Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-8
https://doi.org/10.22049/cco.2026.31379.2838

CCO
Commun. Comb. Optim.

Short Note

A note on the Estrada index of signed graphs

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Received: 29 December 2025; Accepted: 7 January 2026
Published Online: 31 January 2026

Abstract: The Estrada index is a spectral invariant with wide applications in graph theory and network science, and has been studied for both unsigned and signed graphs. In this paper, we investigate its extremal behavior over all signatures of a fixed underlying graph. We prove that for any connected graph balanced signatures are the only ones that maximize the Estrada index.

Keywords: Signed graph, Estrada index, spectral moment, balanced signature, antibalanced signature.

AMS Subject classification: 05C50, 05C22, 15A18

1. Introduction

Signed graphs, introduced by Harary [12], provide a natural framework for studying graphs whose edges carry positive or negative signs. Formally, a signed graph Γ is a pair (G, σ), where G is a graph and $\sigma : E(G) \to \{-1, 1\}$ is a sign function. The adjacency matrix A_Γ is obtained from the adjacency matrix of G by incorporating the edge signs. Since A_Γ is real symmetric, all eigenvalues of Γ are real. A fundamental operation on signed graphs is *switching*, which reverses the signs of all edges across a cut. Switching preserves the spectrum and hence all spectral invariants. A cycle $C \subseteq G$ is said to be positive if the product of the signs of its edges satisfies $\prod_{e\in E(C)} \sigma(e) = 1$, or equivalently if C contains an even number of negative edges (if any). A signed graph is said to be *balanced* if it is switching equivalent to the all-positive signature $(G, +)$; otherwise, it is *unbalanced*. Equivalently, a signed graph is balanced precisely when every cycle is positive.
The Estrada index is a spectral invariant originally introduced for unsigned graphs [7]. For a signed graph Γ with eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$, its Estrada index is defined as

$$EE(\Gamma) = \sum_{i=1}^{n} e^{\mu_i}.$$

The Estrada index depends on the entire spectrum and admits a closed-walk interpretation via the matrix exponential. It has been widely studied in network analysis and mathematical chemistry, where it is often used as a global measure of structural complexity and connectivity in graphs and networks.

For unsigned graphs, the Estrada index has been investigated extensively, and a variety of extremal problems have been considered across different graph families, including trees, cycles, and unicyclic graphs; see, for example, Deng [5], Gutman and Graovac [11], Du and Zhou [6], and de la Peña et al. [4]. Numerous bounds and inequalities have also been established, highlighting the close connection between the Estrada index, spectral moments, and closed walks; see the survey by Estrada [9] for a comprehensive overview.

In the setting of signed graphs, spectral properties are closely related to balance and switching equivalence. Walk-based spectral measures of balance have been explored in applications to social and biological networks; see, for instance, [8, 10]. More recently, the Estrada index itself has been studied for signed graphs under fixed signatures or within specific graph families [13]. At the same time, the spectral theory of signed graphs has developed rapidly; see the survey and problem list of Belardo et al. [1]. A particularly active direction concerns *extremal spectral quantities* in signed graphs, where one seeks to maximize or minimize eigenvalue-based invariants under structural or sign constraints. For example, Brunetti and Stanić [2, 3] characterize unbalanced signed graphs with extremal spectral radius or index within natural classes, while several Turán-type extremal problems for signed graphs have also been investigated via the adjacency spectrum; see, e.g., Wang, Hou and Li [14] and related works.

In this context, the present paper addresses an extremal problem for the Estrada index taken over *all signatures of a fixed underlying graph*. We prove that for any connected graph G, the Estrada index over all signatures of G is maximized by balanced signatures. We also describe the extremal structure in the bipartite case, where balanced and antibalanced signatures attain the same extremal value, while all other signatures yield strictly smaller Estrada indices. These results complement existing work on extremal spectral invariants of signed graphs by focusing on a matrix-exponential-based measure and by allowing the signature to vary freely.

The remainder of the paper is organized as follows. Section 2 establishes preliminary results on spectral moments and closed walks in signed graphs. Section 3 contains the main extremal results for the Estrada index, including the characterization of maximizing signatures and the bipartite case. Section 4 presents illustrative examples. These are consistent with the theoretical results. In all numerical examples, the Estrada index is computed directly from the eigenvalues of the signed adjacency matrix using its defining formula $EE(G, \sigma) = \sum_i e^{\mu_i(G,\sigma)}$.

2. Preliminaries

We fix notation and recall basic definitions.

