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Research Article

Counting the number of domatic partitions of specific graphs

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Abstract: A subset of vertices S of a graph G is a dominating set if every vertex in $V \setminus S$ has at least one neighbor in S. A domatic partition is a partition of the vertices of a graph G into disjoint dominating sets. The domatic number $d(G)$ is the maximum size of a domatic partition. We consider the number of domatic partitions of G with different sizes. Inspired by existing results for trees, this paper extends the analysis to several other important families of graphs. We focus primarily on the coefficient $dp(G, 2)$, which counts domatic 2-partitions. We present some recurrence relations for this coefficient for the cycle graphs C_n and the wheel graphs W_n. Furthermore, we present precise closed-form formulas for the domatic polynomial of star graphs $K_{1,n}$ and friendship graphs F_n. We also derive a formula for $dp(K_{m,n}, 2)$ for complete bipartite graphs. Finally, through a comprehensive computational analysis of all 3-regular graphs of order 10, we observe that the Petersen graph cannot be determined by its domatic polynomial.

Keywords: domatic partition, domatic number, dominating set, polynomial.

AMS Subject classification: 05C69

1. Introduction

Let $G = (V, E)$ be a simple graph. A subset $S \subseteq V$ is a dominating set if every vertex $v \in V \setminus S$ has at least one neighbor in S. A domatic partition of size k for G is a partition of the vertex set V into k disjoint sets $\{V_1, V_2, \ldots, V_k\}$ such that every set V_i is a dominating set. The maximum possible k is called the domatic number, $d(G)$.

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The domatic number problem, $d(G)$, seeks to find the maximum possible quantity of pairwise disjoint dominating sets into which a graph G can be divided. An extension of this, the domatic partition problem, not only requires this maximum number $d(G)$ but also the actual sets forming this partition.

These problems are fundamental classical optimisation problems, first introduced in [3]. Their computational complexity is generally high, as both are NP-complete, a difficulty that persists even for certain graphs. However, they can be efficiently solved in linear time for more constrained graph families, such as interval graphs and strongly chordal graphs [6, 7].

A key application of domatic problems is the strategic placement of facilities or resources within a network. Consider a scenario where various essential facility types (e.g., hospitals, servers, universities) need to be installed in a network such that no network node hosts more than one facility type, and a facility's resources are only accessible to its own node and its immediate neighbours. In this model, the domatic number $d(G)$ answers the question: "What is the maximum number of different facility types that can be located?" The domatic partition then specifies "Where should each facility type be placed?" Beyond infrastructure planning, these problems are also relevant in strategy-related issues in business and military domains [5].

Due to the importance of counting domatic partitions, the domatic polynomial was recently introduced [1] and is defined as:

$$DP(G,x) = \sum_{i=1}^{d(G)} dp(G,i)x^i,$$

where $dp(G,i)$ is the number of domatic partitions of size i. It has been previously shown that for the path graph P_n, the coefficient $dp(P_n,2)$ follows the Fibonacci sequence [1]. In this paper, we continue this analysis by focusing on several other graph families. Our investigation primarily targets the coefficient $dp(G,2)$, for which we derive new recurrence relations and formulas. However, we also successfully compute the domatic polynomial $DP(G,x)$ for specific families where this is achievable, such as star graphs and friendship graphs. We compute the domatic polynomial of all 3-regular graphs of order 10. Finally, we briefly describe the computational methodology used to obtain the domatic polynomial.

2. The closed-form formulas

In this section, we study the number of domatic partition of any size of some specific graphs.

2.1. Star graphs

We consider the star graph $K_{1,n}$ on $n+1$ vertices. Its vertex set V can be partitioned into two disjoint subsets, $V_1 = \{c\}$ and $V_2 = \{l_1, l_2, \ldots, l_n\}$, such that the edge set

E consists of all edges connecting the vertex in V_1 to every vertex in V_2. Formally, $E = \{\{c, l_i\} \mid 1 \leq i \leq n\}$. The vertex c is called the center, and the vertices in V_2 are called the leaves. We need the following theorem:

Theorem 1. [3] *For any graph G, $d(G) \leq \delta + 1$, where δ is the minimum degree of G.*

