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Research Article

On generalized commutative Leonardo quaternions and their generalization

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Abstract: In this paper, we give some properties of the generalized commutative Leonardo quaternions, among others the Binet formula, generating function, and the general bilinear index-reduction formula which imply d'Ocagne, Vajda, Halton, Catalan, and Cassini identities. We also give the matrix representations and some sum formulas of the generalized commutative Leonardo quaternions. Moreover, we present a one-parameter generalization of the generalized commutative Leonardo quaternions and their properties.

Keywords: generalized commutative Leonardo quaternions, generalized Fibonacci-Leonardo numbers, Leonardo numbers, recurrence relations.

AMS Subject classification: 11B37, 11B39

1. Introduction

Let $n \geq 0$ be an integer. The nth Leonardo number Le_n is defined recursively by

$$Le_n = Le_{n-1} + Le_{n-2} + 1, \text{ for } n \geq 2$$

with $Le_0 = 1,\ Le_1 = 1$. This sequence is the sequence of the on-line encyclopedia of integer sequences (https://oeis.org/A001595). The first ten Leonardo numbers are $1, 1, 3, 5, 9, 15, 25, 41, 67, 109$. The sequence of Leonardo numbers can also be defined

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recursively by the formula $Le_n = 2Le_{n-1} - Le_{n-3}$, for $n \geq 3$ with $Le_0 = 1$, $Le_1 = 1$ and $Le_2 = 3$.
The Fibonacci numbers are defined recursively by $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ with the initial terms $F_0 = 0$, $F_1 = 1$. It is clear that $Le_n = 2F_{n+1} - 1$.
Other properties of Leonardo numbers can be found in [6].
In the literature, we can find some generalizations of Leonardo numbers. Generalized Leonardo numbers were defined in [16]. In [7] (available online at: https://www.cs.utexas.edu/users/EWD/transcriptions/EWD07xx/EWD797.html), the author extended and generalized the Leonardo numbers with two sequences. Another generalization of Leonardo numbers are Leonardo polynomials, presented in [18]. The generalized Horadam-Leonardo polynomials were examined in [21].
The generalized Fibonacci-Leonardo numbers $Le(t, n)$, introduced in [1], are given by the recurrence relation

$$Le(t, n) = Le(t, n-1) + Le(t, n-2) + (t-1) \text{ for } n \geq 2 \text{ and } t \geq 1,$$

with initial terms $Le(t, 0) = Le(t, 1) = 1$. As you can see, for $t = 1$ we obtain $Le(1, n) = F_{n+1}$ and for $t = 2$, we have $Le(2, n) = Le_n$. Applications of Leonardo sequence can be found in the theory of hypercomplex numbers. The term "hypercomplex number" is well-known in mathematics. Real numbers can be multiplied by imaginary units to define a variety of number systems. A traditional illustration of such a system is the complex number system. Extensions of complex numbers are called hypercomplex numbers and include quaternions, tessarines, coquaternions, octonions, biquaternions, hybrid numbers, sedenions, etc.
Numerous researches examined Leonardo numbers in relation to quaternions, dual numbers, hybrid numbers, complex numbers, and other concepts. In [14], complex Leonardo numbers were presented and examined. Dual Leonardo numbers and bicomplex Leonardo numbers were studied in [15] and [23], respectively. The Leonardo quaternions were introduced in [2]. The new type of quaternions, called Leonard Gaussian quaternions, was investigated in [9].
The quaternions can also be represented using Pauli matrices. The quaternions formed with the Pauli matrices are called Pauli-quaternions. The Pauli-Leonardo quaternions, i.e. Pauli quaternions with Leonardo coefficients, were defined in [13].
The generalized Leonardo numbers defined in [16] were used to construct the generalized Leonardo quaternions presented in [17]. Information about hyperbolic generalized k-Horadam quaternions we can find in [19].
Complex numbers, hyperbolic numbers, dual numbers and hybrid numbers arise in many areas such as velocity analysis, coordinate transformation, displacement analysis, matrix modeling, rigid body dynamics, static analysis, mechanics, dynamic analysis, transformation, kinematics, physics, biology, mathematics, and geometry [8, 11, 12].
Real quaternions are defined in the following general form

$$\mathbf{h} = h_0 + h_1 i + h_2 j + h_3 k,$$

where $i, j, k \notin \mathbb{R}$ are quaternion units and h_0, h_1, h_2, h_3 are real numbers. Moreover,

$$i^2 = j^2 = k^2 = ijk = -1,$$

$$ij = -ji = k, \quad ki = -ik = j, \quad jk = -kj = i.$$

The quaternions naturally belong to one of the two classes: the commutative quaternions (generalized Segre, dual quaternions, etc.) and the non-commutative quaternions (real, split, semi, split-semi, quasi, hyperbolic, elliptical, hyperbolic split, two-parameter generalized, three-parameter generalized quaternions, etc.). The units of a generalized non-commutative quaternion (two-parameter) satisfy the following equalities:

