn - in\right)\right] \\ &= \frac{1}{n} + \frac{n}{6}\left(m^2 - 1\right) + mn - m - 1. \end{aligned} \tag{4.4}$$

□

Lemma 5. *Let $B\left[K_n, m\right]$, $n > 3$ be the bridge graph over the complete graph K_n. Then, the number of vertices of $B\left[K_n, m\right]$ is mn and the number of edges $\frac{mn(n-1)}{2} + m - 1$.*

Proof. The proof is straightforward, see Figure 7. □

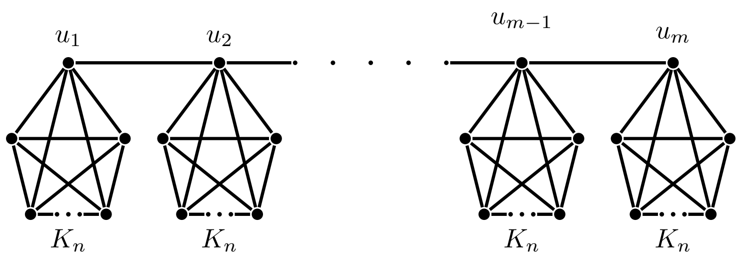

Figure 7. **Bridge graph over complete graph K_n.**

Theorem 9. *Let $B\left[K_n, m\right]$, $n > 3$ be the bridge graph over the complete graph K_n. Then*

$$ISI_2\left(B\left[K_n, m\right]\right) = \frac{mn(n-1)(m-1)}{mn - n + 1} + \frac{1}{2}m(n-1)\left(\frac{n}{2} - 1\right) + \frac{1}{6}mn^2(m^2 - 1).$$

Proof. Considering the complete graph of order n and the bridge graph $B\left[K_n, m\right]$, according to Lemma 5, the number of vertices and the number of edges are equal to mn and $\frac{mn(n-1)}{2} + m - 1$. For calculating the second inverse sum indeg index of this bridge graph, it is necessary to categorize the edges as follows.
The first category of edge grouping includes $m(n-1)$ edges uv for which $n_u = 1$ and $n_v = n(m-1)$.

The second category of edge grouping includes $m(n-1)(\frac{n}{2}-1)$ edges uv for which $n_u = n_v = 1$.
The third category of edge grouping includes $m-1$ edges u_iu_{i+1} for $i = 1, 2, \dots m-1$, for which $n_{u_i} = in$ and $n_{u_{i+1}} = mn - in$. Now, the calculation of the ISI_2 for this bridge graph is as follows:

$$
\begin{aligned}
ISI_2\left(B\left[K_n, m\right]\right) &= \sum_{uv\in E} \frac{n_u\, n_v}{n_u + n_v} \\
&= m(n-1)\left(\frac{n(m-1)}{n(m-1)+1}\right) + \frac{1}{2}\left(m(n-1)(\frac{n}{2}-1)\right) \\
&+ \frac{1}{mn}\left[\sum_{i=1}^{m-1} in\left(mn - in\right)\right] \\
&= \frac{mn(n-1)(m-1)}{mn-n+1} + \frac{1}{2}m(n-1)(\frac{n}{2}-1) + \frac{1}{6}mn^2(m^2-1).
\end{aligned} \tag{4.5}
$$

□

5. Concluding and future work.

In this study, by focusing on the second inverse sum indeg index, significant strides were made towards enriching our knowledge on topological indices, especially those based on distances between vertices. The provision of upper and lower bounds for specific graphs and trees, along with the computation of this index for bridge graphs, not only deepened our understanding of dependence of topological indices on graph structures, but also revealed their potential applications in interdisciplinary fields such as molecular property prediction. However, this research should be considered merely as a starting point for future explorations.
We have included several results on specific classes of bridge graphs such as paths, cycles, and others. These results can be further generalized. Instead of restricting to particular graphs, we could consider trees in general and express the results for general bridge graphs in terms of the ISI_2 index of the corresponding trees. Similarly, this approach can be extended to bipartite graphs by attaching an arbitrary bipartite graph and presenting the results in terms of its ISI_2 index.
Questions regarding the behavior of second inverse sum indeg index in more complex graphs, its relationship with other graph parameters, and its applications in large-scale networks remain a fertile ground for further investigation. Moreover, extending these findings to areas such as computational chemistry and materials science could pave new avenues for understanding and predicting the structure-dependency of physical and chemical properties of molecules.

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