Proposition 1. *The domatic polynomial of $K_{1,n}$ is*

$$DP(K_{1,n}, x) = x + x^2.$$

Proof. Let $V = \{c\} \cup L$ be the vertex set of $K_{1,n}$, where c is the central vertex and $L = \{l_1, l_2, \ldots, l_n\}$ is the set of n leaves. We determine the coefficients of the domatic polynomial, $dp(K_{1,n}, i)$, for all possible values of i. First, for $i = 1$, the only partition of size 1 is the set V itself, which means $dp(K_{1,n}, 1) = 1$. By Theorem 1, $d(K_{1,n}) \leq 2$. So $dp(K_{1,n}, i) = 0$ for all $i > 2$. Now we investigate the case for $i = 2$. Consider the partition of V into two disjoint sets: $V_1 = \{c\}$ and $V_2 = L$. Since V_1 and V_2 are dominating sets of $K_{1,n}$, so $\{V_1, V_2\}$ constitutes a valid domatic partition of size 2. It follows that $d(K_{1,n}) \geq 2$. Therefore $d(K_{1,n}) = 2$.
To determine the coefficient $dp(K_{1,n}, 2)$, we show that this partition is unique. Let $\{S_1, S_2\}$ be any domatic partition of size 2. The central vertex c must belong to one of the sets, say $c \in S_1$. For S_2 to be a dominating set, it must dominate the vertex c. This requires that at least one neighbor of c (i.e., a leaf from L) must be in S_2. Now, consider any leaf $l_j \in L$. If l_j were also in S_1, then S_2 would fail to dominate l_j, as its only neighbor, c, is not in S_1. This would contradict the assumption that S_2 is a dominating set. Therefore, to ensure S_2 is dominating, all leaf vertices must reside in S_2. This forces the partition to be $S_1 = \{c\}$ and $S_2 = L$. Thus, the domatic partition of size 2 is unique, and $dp(K_{1,n}, 2) = 1$. This completes the proof. □

2.2. Complete bipartite graphs

We consider the complete bipartite graph $K_{m,n}$ whose vertex set V can be partitioned into two disjoint and independent sets, A and B, with $|A| = m$ and $|B| = n$. The edge set E contains all possible edges between the vertices of A and B, i.e., $E = \{\{a, b\} \mid a \in A, b \in B\}$.

Theorem 2. *The number of domatic partitions of size 2 of $K_{m,n}$ is*

$$dp(K_{m,n}, 2) = \frac{(2^m - 2)(2^n - 2)}{2} + 1.$$

Proof. Let the vertex set be $V = A \cup B$ with $|A| = m$ and $|B| = n$. First, it is clear that $dp(K_{m,n}, 1) = 1$ as the set V itself is always a dominating set. To find $dp(K_{m,n}, 2)$, we count the number of ways to partition V into two dominating sets, $\{V_1, V_2\}$. We consider two disjoint cases.

Case 1. The natural bipartition: Consider the partition $\{A, B\}$. The set A is a dominating set, because every vertex $b \in B$ is adjacent to all m vertices in A. Similarly, the set B is a dominating set because every vertex $a \in A$ is adjacent to all n vertices in B. Thus, $\{A, B\}$ is a valid domatic partition of size 2. This yields one such partition.

Case 2. All the other partitions: Let $\{V_1, V_2\}$ be any domatic 2-partition other than $\{A, B\}$. We claim that for such a partition, both V_1 and V_2 must have a non-empty intersection with both A and B. Suppose, for contradiction, that $V_1 \cap A = \emptyset$. This implies $A \subseteq V_2$. For V_1 to be a dominating set, it must dominate every vertex in A. Since all neighbors of vertices in A are in B, V_1 must be a non-empty subset of B. However, if $V_1 \subseteq B$, it cannot dominate any other vertex in B (as B is an independent set). This would require $B \setminus V_1 = \emptyset$, which means $V_1 = B$ and thus $V_2 = A$, contradicting our assumption that this is a different partition. Thus, $V_1 \cap A \neq \emptyset$ and, similarly, $V_1 \cap B \neq \emptyset$, $V_2 \cap A \neq \emptyset$, and $V_2 \cap B \neq \emptyset$.
Let $V_1 = A_1 \cup B_1$ and $V_2 = A_2 \cup B_2$, where $A_1 = V_1 \cap A$, $A_2 = V_2 \cap A$, etc. The condition above means that A_1, A_2, B_1, B_2 are all non-empty. This is equivalent to choosing a proper, non-empty subset A_1 from A, and a proper, non-empty subset B_1 from B to form the set V_1. The number of ways to choose a proper non-empty subset A_1 from A is $2^m - 2$. The number of ways to choose a proper non-empty subset B_1 from B is $2^n - 2$. Any such choice of A_1 and B_1 forms a set $V_1 = A_1 \cup B_1$. The remaining vertices form $V_2 = (A \setminus A_1) \cup (B \setminus B_1) = A_2 \cup B_2$. Let's verify that $\{V_1, V_2\}$ is a valid domatic partition.

- Any vertex $a \in A_2$ has neighbors in B. Since B_1 is non-empty, so a is dominated by V_1. Any vertex $b \in B_2$ has neighbors in A. Since A_1 is non-empty, so b is dominated by V_1. Thus, V_1 is a dominating set.
- Any vertex $a \in A_1$ has neighbors in B. Since $B_2 = B \setminus B_1$ is non-empty, a is dominated by V_2. Any vertex $b \in B_1$ has neighbors in A. Since $A_2 = A \setminus A_1$ is non-empty, b is dominated by V_2. Thus, V_2 is a dominating set.