$$i^2 = -\alpha, \quad j^2 = -\beta, \quad k^2 = -\alpha \cdot \beta,$$

$$ij = -ji = k, \quad ki = -ik = \alpha \cdot j,$$

$$jk = -kj = \beta \cdot i, \quad ijk = -\alpha \cdot \beta,$$

where α and β are real numbers. The two-parameter generalized non-commutative quaternions are analogous to real quaternions for $\alpha = \beta = 1$, to split quaternions for $\alpha = 1$, $\beta = -1$, to semi quaternions for $\alpha = 1$, $\beta = 0$, to split-semi quaternions for $\alpha = -1$, $\beta = 0$, and to quasi quaternions for $\alpha = \beta = 0$. The units of a generalized non-commutative quaternion (three-parameter) satisfy the following equalities:

$$i^2 = -a_1 a_2, \quad j^2 = -a_1 a_3, \quad k^2 = -a_2 a_3,$$

$$ij = -ji = a_1 \cdot k, \quad ki = -ik = a_2 \cdot j,$$

$$jk = -kj = a_3 \cdot i, \quad ijk = -a_1 a_2 a_3,$$

where a_1, a_2 and a_3 are real numbers. If we take conditions $a_1 = 1$, $a_2 = \alpha$ and $a_3 = \beta$ on the three-parameter generalized non-commutative quaternions, we have the two-parameter generalized non-commutative quaternions.
A generalized commutative quaternion $\mathbf{x}$ is a vector of the form

$$\mathbf{x} = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3,$$

where quaternionic units e_1, e_2, e_3 satisfy the equalities

$$\begin{aligned} e_1^2 = \alpha, \quad e_2^2 = \beta, \quad e_3^2 = \alpha \cdot \beta, \\ e_1 e_2 = e_2 e_1 = e_3, \quad e_3 e_1 = e_1 e_3 = \alpha \cdot e_2, \\ e_2 e_3 = e_3 e_2 = \beta \cdot e_1, \quad e_1 e_2 e_3 = \alpha \cdot \beta, \end{aligned} \tag{1.1}$$

where α and β are real numbers. For special α and β there are following subclasses of generalized commutative quaternions: for $\alpha < 0$, $\beta = 1$ elliptic quaternions, for $\alpha = 0$,

$\beta = 1$ parabolic quaternions, for $\alpha > 0$, $\beta = 1$ hyperbolic quaternions, for $\alpha = -1$, $\beta = 1$ complex hyperbolic (tessarines) numbers, for $\alpha = -1$, $\beta = -1$ bicomplex numbers, and for $\alpha = 1$, $\beta = -1$ hyperbolic complex numbers.
In [22], the authors first constructed generalized commutative quaternions, and then they defined and investigated generalized commutative quaternions of the Fibonacci type.
In [4], generalized commutative Pell and generalized commutative Pell-Lucas quaternions were examined. The authors discussed some of these numbers' properties as well as their relationships. There were provided some examples of matrix generators for generalized commutative Jacobsthal quaternions in [5] and other properties in [3].
In this paper, we define generalized commutative Leonardo quaternions and present some of their properties.

2. Generalized commutative Leonardo quaternions

Let $n \geq 0$ be an integer. The nth generalized commutative Leonardo quaternion $gcLe_n$ is defined as

$$gcLe_n = Le_n + Le_{n+1}e_1 + Le_{n+2}e_2 + Le_{n+3}e_3, \tag{2.1}$$

where Le_n is the nth Leonardo number and e_1, e_2, e_3 are units which satisfy (1.1), where α, $\beta \in \mathbb{R}$.
The generalized commutative Leonardo quaternions starting from $n = 0$ can be written as

$$\begin{aligned} gcLe_0 &= 1 + e_1 + 3e_2 + 5e_3, \\ gcLe_1 &= 1 + 3e_1 + 5e_2 + 9e_3, \\ gcLe_2 &= 3 + 5e_1 + 9e_2 + 15e_3, \end{aligned}$$

and

$$gcLe_n = gcLe_{n-1} + gcLe_{n-2} + \hat{\mathbf{1}}, \text{ for } n \geq 2$$

where $\hat{\mathbf{1}} = 1 + e_1 + e_2 + e_3$ or

$$gcLe_n = 2gcLe_{n-1} - gcLe_{n-3}, \text{ for } n \geq 3.$$

Similarly to any sequence defined by the recurrence relations, we can also define generalized commutative Leonardo quaternions with negative indices. Let $n \geq 0$ be an integer. The generalized commutative Leonardo quaternion $gcLe_{-n}$ is defined by

$$gcLe_{-n} = gcLe_{-n+2} - gcLe_{-n+1} - \hat{\mathbf{1}},$$

or

$$gcLe_{-n} = 2gcLe_{-n+2} - gcLe_{-n+3}.$$

Using Leonardo numbers, we can also write $gcLe_{-n}$ as

$$\begin{aligned} gcLe_{-n} = (-1)^n \, [& (Le_{n-2}+1) - (Le_{n-3}+1)e_1 \\ & +(Le_{n-4}+1)e_2 - (Le_{n-5}+1)e_3] - \hat{\mathbf{1}}. \end{aligned}$$