Throughout the paper, $\sigma : E(G) \to \{\pm 1\}$ denotes a signature of a simple graph

$G = (V, E)$ on n vertices, and A_σ denotes the corresponding signed adjacency matrix. Since A_σ is real symmetric, all its eigenvalues are real; we denote them by $\mu_1(G,\sigma), \mu_2(G,\sigma), \dots, \mu_n(G,\sigma)$.
For $k \geq 0$, the k-th spectral moment of (G,σ) is

$$M_k(G,\sigma) = \operatorname{tr}(A_\sigma^k) = \sum_{i=1}^{n} \mu_i(G,\sigma)^k. \tag{2.1}$$

The Estrada index admits the series representation

$$EE(G,\sigma) = \sum_{i=1}^{n} e^{\mu_i(G,\sigma)} = \sum_{k=0}^{\infty} \frac{M_k(G,\sigma)}{k!}. \tag{2.2}$$

Note that $M_0(G,\sigma) = n = W_0(G)$.
Each closed walk of length k in G contributes the product of the signs of its edges to $M_k(G,\sigma)$. We call a closed walk *positive* (resp. *negative*) if this product equals $+1$ (resp. -1). Writing $W_k^+(G,\sigma)$ and $W_k^-(G,\sigma)$ for the numbers of positive and negative closed walks of length k, respectively, we have

$$M_k(G,\sigma) = W_k^+(G,\sigma) - W_k^-(G,\sigma), \tag{2.3}$$

and the total number of closed walks of length k in the underlying graph G is

$$W_k(G) = W_k^+(G,\sigma) + W_k^-(G,\sigma), \tag{2.4}$$

which is independent of the signature. (If G is bipartite then $W_k(G) = 0$ for every odd k.)

Switching. For a function $\tau : V \to \{\pm 1\}$ define the switched signature σ^τ by

$$\sigma^\tau(uv) = \tau(u)\,\sigma(uv)\,\tau(v) \qquad (uv \in E).$$

Equivalently, writing $D_\tau = \operatorname{diag}(\tau(v) : v \in V)$, the signed adjacency matrices satisfy

$$A_{\sigma^\tau} = D_\tau A_\sigma D_\tau.$$

Thus switching acts by conjugation. Since conjugation by a diagonal matrix with entries ± 1 preserves traces and eigenvalues, switching preserves every spectral moment and hence the Estrada index:

$$M_k(G,\sigma^\tau) = M_k(G,\sigma) \quad \text{for all } k, \qquad EE(G,\sigma^\tau) = EE(G,\sigma).$$

Two signatures are called *switching equivalent* if one is obtained from the other by switching.

Balanced and antibalanced signatures. A signed graph (G,σ) is balanced if it is switching equivalent to the all-positive signature $(G,+)$. It can be proved (see for instance [15]) that (G,σ) is balanced if and only if G does not contain negative cycles. A signature σ is antibalanced if $-\sigma$ is balanced. It is straightforward to verify that σ is antibalanced if it is switching equivalent to the all-negative signature $(G,-)$.

3. Results

In this section, we establish the main extremal results of the paper. We first relate spectral moments of signed graphs to signed closed walks, which allows us to compare Estrada indices across different signatures. Using these moment inequalities, we then prove that balanced signatures maximize the Estrada index of any connected graph, and we characterize the extremal structure in the bipartite case.

Lemma 1. *Let (G, σ) be a signed graph. Then, for all $k \geq 0$,*

$$M_k(G, \sigma) \leq W_k(G),$$

with equality if and only if $W_k^-(G, \sigma) = 0$.

Proof. Each closed walk of length k contributes the product of the signs of its edges to the diagonal entries of A_σ^k, and summing over vertices yields

$$M_k(G, \sigma) = \mathrm{tr}(A_\sigma^k) = W_k^+(G, \sigma) - W_k^-(G, \sigma).$$

Since $W_k^\pm(G, \sigma) \geq 0$ and $W_k(G) = W_k^+(G, \sigma) + W_k^-(G, \sigma)$, it follows that

$$M_k(G, \sigma) = W_k^+(G, \sigma) - W_k^-(G, \sigma) \leq W_k^+(G, \sigma) + W_k^-(G, \sigma) = W_k(G),$$

and equality holds precisely when $W_k^-(G, \sigma) = 0$. □

Lemma 2. *Let G be a graph and let $\sigma : E(G) \to \{\pm 1\}$ be a signature. If σ is unbalanced, then there exists an integer $k \geq 1$ such that*

$$M_k(G, \sigma) < W_k(G).$$

Proof. Since σ is unbalanced, there exists a cycle C in G whose sign is negative. Let $\ell = |C|$ be the length of this cycle. Traversing C once yields a negative closed walk of length ℓ, and hence