The total number of ways to construct an ordered pair of sets (V_1, V_2) in this way is $(2^m - 2)(2^n - 2)$. Since a domatic partition $\{V_1, V_2\}$ is an unordered pair of sets, and each choice of V_1 uniquely determines V_2, we have counted each partition twice (once by forming V_1 and once by forming its complement V_2). Therefore, the number of such partitions is $\frac{(2^m-2)(2^n-2)}{2}$. Therefore,

$$dp(K_{m,n}, 2) = 1 + \frac{(2^m - 2)(2^n - 2)}{2}.$$

□

2.3. Cycle graphs

We consider the cycle graph C_n with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$, and the edge set $E = \{\{v_i, v_{i+1}\} \mid 1 \leq i \leq n-1\} \cup \{\{v_n, v_1\}\}$. We recall that the Lucas number L_n

satisfies the recurrence relation $L_n = L_{n-1} + L_{n-2}$, with two Lucas numbers $L_1 = 1$ and $L_2 = 3$ (see [4]). The following theorem gives the number of domatic partition of size two of C_n. We recall that the trace of a square matrix A, denoted $\mathrm{Tr}(A)$, is the sum of the elements on its main diagonal.

Theorem 3. *Let L_n be nth term of Lucas sequence. For $n \geq 5$,*

$$dp(C_n, 2) = \frac{1}{2}L_n + \cos\left(\frac{2\pi n}{3}\right).$$

Proof. A domatic 2-partition of C_n corresponds to a 2-coloring of its vertices (e.g., with colors Black (B) and White (W)) such that each color class is a dominating set. This condition is met if and only if every vertex has a neighbor of the opposite color. For a cycle graph, this is equivalent to ensuring that no three consecutive vertices have the same color. Let c_n be the number of such valid 2-colorings. Since swapping the two colors results in a valid coloring that yields the same partition, the number of domatic partitions is $dp(C_n, 2) = \frac{1}{2}c_n$. We use the transfer matrix method to count these valid colorings. To decide the color of a vertex in C_n, we only need to know the colors of the previous two vertices. This suggests the four states for a path coloring:

- State 1: The path ends in a single Black vertex preceded by White (...WB).
- State 2: The path ends in two consecutive Black vertices (...WBB).
- State 3: The path ends in a single White vertex preceded by Black (...BW).
- State 4: The path ends in two consecutive White vertices (...BWW).

The transitions between these states can be represented by a transfer matrix M. For instance, from state 2 (...WBB), the next vertex must be White to avoid a 'BBB' sequence, leading to state 3 (...BBW). The complete matrix is:

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The entry M_{ij} of the matrix represents a valid transition from state j to state i. Consequently, the entry (i, j) of M^k counts the number of valid colored paths of length $k + 1$ that start in state j and end in state i. For a path graph, we would sum all entries of M^{n-1}. However, for the cycle graph C_n, the coloring must form a closed loop. This imposes the constraint that the state of the graph after coloring all n vertices must be compatible with the initial state. This is equivalent to counting closed walks of length n in the state graph. The total number of such closed walks is precisely the trace of M^n. So the number of valid colorings is given by the sum of the n-th powers of the eigenvalues (λ_i) of M:

$$c_n = \mathrm{Tr}(M^n) = \sum_{i=1}^{4} \lambda_i^n.$$

The characteristic polynomial of M is $\det(M - \lambda I) = \lambda^4 - \lambda^2 - 2\lambda - 1 = 0$. The four eigenvalues are the roots of this polynomial:

$$\phi = \frac{1+\sqrt{5}}{2}, \quad \psi = \frac{1-\sqrt{5}}{2}, \quad \omega = e^{2\pi i/3}, \quad \bar{\omega} = e^{-2\pi i/3}.$$

The first two eigenvalues, ϕ and ψ, are the roots of $x^2 - x - 1 = 0$. A well-known identity states that the sum of their n-th powers yields the n-th Lucas number, $L_n = \phi^n + \psi^n$ (see e.g. [4]). The last two eigenvalues are the non-real cube roots of unity. Using Euler's formula, their sum is $\omega^n + \bar{\omega}^n = 2\cos\left(\frac{2\pi n}{3}\right)$.
Combining these results, we obtain the closed-form expression for c_n:

$$c_n = (\phi^n + \psi^n) + (\omega^n + \bar{\omega}^n) = L_n + 2\cos\left(\frac{2\pi n}{3}\right).$$

Therefore, the number of domatic 2-partitions is given by the exact formula:

$$dp(C_n, 2) = \frac{c_n}{2} = \frac{1}{2}L_n + \cos\left(\frac{2\pi n}{3}\right).$$

□

Remark 1. It is easy to see that the value of $dp(C_n, 2)$ in Theorem 3 satisfies the following recurrence relation:

$$dp(C_n, 2) = dp(C_{n-1}, 2) + dp(C_{n-2}, 2) + f(n),$$

where $f(n) = 2$ if $n \equiv 0 \pmod 3$ and $f(n) = -1$ otherwise, with initial conditions $dp(C_3, 2) = 3, dp(C_4, 2) = 3$.