Lemma 1. [6] *(Binet formula for Leonardo numbers) For $n \geq 0$*

$$Le_n = 2\left(\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta}\right) - 1 = \frac{\gamma(2\gamma^n - 1) - \delta(2\delta^n - 1)}{\gamma-\delta},$$

where

$$\gamma = \frac{1+\sqrt{5}}{2}, \quad \delta = \frac{1-\sqrt{5}}{2}. \tag{2.2}$$

Theorem 1. *(Binet formula for generalized commutative Leonardo quaternions) Let $n \geq 0$ be an integer. Then*

$$gcLe_n = \frac{2\gamma^{n+1}}{\gamma-\delta}\hat{\gamma} - \frac{2\delta^{n+1}}{\gamma-\delta}\hat{\delta} - \hat{\mathbf{1}}, \tag{2.3}$$

where γ, δ are given by (2.2) and

$$\hat{\gamma} = 1+\gamma e_1+\gamma^2 e_2+\gamma^3 e_3, \;\; \hat{\delta} = 1+\delta e_1+\delta^2 e_2+\delta^3 e_3, \;\; \hat{\mathbf{1}} = 1+e_1+e_2+e_3. \tag{2.4}$$

Proof. By Lemma 1, we get

$$\begin{aligned} gcLe_n &= \left(2\left(\frac{\gamma^{n+1}-\delta^{n+1}}{\gamma-\delta}\right)-1\right) + \left(2\left(\frac{\gamma^{n+2}-\delta^{n+2}}{\gamma-\delta}\right)-1\right)e_1 \\ &\quad + \left(2\left(\frac{\gamma^{n+3}-\delta^{n+3}}{\gamma-\delta}\right)-1\right)e_2 + \left(2\left(\frac{\gamma^{n+4}-\delta^{n+4}}{\gamma-\delta}\right)-1\right)e_3 \\ &= \frac{2\gamma^{n+1}}{\gamma-\delta}\left(1+\gamma e_1+\gamma^2 e_2+\gamma^3 e_3\right) - \frac{2\delta^{n+1}}{\gamma-\delta}\left(1+\delta e_1+\delta^2 e_2+\delta^3 e_3\right) \\ &\quad - \left(1+e_1+e_2+e_3\right), \end{aligned}$$

which ends the proof. □

Using (2.3), we can prove the following theorem.

Theorem 2. *(General bilinear index-reduction formula for generalized commutative Leonardo quaternions) Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b=c+d$. Then*

$$\begin{aligned}
&gcLe_a \cdot gcLe_b - gcLe_c \cdot gcLe_d \\
&= \frac{4}{5}\left(\gamma^a\delta^b + \delta^a\gamma^b - \gamma^c\delta^d - \delta^c\gamma^d\right)\hat{\gamma}\hat{\delta} \\
&\quad + \frac{\sqrt{5}+5}{5}\left(\gamma^c + \gamma^d - \gamma^a - \gamma^b\right)\hat{\gamma}\hat{\mathbf{1}} \\
&\quad + \frac{\sqrt{5}-5}{5}\left(\delta^a + \delta^b - \delta^c - \delta^d\right)\hat{\delta}\hat{\mathbf{1}},
\end{aligned}$$

where γ, δ and $\hat{\gamma}$, $\hat{\delta}$, $\hat{\mathbf{1}}$ are given by (2.2) and (2.4), respectively.

Proof. By (2.3), we have

$$\begin{aligned}
&gcLe_a \cdot gcLe_b - gcLe_c \cdot gcLe_d \\
&= \left(\frac{2\gamma^{a+1}}{\gamma-\delta}\hat{\gamma} - \frac{2\delta^{a+1}}{\gamma-\delta}\hat{\delta} - \hat{\mathbf{1}}\right)\cdot\left(\frac{2\gamma^{b+1}}{\gamma-\delta}\hat{\gamma} - \frac{2\delta^{b+1}}{\gamma-\delta}\hat{\delta} - \hat{\mathbf{1}}\right) \\
&\quad - \left(\frac{2\gamma^{c+1}}{\gamma-\delta}\hat{\gamma} - \frac{2\delta^{c+1}}{\gamma-\delta}\hat{\delta} - \hat{\mathbf{1}}\right)\cdot\left(\frac{2\gamma^{d+1}}{\gamma-\delta}\hat{\gamma} - \frac{2\delta^{d+1}}{\gamma-\delta}\hat{\delta} - \hat{\mathbf{1}}\right) \\
&= \frac{-4\gamma^{a+1}\delta^{b+1} - 4\delta^{a+1}\gamma^{b+1} + 4\gamma^{c+1}\delta^{d+1} + 4\delta^{c+1}\gamma^{d+1}}{(\gamma-\delta)^2}\hat{\gamma}\hat{\delta} \\
&\quad + \frac{-2\gamma^{a+1} - 2\gamma^{b+1} + 2\gamma^{c+1} + 2\gamma^{d+1}}{\gamma-\delta}\hat{\gamma}\hat{\mathbf{1}} \\
&\quad + \frac{2\delta^{a+1} + 2\delta^{b+1} - 2\delta^{c+1} - 2\delta^{d+1}}{\gamma-\delta}\hat{\delta}\hat{\mathbf{1}} \\
&= \frac{4}{5}\left(\gamma^a\delta^b + \delta^a\gamma^b - \gamma^c\delta^d - \delta^c\gamma^d\right)\hat{\gamma}\hat{\delta} \\
&\quad + \frac{2}{\sqrt{5}}\left(-\gamma^{a+1} - \gamma^{b+1} + \gamma^{c+1} + \gamma^{d+1}\right)\hat{\gamma}\hat{\mathbf{1}} \\
&\quad + \frac{2}{\sqrt{5}}\left(\delta^{a+1} + \delta^{b+1} - \delta^{c+1} - \delta^{d+1}\right)\hat{\delta}\hat{\mathbf{1}} \\
&= \frac{4}{5}\left(\gamma^a\delta^b + \delta^a\gamma^b - \gamma^c\delta^d - \delta^c\gamma^d\right)\hat{\gamma}\hat{\delta} \\
&\quad + \frac{2\gamma}{\sqrt{5}}\left(\gamma^c + \gamma^d - \gamma^a - \gamma^b\right)\hat{\gamma}\hat{\mathbf{1}} \\
&\quad + \frac{2\delta}{\sqrt{5}}\left(\delta^a + \delta^b - \delta^c - \delta^d\right)\hat{\delta}\hat{\mathbf{1}}
\end{aligned}$$