$$W_\ell^-(G, \sigma) \geq 1.$$

By the closed-walk interpretation of spectral moments,

$$M_\ell(G, \sigma) = W_\ell^+(G, \sigma) - W_\ell^-(G, \sigma).$$

Since $W_\ell(G) = W_\ell^+(G, \sigma) + W_\ell^-(G, \sigma)$, it follows that

$$M_\ell(G, \sigma) < W_\ell^+(G, \sigma) + W_\ell^-(G, \sigma) = W_\ell(G).$$

Thus, taking $k = \ell$ yields the desired inequality. □

Theorem 1. *Let G be a connected graph. Among all signatures $\sigma : E(G) \to \{\pm 1\}$ of G, the Estrada index $EE(G,\sigma)$ is maximized by a balanced signature. Moreover, equality holds if and only if σ is switching equivalent to the all-positive signature $(G,+)$.*

Proof. By Lemma 1, for any signature σ and for all $k \geq 0$,

$$M_k(G,\sigma) \leq W_k(G),$$

where $W_k(G)$ denotes the total number of closed walks of length k in G. Consequently,

$$EE(G,\sigma) = \sum_{k=0}^{\infty} \frac{M_k(G,\sigma)}{k!} \leq \sum_{k=0}^{\infty} \frac{W_k(G)}{k!}.$$

By the characterization of balanced signatures, $M_k(G,\sigma) = W_k(G)$ for all $k \geq 0$ holds if and only if σ is balanced, that is, switching equivalent to $(G,+)$. In this case, $EE(G,\sigma) = \sum_{k=0}^{\infty} \frac{W_k(G)}{k!} = EE(G,+)$. □

Corollary 1. *Let G be a connected bipartite graph. Then*

$$EE(G,+) = EE(G,-).$$

Moreover, among all signatures $\sigma : E(G) \to \{\pm 1\}$, the Estrada index $EE(G,\sigma)$ is maximized precisely by balanced and antibalanced signatures. If σ is neither balanced nor antibalanced, then

$$EE(G,\sigma) < EE(G,+) = EE(G,-).$$

Proof. Let G be a bipartite graph. As observed in [1, Section 3.1], every signature σ on G is switching equivalent to $-\sigma$. Consequently, the signed graphs $(G,+)$ and $(G,-)$ have the same spectrum. The statement now follows directly from Theorem 1. □

4. Illustrative examples

We illustrate the extremal behavior of the Estrada index through a series of explicit examples that are consistent with the theoretical results proved in Section 3.

A bipartite graph.

Consider the cycle graph C_4. Under the all-positive signature $(C_4,+)$, the eigenvalues of the adjacency matrix are $\{2,0,0,-2\}$, and hence

$$EE(C_4,+) = e^2 + 2 + e^{-2}.$$

Now assign a negative sign to exactly one edge of C_4. The resulting signature is unbalanced, and the corresponding eigenvalues are $\{\sqrt{2}, \sqrt{2}, -\sqrt{2}, -\sqrt{2}\}$, yielding

$$EE(C_4, \sigma) = 2e^{\sqrt{2}} + 2e^{-\sqrt{2}}.$$

Since $e^2 + e^{-2} > 2e^{\sqrt{2}} + 2e^{-\sqrt{2}}$, this example is consistent with what we expect after Theorem 1 and Corollary 1.

A non-bipartite graph.

To illustrate the behavior in the non-bipartite setting, consider the triangle graph K_3. Under the all-positive signature, the eigenvalues are $\{2, -1, -1\}$, yielding

$$EE(K_3, +) = e^2 + 2e^{-1}.$$

If one edge is assigned a negative sign, the resulting signature is unbalanced and produces eigenvalues $\{1, 1, -2\}$, giving

$$EE(K_3, \sigma) = 2e^1 + e^{-2}.$$

As expected, $EE(K_3, +) > EE(K_3, \sigma) = EE(K_3, -)$.

A larger bipartite graph.