3. Recurrence relations

In this section, we obtain recurrence relations for the number of domatic partition of any size of some specific graphs.

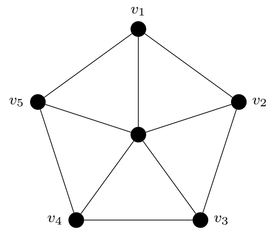

Figure 1. **An example of a wheel graph, W_6.**

3.1. Wheel graphs (W_n)

We consider the wheel graph W_n, formed by connecting a single central vertex (the hub) to all vertices of a cycle graph C_{n-1} (the rim). We denote the hub as v_0 and the vertices on the rim as $v_1, v_2, \ldots, v_{n-1}$ in order. The graph W_6 consists of a central hub connected to the 5 vertices of a C_5 cycle, drawn in Figure 1.

Theorem 4. *Let $a_n = dp(W_n, 2)$. For $n \geq 7$, the sequence $\{a_n\}$ satisfies the recurrence relation $a_n = a_{n-1} + a_{n-2} + a_{n-3}$ with initial conditions $a_4 = 7$, $a_5 = 11$, $a_6 = 21$.*

Proof. The wheel graph W_n is constructed by joining the cycle graph C_{n-1} and a single vertex K_1, i.e., $W_n = C_{n-1} \vee K_1$. We transform the problem of counting domatic 2-partitions of W_n to counting a specific type of 2-coloring of the vertices of C_{n-1}. The number of domatic 2-partitions of W_n, a_n, is equal to the number of valid 2-colorings (black and white) of the vertices of W_n where the color of the central vertex v_0 is fixed to black. A coloring is valid if every vertex has a neighbor of the opposite color. Any white vertex on the rim C_{n-1} is adjacent to the black hub v_0, so its domination is always satisfied. The black hub v_0 must have at least one white neighbor on the rim. Thus, the rim cannot be colored all black. Any black vertex on the rim is adjacent to the black hub v_0, so it must have a white neighbor on the rim. This means no three consecutive vertices on the rim can be colored black. So, a_n is equal to the number of 2-colorings of C_{n-1} that have no cyclic 'BBB' (three consecutive blacks) substring. Let us denote this number by c'_k for a cycle C_k. Then $a_n = c'_{n-1}$. The condition that the rim is not all black is implicitly satisfied by the "no BBB" rule for any cycle of length $n - 1 \geq 3$. To derive the recurrence for c'_k, we use the transfer matrix method. We define states based on the number of consecutive black vertices at the end of a path.

- State 0: The sequence ends with a white vertex ($\ldots$W).
- State 1: The sequence ends with a single black vertex ($\ldots$WB).
- State 2: The sequence ends with two consecutive black vertices ($\ldots$WBB).

The transfer matrix M, where M_{ij} is the number of ways to transition from state j to state i, is constructed based on allowed colorings:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

For example, from State 0 (...W), one can add a W (staying in State 0) or a B (moving to State 1), giving the first column. From State 1 (...WB), one can add a W (moving to State 0) or a B (moving to State 2), giving the second column. From State 2 (...WBB), one can only add a W (moving to State 0).
For the cycle C_k, the number of valid colorings is given by $c'_k = \mathrm{Tr}(M^k)$. The characteristic polynomial of M is $\det(M - \lambda I) = \lambda^3 - \lambda^2 - \lambda - 1 = 0$. Therefore, the sequence c'_k must satisfy the recurrence relation:

$$c'_k = c'_{k-1} + c'_{k-2} + c'_{k-3}.$$

The initial values can be verified by direct enumeration: $c'_3 = 7$ (all but BBB), $c'_4 = 11$ (all but BBBB and rotations of BBBW), and $c'_5 = 21$. This recurrence is known to hold for $k \geq 6$.
Substituting $k = n - 1$ and using the fact that $a_n = c'_{n-1}$, we have $a_n = c'_{n-1}$, $a_{n-1} = c'_{n-2}$, and so on. This gives the desired recurrence for the wheel graph:

$$a_n = a_{n-1} + a_{n-2} + a_{n-3}.$$

This relation is valid for $n \geq 7$. This completes the proof. □

3.2. Friendship graphs

The friendship graph F_n is a graph with $2n+1$ vertices and $3n$ edges, constructed by joining n copies of the cycle graph C_3 at a single common vertex. This central vertex is adjacent to all other $2n$ vertices. For example, the graph F_2, also known as the butterfly graph, consists of two triangles joined at a central vertex. It has 5 vertices and 6 edges.