since $a+b=c+d$, $\gamma\cdot\delta=-1$ and $\gamma-\delta=\sqrt{5}$. □

For special values of a, b, c, d, by Theorem 2, we can obtain some identities for generalized commutative Leonardo quaternions:

- first Halton type identity – for $a = m + r$, $b = n$, $c = r$, $d = m + n$,
- second Halton type identity – for $a = n + k$, $b = n - k$, $c = n + s$, $d = n - s$,
- Vajda type identity – for $a = m + r$, $b = n - r$, $c = m$, $d = n$,
- d'Ocagne type identity – for $a = n$, $b = m + 1$, $c = n + 1$, $d = m$,
- Catalan type identity – for $a = n + r$, $b = n - r$, $c = d = n$,
- Cassini type identity – for $a = n + 1$, $b = n - 1$, $c = d = n$.

In a similar way, using Binet formula, we can prove two additional identities: the Ruggles type identity and the Honsberger type identity.

Theorem 3. *(Ruggles type identity for generalized commutative Leonardo quaternions) Let $n \geq m$, $m \geq 1$ be integers. Then*

$$\begin{aligned}
&gcLe_{m-1} \cdot gcLe_{n-m} + gcLe_m \cdot gcLe_{n-m+1} \\
&= \frac{4}{5}\left(\gamma^{n+1} + \gamma^{n+3}\right)\hat{\gamma}^2 + \frac{4}{5}\left(\delta^{n+1} + \delta^{n+3}\right)\hat{\delta}^2 \\
&\quad - \frac{2}{\sqrt{5}}(1+\gamma)\left(\gamma^m + \gamma^{n-m+1}\right)\hat{\gamma}\hat{\mathbf{1}} + \frac{2}{\sqrt{5}}(1+\delta)\left(\delta^m + \delta^{n-m+1}\right)\hat{\delta}\hat{\mathbf{1}} + 2(\hat{\mathbf{1}})^2,
\end{aligned}$$

where γ, δ and $\hat{\gamma}$, $\hat{\delta}$, $\hat{\mathbf{1}}$ are given by (2.2) and (2.4), respectively.