Consider the complete bipartite graph $K_{3,3}$. Under the all-positive signature, its spectrum is $\{3, -3, 0, 0, 0, 0\}$, and hence

$$EE(K_{3,3}, +) = e^3 + e^{-3} + 4.$$

Since $K_{3,3}$ is bipartite, the all-negative signature produces the same spectrum and the same Estrada index.
Up to switching equivalence, unbalanced signatures of $K_{3,3}$ give rise to two distinct adjacency spectra, which can be obtained by direct computation. These spectra are

$$S_1 = \{2^{(2)}, 1, -1, -2^{(2)}\}$$

and

$$S_2 = \left\{ -\sqrt{\frac{9+\sqrt{17}}{2}}, -\sqrt{\frac{9-\sqrt{17}}{2}}, 0^{(2)}, \sqrt{\frac{9-\sqrt{17}}{2}}, \sqrt{\frac{9+\sqrt{17}}{2}} \right\}.$$

For the spectrum S_1, the Estrada index is given by

$$EE(S_1) = 2e^2 + e^1 + e^{-1} + 2e^{-2} \approx 18.1349.$$

For the spectrum S_2, we obtain

$$EE(S_2) = e^{\sqrt{\frac{9+\sqrt{17}}{2}}} + e^{-\sqrt{\frac{9+\sqrt{17}}{2}}} + e^{\sqrt{\frac{9-\sqrt{17}}{2}}} + e^{-\sqrt{\frac{9-\sqrt{17}}{2}}} + 2 \approx 20.0102.$$

In this case, the smallest Estrada index corresponds to the signatures with the smallest index.
We conclude with the following conjecture.

Conjecture. Let G be a connected graph. Among all signatures $\sigma : E(G) \to \{\pm 1\}$, the minimum Estrada index $EE(G, \sigma)$ is always achieved by a signature that gives rise to the smallest index.

Acknowledgements: The author gratefully acknowledges the support provided by JIS University through seed funding under grant number RFMO/SF/JISU/24-25/005.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] F. Belardo, S.M. Cioabă, J.H. Koolen, and Q. Wang, *Open problems in the spectral theory of signed graphs*, The Art of Discrete and Applied Mathematics **1** (2018), no. 2, Article P2.10.
https://doi.org/10.26493/2590-9770.1286.d7b.

[2] M. Brunetti and Z. Stanić, *Ordering signed graphs with large index*, Ars Math. Contemp. **22** (2022), no. 4, P4.01.
https://doi.org/10.26493/1855–3974.2714.9b3.

[3] ———, *Unbalanced signed graphs with extremal spectral radius or index*, Comput. Appl. Math. **41** (2022), no. 3, 118.
https://doi.org/10.1007/s40314-022-01814-5.

[4] J. A. De La Peña, I. Gutman, and J. Rada, *Estimating the Estrada index*, Linear Algebra. Appl. **427** (2007), no. 1, 70–76.
https://doi.org/10.1016/j.laa.2007.06.020.

[5] H. Deng, *A proof of a conjecture on the Estrada index*, MATCH Commun. Math. Comput. Chem. **62** (2009), no. 3, 599–606.

[6] Z. Du and B. Zhou, *The Estrada index of unicyclic graphs.*, Linear Algebra. Appl. **436** (2012), no. 9, 3149–3159
https://doi.org/10.1016/j.laa.2011.10.020.

[7] E. Estrada, *Characterization of 3D molecular structure*, Chemical Physics Lett. **319** (2000), no. (5-6), 713–718.
https://doi.org/10.1016/S0009-2614(00)00158-5.

[8] ———, *Rethinking structural balance in signed social networks*, Discrete Appl. Math. **268** (2019), 70–90.
https://doi.org/10.1016/j.dam.2019.04.019.

[9] ———, *The many facets of the Estrada indices of graphs and networks*, SeMA J. **79** (2022), no. 1, 57–125.
https://doi.org/10.1007/s40324-021-00275–w.

[10] E. Estrada and M. Benzi, *A walk-based measure of balance in signed networks: Detecting lack of balance in social networks*, Physical Review E **90** (2014), no. 4, 042802.
https://doi.org/10.1103/PhysRevE.90.042802.

[11] I. Gutman and A. Graovac, *Estrada index of cycles and paths*, Chemical Physics Lett. **436** (2007), no. 1-3, 294–296.
https://doi.org/10.1016/j.cplett.2007.01.044.

[12] F. Harary, *On the notion of balance of a signed graph.*, Michigan Math. J. **2** (1953), no. 2, 143–146.
https://doi.org/10.1307/mmj/1028989917.

[13] T. Shamsher, S. Pirzada, and M.A. Bhat, *On the Estrada index of signed graphs*, Filomat **38** (2024), no. 19, 6851–6861.
https://doi.org/10.2298/FIL2419851S.

[14] D. Wang, Y. Hou, and D. Li, *Extremal results for C_3^--free signed graphs*, Linear Algebra. Appl. **681** (2024), 47–65.
https://doi.org/10.1016/j.laa.2023.10.024.

[15] T. Zaslavsky, *Signed graphs*, Discrete Appl. Math. **4** (1982), no. 1, 47–74.
https://doi.org/10.1016/0166-218X(82)90033-6.