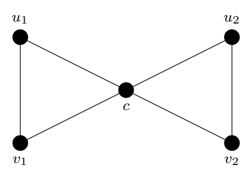

Figure 2. **The friendship graph F_2 (butterfly graph).**

We state and prove the following theorem which is a closed-form formula for the domatic polynomial of the friendship graph.

Theorem 5. *The domatic polynomial of the friendship graph F_n is:*

$$DP(F_n, x) = x + 3^n x^2 + 2^{n-1} x^3.$$

Proof. The coefficient $dp(F_n, 1) = 1$ is trivial for any connected graph. The minimum degree $\delta(F_n) = 2$, implies $d(F_n) \leq \delta(F_n) + 1 = 3$. Therefore, $dp(F_n, k) = 0$ for all $k \geq 4$. We find the number of domatic 2-partitions by counting the number of valid 2-colorings (using colors Black and White) and dividing the result by 2. A coloring is valid if every vertex has at least one neighbor of the opposite color. Let c be the central vertex. We first consider the case where c is colored Black. Any White vertex is adjacent to c, so its domination is guaranteed. The central vertex c must be dominated, so at least one of its neighbors must be White. Consider any non-central Black vertex, say u_i from the i-th triangle. Its neighbors are c (which is Black) and its triangle partner v_i. For u_i to be dominated, its partner v_i must be colored White. So any pair of non-central vertices $\{u_i, v_i\}$ in a triangle cannot both be Black. Therefore, for each of the n triangles, there are exactly 3 ways to color its two non-central vertices. Since these choices are independent for each triangle, the total number of valid colorings when c is Black is 3^n.

Similarly, if the central vertex c is colored White, there are also 3^n valid colorings and so the total number of valid 2-colorings is $2 \cdot 3^n$. Since a domatic partition is an unordered pair of sets $\{V_{\text{Black}}, V_{\text{White}}\}$, we divide this number by 2 and so,

$$dp(F_n, 2) = \frac{2 \cdot 3^n}{2} = 3^n.$$

Now, let $\{V_1, V_2, V_3\}$ be a domatic 3-partition of F_n. The central vertex c must belong to one of these sets, say $c \in V_1$. For V_1 to be a dominating set, it is sufficient that $V_1 = \{c\}$, since c is adjacent to all other vertices in the graph. For V_2 and V_3 to be the dominating sets, they must dominate c. This requires that both V_2 and V_3 contain at least one neighbor of c. The remaining $2n$ non-central vertices must be partitioned into V_2 and V_3. Consider an arbitrary non-central vertex $u_i \in V_2$. For the set V_3 to dominate u_i, it must contain one of its neighbors, which are $\{c, v_i\}$. Since $c \notin V_3$, it is necessary that $v_i \in V_3$.

This establishes a strict rule for the partition: for each non-central pair $\{u_i, v_i\}$ from a triangle, one vertex must be in V_2 and the other must be in V_3. We now count the number of ways to form such a partition. For each of the n pairs, there are 2 choices for assigning its vertices to the sets V_2 and V_3. This yields 2^n ways to construct an ordered pair of sets (V_2, V_3). Since the domatic partition $\{V_1, V_2, V_3\}$ is the same as $\{V_1, V_3, V_2\}$, the pair of sets (V_2, V_3) is unordered. We must therefore divide by 2. So $dp(F_n, 3) = 2^{n-1}$. This completes the proof. □

3.3. Generalized fan graphs

The generalized fan graph $F_{m,n}$ is defined as the graph join of an empty graph on m vertices $\overline{K_m}$ and a path graph on n vertices P_n. Formally, $F_{m,n} = \overline{K_m} \vee P_n$. Its

vertex set can be partitioned into an independent set of m vertices, which we call the center, and a set of n vertices forming a path, which we call the rim. Every vertex in the center is connected to every vertex on the rim.
The graph $F_{2,3}$ consists of a center with 2 vertices and a rim with 3 vertices forming a path, has shown in Figure 3.