Proof. By (2.3), we have

$$\begin{aligned}
&gcLe_{m-1} \cdot gcLe_{n-m} + gcLe_m \cdot gcLe_{n-m+1} \\
&= \left(\frac{2\gamma^m}{\gamma-\delta}\hat{\gamma} - \frac{2\delta^m}{\gamma-\delta}\hat{\delta} - \hat{\mathbf{1}}\right) \cdot \left(\frac{2\gamma^{n-m+1}}{\gamma-\delta}\hat{\gamma} - \frac{2\delta^{n-m+1}}{\gamma-\delta}\hat{\delta} - \hat{\mathbf{1}}\right) \\
&\quad + \left(\frac{2\gamma^{m+1}}{\gamma-\delta}\hat{\gamma} - \frac{2\delta^{m+1}}{\gamma-\delta}\hat{\delta} - \hat{\mathbf{1}}\right) \cdot \left(\frac{2\gamma^{n-m+2}}{\gamma-\delta}\hat{\gamma} - \frac{2\delta^{n-m+2}}{\gamma-\delta}\hat{\delta} - \hat{\mathbf{1}}\right) \\
&= \frac{4\gamma^{n+1}\hat{\gamma}^2 + 4\gamma^{n+3}\hat{\gamma}^2 + 4\delta^{n+1}\hat{\delta}^2 + 4\delta^{n+3}\hat{\delta}^2}{(\gamma-\delta)^2} \\
&\quad - \frac{4\gamma^m\delta^{n-m+1} + 4\gamma^{n-m+1}\delta^m + 4\gamma^{m+1}\delta^{n-m+2} + 4\gamma^{n-m+2}\delta^{m+1}}{(\gamma-\delta)^2}\hat{\gamma}\hat{\delta} \\
&\quad - \frac{2\gamma^m + 2\gamma^{n-m+1} + 2\gamma^{m+1} + 2\gamma^{n-m+2}}{\gamma-\delta}\hat{\gamma}\hat{\mathbf{1}} \\
&\quad + \frac{2\delta^m + 2\delta^{n-m+1} + 2\delta^{m+1} + 2\delta^{n-m+2}}{\gamma-\delta}\hat{\delta}\hat{\mathbf{1}} + 2(\hat{\mathbf{1}})^2 \\
&= \frac{4}{5}\left(\gamma^{n+1} + \gamma^{n+3}\right)\hat{\gamma}^2 + \frac{4}{5}\left(\delta^{n+1} + \delta^{n+3}\right)\hat{\delta}^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{4}{5}\left(\gamma^{m-1}\delta^{n-m}+\gamma^{n-m}\delta^{m-1}+\gamma^{m}\delta^{n-m+1}+\gamma^{n-m+1}\delta^{m}\right)\hat{\gamma}\hat{\delta}\\
&-\frac{2}{\sqrt{5}}\left(\gamma^{m}+\gamma^{n-m+1}+\gamma^{m+1}+\gamma^{n-m+2}\right)\hat{\gamma}\hat{\mathbf{1}}\\
&+\frac{2}{\sqrt{5}}\left(\delta^{m}+\delta^{n-m+1}+\delta^{m+1}+\delta^{n-m+2}\right)\hat{\delta}\hat{\mathbf{1}}+2(\hat{\mathbf{1}})^{2}\\
&=\frac{4}{5}\left(\gamma^{n+1}+\gamma^{n+3}\right)\hat{\gamma}^{2}+\frac{4}{5}\left(\delta^{n+1}+\delta^{n+3}\right)\hat{\delta}^{2}\\
&+\frac{4}{5}\left(\left(\frac{1}{\gamma}+\delta\right)\gamma^{m}\delta^{n-m}+\left(\frac{1}{\delta}+\gamma\right)\gamma^{n-m}\delta^{m}\right)\hat{\gamma}\hat{\delta}\\
&-\frac{2}{\sqrt{5}}\left((1+\gamma)\gamma^{m}+(1+\gamma)\gamma^{n-m+1}\right)\hat{\gamma}\hat{\mathbf{1}}\\
&+\frac{2}{\sqrt{5}}\left((1+\delta)\delta^{m}+(1+\delta)\delta^{n-m+1}\right)\hat{\delta}\hat{\mathbf{1}}+2(\hat{\mathbf{1}})^{2}
\end{aligned}
$$

since $\gamma\cdot\delta=-1$ and $\gamma-\delta=\sqrt{5}$. □

In the same way, we can prove the following theorem.

Theorem 4. *(Honsberger type identity for generalized commutative Leonardo quaternions) Let $n\geq 0$, $m\geq 1$ be integers. Then*

$$
\begin{aligned}
&gcLe_{m-1}\cdot gcLe_{n}+gcLe_{m}\cdot gcLe_{n+1}\\
&=\frac{4}{5}\left(\gamma^{m+n+1}+\gamma^{m+n+3}\right)\hat{\gamma}^{2}+\frac{4}{5}\left(\delta^{m+n+1}+\delta^{m+n+3}\right)\hat{\delta}^{2}\\
&-\frac{2}{\sqrt{5}}(1+\gamma)\left(\gamma^{m}+\gamma^{n+1}\right)\hat{\gamma}\hat{\mathbf{1}}+\frac{2}{\sqrt{5}}(1+\delta)\left(\delta^{m}+\delta^{n+1}\right)\hat{\delta}\hat{\mathbf{1}}+2(\hat{\mathbf{1}})^{2},
\end{aligned}
$$

where γ, δ and $\hat{\gamma}$, $\hat{\delta}$, $\hat{\mathbf{1}}$ are given by (2.2) and (2.4), respectively.

Theorem 5. *The generating function for generalized commutative Leonardo quaternions is*

$$
\sum_{n=0}^{\infty} gcLe_{n}x^{n}=\frac{\left(1-x+x^{2}\right)+\left(1+x-x^{2}\right)e_{1}+\left(3-x-x^{2}\right)e_{2}+\left(5-x-3x^{2}\right)e_{3}}{1-2x+x^{3}}.
$$

Proof. Let $h(x)$ be the generating function for the generalized commutative Leonardo quaternions as $\sum_{n=0}^{\infty} gcLe_{n}x^{n}$. We get the following equations

$$
2xh(x)=2\sum_{n=0}^{\infty} gcLe_{n}x^{n+1}
$$

and

$$
x^{3}h(x)=\sum_{n=0}^{\infty} gcLe_{n}x^{n+3}.
$$

After the needed calculations, the generating function for the generalized commutative Leonardo quaternions is obtained as

$$\sum_{n=0}^{\infty} gcLe_n x^n = \frac{(gcLe_2 - 2gcLe_1)\, x^2 + (gcLe_1 - 2gcLe_0)\, x + gcLe_0}{1 - 2x + x^3},$$

and consequently

$$\sum_{n=0}^{\infty} gcLe_n x^n = \frac{\left(1 - x + x^2\right) + \left(1 + x - x^2\right) e_1 + \left(3 - x - x^2\right) e_2 + \left(5 - x - 3x^2\right) e_3}{1 - 2x + x^3}.$$

□

Theorem 6. *Let $n > 0$ be an integer. The following equalities hold*

$$a) \begin{bmatrix} gcLe_{n+3} & gcLe_{n+2} & gcLe_{n+1} \\ gcLe_{n+2} & gcLe_{n+1} & gcLe_{n} \\ gcLe_{n+1} & gcLe_{n} & gcLe_{n-1} \end{bmatrix} = \begin{bmatrix} gcLe_3 & gcLe_2 & gcLe_1 \\ gcLe_2 & gcLe_1 & gcLe_0 \\ gcLe_1 & gcLe_0 & gcLe_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^n,$$

$$b) \begin{bmatrix} gcLe_{-n+3} & gcLe_{-n+2} & gcLe_{-n+1} \\ gcLe_{-n+2} & gcLe_{-n+1} & gcLe_{-n} \\ gcLe_{-n+1} & gcLe_{-n} & gcLe_{-n-1} \end{bmatrix} = \begin{bmatrix} gcLe_3 & gcLe_2 & gcLe_1 \\ gcLe_2 & gcLe_1 & gcLe_0 \\ gcLe_1 & gcLe_0 & gcLe_{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}^n.$$

Proof. a) For the proof, we use induction method on n. The equality holds for $n = 1$.

$$\begin{bmatrix} gcLe_3 & gcLe_2 & gcLe_1 \\ gcLe_2 & gcLe_1 & gcLe_0 \\ gcLe_1 & gcLe_0 & gcLe_{-1} \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2gcLe_3 - gcLe_1 & gcLe_3 & gcLe_2 \\ 2gcLe_2 - gcLe_0 & gcLe_2 & gcLe_1 \\ 2gcLe_1 - gcLe_{-1} & gcLe_1 & gcLe_0 \end{bmatrix}$$

$$= \begin{bmatrix} gcLe_4 & gcLe_3 & gcLe_2 \\ gcLe_3 & gcLe_2 & gcLe_1 \\ gcLe_2 & gcLe_1 & gcLe_0 \end{bmatrix}.$$

Now suppose that the equality is true for $n>1$. Then, we can verify for $n+1$ as follows

$$\begin{aligned}
&\begin{bmatrix} gcLe_3 & gcLe_2 & gcLe_1 \\ gcLe_2 & gcLe_1 & gcLe_0 \\ gcLe_1 & gcLe_0 & gcLe_{-1} \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^{n+1} \\
&= \begin{bmatrix} gcLe_3 & gchLe_2 & gcLe_1 \\ gcLe_2 & gcLe_1 & gcLe_0 \\ gcLe_1 & gcLe_0 & gcLe_{-1} \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}^{n}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} gcLe_{n+3} & gcLe_{n+2} & gcLe_{n+1} \\ gcLe_{n+2} & gcLe_{n+1} & gcLe_n \\ gcLe_{n+1} & gcLe_n & gcLe_{n-1} \end{bmatrix}\begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} gcLe_{n+4} & gcLe_{n+3} & gcLe_{n+2} \\ gcLe_{n+3} & gcLe_{n+2} & gcLe_{n+1} \\ gcLe_{n+2} & gcLe_{n+1} & gcLe_n \end{bmatrix}.
\end{aligned}$$

Thus, the theorem can be proved easily.
b) Similarly, the proof is seen by induction on n.

□

Lemma 2. [6] *Let Le_n be the nth Leonardo number. Then*

$$\sum_{m=0}^{n} Le_m = Le_{n+2} - (n+2),$$

$$\sum_{m=0}^{n} Le_{2m} = Le_{2n+1} - n,$$

$$\sum_{m=0}^{n} Le_{2m+1} = Le_{2n+2} - (n+2).$$

In the next theorem, we can give the sum of the finite, finite odd, and finite even terms of the generalized commutative Leonardo quaternions.

Theorem 7. *Let $gcLe_n$ be the nth generalized commutative Leonardo quaternion. Then*

$$\sum_{m=0}^{n} gcLe_m = gcLe_{n+2} - \left(n\cdot\hat{\mathbf{1}} + gcLe_2 - gcLe_0\right),$$

$$\sum_{m=0}^{n} gcLe_{2m} = gcLe_{2n+1} - \left(n\cdot\hat{\mathbf{1}} + gcLe_1 - gcLe_0\right),$$

$$\sum_{m=0}^{n} gcLe_{2m+1} = gcLe_{2n+2} - \left(n\cdot\hat{\mathbf{1}} + gcLe_2 - gcLe_1\right).$$

where $\hat{\mathbf{1}}$ is given by (2.4).