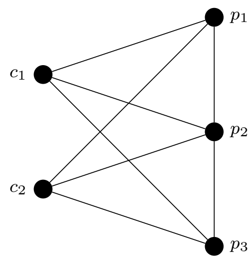

Figure 3. **The generalized fan graph,** $F_{2,3}$**.**

Theorem 6. *For* $m, n \geq 1$*,*

$$dp(F_{m,n}, 2) = c_n + \frac{(2^m - 2)(2^n - 2)}{2},$$

where the sequence c_n *satisfies* $c_n = c_{n-1} + c_{n-2} + c_{n-3}$ *for* $n \geq 4$*, with initial conditions* $c_1 = 1, c_2 = 3,$ *and* $c_3 = 5$*.*

Proof. We count the number of domatic 2-partitions by counting the number of valid 2-colorings (using colors Black and White) and then dividing by 2. A 2-coloring is valid if and only if every vertex has a neighbor of the opposite color. Let M be the set of m vertices in $\overline{K_m}$ and P be the set of n vertices in P_n. We partition the set of all 2^{m+n} colorings into two disjoint cases based on the coloring of the central vertices in M.

Case 1. The coloring of M is monochromatic.
There are two sub-cases: all vertices in the center M are Black, or all are White. We first analyze the case where all vertices in M are colored Black. For a 2-coloring of the vertices on the rim P to form a valid domatic partition with the monochromatic center, two conditions related to the color classes must be satisfied. To ensure the Black vertices in M are dominated, there must be at least one White vertex in P. Thus, the coloring of P cannot be monochromatic Black. To ensure any Black vertex $v \in P$ is dominated, it must have a White neighbor. Since all its neighbors in M are also Black, v must have a White neighbor within the path P. This implies that the coloring of P cannot have a substring of three consecutive Black vertices ('BBB'). Moreover, for the endpoints of P (which have only one neighbor in P), if an endpoint is Black, its single neighbor in P must be White, implying no 'BB' at the start or end of the coloring.

Let the sequence c_n be the number of 2-colorings of P_n that avoid the 'BBB' substring, do not start with 'BB', do not end with 'BB', and are not monochromatic Black. This sequence c_n counts colorings on a *path* P_n with specific boundary constraints, but it is governed by the same underlying 'no BBB' rule discussed in Section 3.0.1. The transfer matrix M for this 'no BBB' constraint was introduced in the proof of Theorem 3.1. As is standard for linear recurrences, any sequence generated by this system (such as the total number of paths, or paths with specific end-point constraints like c_n) must satisfy the recurrence relation derived from its characteristic polynomial $\lambda^3 - \lambda^2 - \lambda - 1 = 0$. This sequence satisfies the recurrence $c_n = c_{n-1} + c_{n-2} + c_{n-3}$ for $n \geq 4$, with initial conditions $c_1 = 1$ (only White), $c_2 = 3$ (BW, WB, WW), and $c_3 = 5$ (BWB, BWW, WBW, WWB, WWW). The colorings counted by c_n satisfy both conditions above. Similarly, if the center M is all White, the number of valid colorings for P is also c_n (no 'WWW', no start/end 'WW', not all White). Thus, the total number of valid colorings in this case is $2c_n$.

Case 2. The coloring of M is not monochromatic.
There are $2^m - 2$ ways to color the vertices of M with a mix of Black and White colors. Consider any vertex $v \in P$. Since v is adjacent to all vertices in M, it has both Black and White neighbors. Therefore, the domination of every vertex in P is always guaranteed, regardless of its color. The only remaining condition is that the vertices in M must be dominated. A Black vertex in M is dominated if there is at least one White vertex in P. A White vertex in M is dominated if there is at least one Black vertex in P. Thus, the coloring of P must simply be non-monochromatic. The number of non-monochromatic colorings of P_n is $2^n - 2$. The total number of valid colorings in this case is $(2^m - 2)(2^n - 2)$.
Summing the counts from both disjoint cases gives the total number of valid 2- colorings which is $2c_n + (2^m - 2)(2^n - 2)$. Since a domatic partition is an unordered pair of sets $\{V_{\text{Black}}, V_{\text{White}}\}$, we divide the total number of valid colorings by 2 to get the number of partitions. Therefore

$$dp(F_{m,n}, 2) = c_n + \frac{(2^m - 2)(2^n - 2)}{2}.$$

This completes the proof. □

3.4. Ladder graphs (L_n)

Here we consider the ladder graph L_n constructed as the Cartesian product of the path graph P_n and the graph K_2. It can be visualized as two paths of length $n-1$ with vertices $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$, where corresponding vertices are connected by edges (the "rungs" of the ladder). Thus, its edge set is $\{\{u_i, u_{i+1}\}, \{v_i, v_{i+1}\} \mid 1 \leq i < n\} \cup \{\{u_i, v_i\} \mid 1 \leq i \leq n\}$. The graph L_4 is a ladder with 4 rungs, consisting of 8 vertices and 11 edges.
Using the computational method described in Algorithm 1, we generated the exact values of $dp(L_n, 2)$ for small values of n. By analyzing the generated integer sequence,

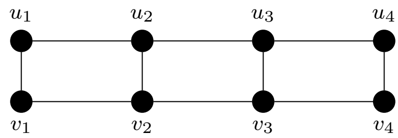

Figure 4. **An example of a ladder graph, L_4.**

we identified a linear recurrence relation, which holds for all computed terms. While a formal combinatorial proof remains an open problem, the numerical evidence strongly supports the validity of this recursive sequence. We close this section by the following conjecture:

Conjecture 7. Let $a_n = dp(L_n, 2)$. For $n \geq 5$, this sequence satisfies the following recurrence relation:

$$a_n = 3a_{n-1} + a_{n-2} - 2a_{n-4},$$

with initial conditions $a_1 = 1, a_2 = 3, a_3 = 11, a_4 = 34$.