Proof. Using (2.1) and Lemma 2, we have

$$\begin{aligned}
\sum_{m=0}^{n} gcLe_m &= \sum_{m=0}^{n} (Le_m + Le_{m+1}e_1 + Le_{m+2}e_2 + Le_{m+3}e_3) \\
&= \sum_{m=0}^{n} Le_m + e_1 \sum_{m=0}^{n} Le_{m+1} + e_2 \sum_{m=0}^{n} Le_{m+2} + e_3 \sum_{m=0}^{n} Le_{m+3} \\
&= (Le_{n+2} - (n+2)) + (Le_{n+3} - (n+4))e_1 \\
&\quad + (Le_{n+4} - (n+6))e_2 + (Le_{n+5} - (n+10))e_3 \\
&= gcLe_{n+2} - (n(1 + e_1 + e_2 + e_3) + 2 + 4e_1 + 6e_2 + 10e_3) \\
&= gcLe_{n+2} - \left(n \cdot \hat{\mathbf{1}} + gcLe_2 - gcLe_0\right).
\end{aligned}$$

Other sum formulas are proven using the same method. □

3. One-parameter generalization of generalized commutative Leonardo quaternions

Let $n \geq 0$, $t \geq 1$ be integers. The nth generalized commutative Fibonacci-Leonardo quaternion $gcLe(t,n)$ is defined as

$$gcLe(t,n) = Le(t,n) + Le(t,n+1)e_1 + (t,n+2)e_2 + Le(t,n+3)e_3, \tag{3.1}$$

where $Le(t,n)$ is the nth generalized Fibonacci-Leonardo number and e_1, e_2, e_3 are units which satisfy (1.1), where α, $\beta \in \mathbb{R}$.

Lemma 3. [1] (*Binet formula for generalized Fibonacci–Leonardo numbers*) *Let $n \geq 0$, $t \geq 1$ be integers. Then*

$$Le(t,n) = t\frac{\gamma^{n+1} - \delta^{n+1}}{\gamma - \delta} - (t-1),$$

where γ, δ are given by (2.2).

Theorem 8. (*Binet formula for generalized commutative Fibonacci–Leonardo quaternions*) *Let $n \geq 0$, $t \geq 1$ be integers. Then*

$$gcLe(t,n) = \frac{t\gamma^{n+1}}{\gamma - \delta}\hat{\gamma} - \frac{t\delta^{n+1}}{\gamma - \delta}\hat{\delta} - (t-1)\hat{\mathbf{1}},$$

where γ, δ and $\hat{\gamma}$, $\hat{\delta}$, $\hat{\mathbf{1}}$ are given by (2.2) and (2.4), respectively.

Proof. The proof of Theorem 8 is analogous to the proof of Theorem 1. □

Using (3.1) and Theorem 8, we get general bilinear index-reduction formulas for generalized commutative Fibonacci–Leonardo quaternions and other identities (Catalan, Cassini, Ruggles, etc.). The proofs of Theorems 9-11 are analogous to the proofs of Theorems 2-3, so we omit them.

Theorem 9. *(General bilinear index-reduction formula for generalized commutative Fibonacci–Leonardo quaternions) Let $a \geq 0$, $b \geq 0$, $c \geq 0$, $d \geq 0$ be integers such that $a+b=c+d$. Then*

$$\begin{aligned} & gcLe(t,a)\cdot gcLe(t,b) - gcLe(t,c)\cdot gcLe(t,d) \\ &= \frac{t^2}{5}\left(\gamma^a\delta^b + \delta^a\gamma^b - \gamma^c\delta^d - \delta^c\gamma^d\right)\hat{\gamma}\hat{\delta} \\ &+ \frac{t(t-1)\gamma}{\sqrt{5}}\left(\gamma^c + \gamma^d - \gamma^a - \gamma^b\right)\hat{\gamma}\hat{\mathbf{1}} \\ &+ \frac{t(t-1)\delta}{\sqrt{5}}\left(\delta^a + \delta^b - \delta^c - \delta^d\right)\hat{\delta}\hat{\mathbf{1}}, \end{aligned}$$

where γ, δ and $\hat{\gamma}$, $\hat{\delta}$, $\hat{\mathbf{1}}$ are given by (2.2) and (2.4), respectively.

Theorem 10. *(Ruggles type identity for generalized commutative Fibonacci–Leonardo quaternions) Let $n \geq m$, $m \geq 1$ be integers. Then*

$$\begin{aligned} & gcLe(t,m-1)\cdot gcLe(t,n-m) + gcLe(t,m)\cdot gcLe(t,n-m+1) \\ &= \frac{t^2}{5}\left(\gamma^{n+1}+\gamma^{n+3}\right)\hat{\gamma}^2 + \frac{t^2}{5}\left(\delta^{n+1}+\delta^{n+3}\right)\hat{\delta}^2 \\ &- \frac{t(t-1)}{\sqrt{5}}(1+\gamma)\left(\gamma^m+\gamma^{n-m+1}\right)\hat{\gamma}\hat{\mathbf{1}} + \frac{t(t-1)}{\sqrt{5}}(1+\delta)\left(\delta^m+\delta^{n-m+1}\right)\hat{\delta}\hat{\mathbf{1}} + 2(\hat{\mathbf{1}})^2, \end{aligned}$$

where γ, δ and $\hat{\gamma}$, $\hat{\delta}$, $\hat{\mathbf{1}}$ are given by (2.2) and (2.4), respectively.

Theorem 11. *(Honsberger type identity for generalized commutative Fibonacci–Leonardo quaternions) Let $n \geq 0$, $m \geq 1$ be integers. Then*

$$\begin{aligned} & gcLe(t,m-1)\cdot gcLe(t,n) + gcLe(t,m)\cdot gcLe(t,n+1) \\ &= \frac{t^2}{5}\left(\gamma^{m+n+1}+\gamma^{m+n+3}\right)\hat{\gamma}^2 + \frac{t^2}{5}\left(\delta^{m+n+1}+\delta^{m+n+3}\right)\hat{\delta}^2 \\ &- \frac{t(t-1)}{\sqrt{5}}(1+\gamma)\left(\gamma^m+\gamma^{n+1}\right)\hat{\gamma}\hat{\mathbf{1}} + \frac{t(t-1)}{\sqrt{5}}(1+\delta)\left(\delta^m+\delta^{n+1}\right)\hat{\delta}\hat{\mathbf{1}} + 2(\hat{\mathbf{1}})^2, \end{aligned}$$

where γ, δ and $\hat{\gamma}$, $\hat{\delta}$, $\hat{\mathbf{1}}$ are given by (2.2) and (2.4), respectively.

Theorem 12. *The generating function for the generalized commutative Fibonacci-Leonardo quaternions is*

$$\begin{aligned}&\sum_{n=0}^{\infty} gcLe(t,n)x^n\\&=\frac{\left(1-x+tx^2-x^2\right)+\left(1-x+tx-x^2\right)e_1+\left(1+t-x-x^2\right)e_2}{1-2x+x^3}\\&+\frac{\left(1+2t-x-x^2-tx^2\right)e_3}{1-2x+x^3}.\end{aligned}$$

Proof. The proof of Theorem 12 is analogous to the proof of Theorem 5. □

Lemma 4. [1] *Let $Le(t,n)$ be the nth generalized Fibonacci-Leonardo number. Then*

$$\sum_{m=0}^{n} Le(t,m)=Le(t,n+2)-(t-1)n-t,$$

$$\sum_{m=0}^{n} Le(t,2m)=Le(t,2n+1)-(t-1)n,$$

$$\sum_{m=0}^{n} Le(t,2m+1)=Le(t,2n+2)-(t-1)n-t.$$

In the next theorem, we can give the sum of the finite, finite odd, and finite even terms of the generalized commutative Fibonacci-Leonardo quaternions.

Theorem 13. *Let $gcLe(t,n)$ be the nth generalized commutative Fibonacci-Leonardo quaternion. Then*

$$\sum_{m=0}^{n} gcLe(t,m)=gcLe(t,n+2)-\left(n(t-1)\cdot\hat{\mathbf{1}}+gcLe(t,2)-gcLe(t,0)\right),$$

$$\sum_{m=0}^{n} gcLe(t,2m)=gcLe(t,2n+1)-\left(n(t-1)\cdot\hat{\mathbf{1}}+gcLe(t,1)-gcLe(t,0)\right),$$

$$\sum_{m=0}^{n} gcLe(t,2m+1)=gcLe(t,2n+2)-\left(n(t-1)\cdot\hat{\mathbf{1}}+gcLe(t,2)-gcLe(t,1)\right).$$

Proof. Using (3.1) and Lemma 4, we have

$$\begin{aligned}&\sum_{m=0}^{n} gcLe(t,m)\\&=\sum_{m=0}^{n}\left(Le(t,m)+Le(t,m+1)e_1+Le(t,m+2)e_2+Le(t,m+3)e_3\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{n} Le(t,m) + e_1 \sum_{m=0}^{n} Le(t,m+1) + e_2 \sum_{m=0}^{n} Le(t,m+2) + e_3 \sum_{m=0}^{n} Le(t,m+3) \\
&= (Le(t,n+2) - (t-1)n - t) + (Le(t,n+3) - (t-1)(n+1) - t - 1)e_1 \\
&\ + (Le(t,n+4) - (t-1)(n+2) - t - 2)e_2 \\
&\ + (Le(t,n+5) - (t-1)(n+3) - 2t - 3)e_3 \\
&= gcLe(t,n+2) - (n(t-1)(1+e_1+e_2+e_3) + t + 2te_1 + 3te_2 + 5te_3) \\
&= gcLe(t,n+2) - \big(n(t-1)\cdot \hat{\mathbf{1}} + gcLe(t,2) - gcLe(t,0)\big)\,.
\end{aligned}
$$

Other sum formulas are proven using the same method. □

Concluding Remarks

In this paper, we introduced and studied generalized commutative Leonardo numbers and then one-parameter generalized commutative Fibonacci-Leonardo numbers. In the literature, we can find other generalizations of Leonardo numbers or Leonardo-type numbers, e.g. Leonardo-Alwyn numbers ([10]), generalized k-Leonardo numbers ([20]), etc. Using these generalizations, we can define and investigate new generalizations of commutative Leonardo quaternions.

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