3.5. 3-Regular graphs of order 10

In this section, we study the domatic polynomial of cubic graphs of order 10. In particular, we obtain the domatic polynomial of the Petersen graph.

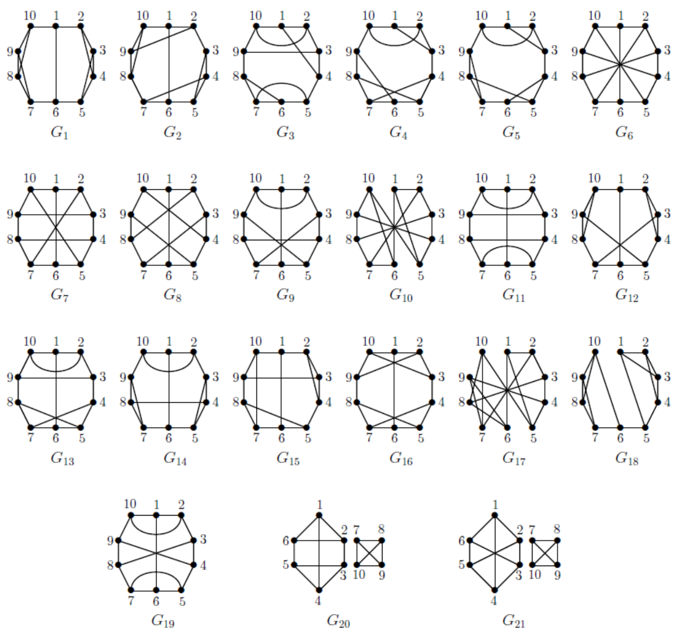

Figure 5. **Cubic graphs of order 10.**

There are exactly 21 cubic graphs of order 10 denoted by $G_1, G_2, ..., G_{21}$ in Figure 5 (see [2]). In particular, the graph G_{17} is isomorphic to the Petersen graph P.
We computed the domatic polynomials for all 21 graphs shown in Figure 5 using a brute-force algorithm in Python with the NetworkX library. The full list of polynomials for all 21 graphs is presented in Table 3.5.

Graph Name	**Domatic Polynomial** $DP(G,x)$
G_1, G_2, G_6	$x + 181x^2 + 25x^3$
G_{11}, G_{16}, G_{19}	$x + 181x^2 + 25x^3$
G_3, G_7, G_{12}	$x + 193x^2 + 24x^3$
G_4, G_8, G_{15}	$x + 165x^2 + 9x^3$
G_5, G_9	$x + 213x^2 + 56x^3$
G_{13}, G_{14}, G_{18}	$x + 157x^2 + 2x^3$
G_{10}	$x + 141x^2$
G_{17} (Petersen)	$x + 141x^2$
G_{20}, G_{21}	$x + 141x^2$

Table 1. **Domatic polynomials for all 21 cubic graphs of order 10 from Figure 5.**

Our comprehensive analysis revealed that these 21 graphs fall into 7 distinct groups based on their domatic polynomials (likely corresponding to non-isomorphic classes). The computations confirm a key finding: the Petersen graph (G_{17}) has the domatic polynomial $DP(P, x) = x + 141x^2$.
More importantly, we found this polynomial is not unique. The graphs G_{10}, G_{20}, and G_{21} are non-isomorphic to the Petersen graph but share the exact same domatic polynomial. This leads to the following observation.
Now we close the paper by the following observation.

Observation 8. The Petersen graph is not determined uniquely by its domatic polynomial. Our comprehensive analysis shows that at least three other 3-regular graphs of order 10 (G_{10}, G_{20}, and G_{21}) share the same polynomial $x + 141x^2$.

4. Computational methodology

We briefly describe the methodology used to discover several of the recurrence relations and to compute the polynomials for the cubic graphs.
Our approach was computational-experimental. We first developed a brute-force algorithm to compute the exact coefficients $dp(G, k)$ of the domatic polynomial for a given graph G and small values of k. This algorithm works by enumerating all possible partitions of the vertex set V into k disjoint subsets, and then checking if every subset in a given partition is a dominating set.
The core logic of this algorithm is presented in Algorithm 1. We implemented this in Python. This algorithm has an exponential complexity, roughly $O(k^n \cdot n^2)$, and is therefore only feasible for small graphs.

Algorithm 1 Brute-Force Computation of $dp(G, k)$

```
function IsDominating(S, G = (V, E))
    dominated_set ← S
    for all u ∈ S do
        for all v ∈ Neighbors(u) do
            dominated_set ← dominated_set ∪ {v}
        end for
    end for
    if dominated_set = V then
        return true
    else
        return false
    end if
end function

function CalculateDPCoefficient(G = (V, E), k)
    V_list ← List(V)
    n ← |V|
    labeled_partition_count ← 0
                                ▷ Iterate through all k^n assignments of n vertices into k bins
    for all assignment A in product({1, ..., k}, repeat = n) do
        Partitions ← [ {} for i in 1 ... k ]
        for i ← 1 to n do
            j ← A[i]                                        ▷ Get bin index for vertex v_i
            Partitions[j] ← Partitions[j] ∪ {V_list[i]}
        end for
        if all(S ≠ ∅ for S ∈ Partitions) then               ▷ Ensure no set is empty
            all_are_dominating ← true
            for S ∈ Partitions do
                if not IsDominating(S, G) then
                    all_are_dominating ← false
                    break
                end if
            end for
            if all_are_dominating then
                labeled_partition_count ← labeled_partition_count + 1
            end if
        end if
    end for
    unlabeled_count ← labeled_partition_count/k!
    return unlabeled_count
end function
```

By running this algorithm on graph families for increasing orders (e.g., W_n, L_n for $n = 3, 4, 5, \ldots$), we generated sequences of coefficients (e.g., the sequence for $dp(L_n, 2)$). We then analyzed these integer sequences to hypothesize the linear recurrence relations presented in Section 3. The same algorithm was used to generate the data for the 3-regular graphs of order 10, where $n = 10$ and k is small.

While the brute-force algorithm presented here suffices for small graphs, processing larger graphs requires more efficient approaches to overcome the exponential complexity. Future investigations could utilize dynamic programming on tree decompositions, which is particularly effective for graph families with bounded treewidth. Additionally, mapping the domatic partition problem to a SAT (Satisfiability) instance and employing modern SAT solvers, or using heuristic search methods such as backtracking with pruning, could significantly extend the range of computable graphs.

5. Conclusion and future work

In this paper, we investigated the domatic polynomial $DP(G, x)$ for several important families of graphs. By combining combinatorial analysis, the transfer matrix method,

and direct computational enumeration, we successfully derived new recurrence relations and closed-form formulas.
Our main contributions include:

- Precise closed-form formulas for the entire domatic polynomial of the star graphs $K_{1,n}$ and the friendship graphs F_n.
- Recurrence relations and formulas for the coefficient $dp(G, 2)$ (the number of domatic 2-partitions) for cycle C_n, wheel W_n, complete bipartite $K_{m,n}$, and generalized fan $F_{m,n}$ graphs.
- A comprehensive computational analysis of all 3-regular graphs of order 10. This analysis led to the significant observation that the Petersen graph (G_{17}) is not determined by its domatic polynomial, as it shares its polynomial $x + 141x^2$ with at least three other non-isomorphic graphs (G_{10}, G_{20}, G_{21}).

This work leaves several natural open problems and directions for future research:

1. **Proving Conjecture 3.4:** The most immediate open problem is to provide a formal proof for the recurrence relation $a_n = 3a_{n-1} + a_{n-2} - 2a_{n-4}$ for the $dp(L_n, 2)$ coefficient of the ladder graph L_n. Our computational evidence supports this conjecture, but a theoretical proof is still required.
2. **Higher-Order Coefficients:** While this paper made significant progress on $dp(G, 2)$, the coefficients $dp(G, k)$ for $k \geq 3$ remain unknown for most families studied, such as wheel graphs (W_n) and complete bipartite graphs ($K_{m,n}$). Finding complete polynomial formulas for these graphs is a challenging next step.
3. **Domatic Uniqueness (DP-uniqueness):** Our finding that the Petersen graph is not "DP-unique" (i.e., $DP(G, x) = DP(H, x)$ does not imply $G \cong H$) motivates a new line of inquiry. Which graphs, if any, *are* uniquely determined by their domatic polynomials? Characterizing DP-unique graphs or finding other non-isomorphic "DP-cospectral" graphs would be a valuable contribution to the field.
4. **Other Graph Products:** The analysis of ladder graphs ($L_n = P_n \times K_2$) could be extended to other Cartesian products, such as grid graphs ($P_n \times P_m$) or torus graphs ($C_n \times C_m$), to explore how graph operations affect the domatic polynomial.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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