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Research Article

Sharp upper bounds on KG-Sombor index of graphs

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Abstract: The KG-Sombor index of a graph G is defined as

$$KG(G) = \sum_{\substack{u \in V(G) \\ e \in E(G) \\ u \sim e}} \sqrt{d_G^2(u) + d_G^2(e)},$$

where the summation goes over pairs of vertices u and edges e such that e is incident to u. In this paper, we establish sharp upper bounds for the KG-Sombor index in three classes of graphs: graphs of order n with k pendent vertices, graphs of order n with k cut edges, and unicyclic graphs of order n with girth g. Moreover, in each case, the extremal graphs attaining these bounds are completely characterized.

Keywords: Sombor index, KG-Sombor index, cut edge, pendent vertex.

AMS Subject classification: 05C35, 05C09, 05C07

1. Introduction

Consider a simple, connected graph G with a vertex set $V(G)$ and edge set $E(G)$. If $uv \in E(G)$, we say that a vertex u is adjacent to v (and vice versa). The degree $d_G(u)$ of a vertex $u \in V(G)$ is the number of vertices adjacent to u. A vertex of degree one is referred to as a pendent vertex, and an edge incident to such a vertex (pendent vertex)

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is called a pendent edge. A cut edge (also called a bridge) is precisely an edge whose removal disconnects the graph or increases the number of connected components. A cut vertex is a vertex whose removal increases the number of connected components of a graph.
If $e = uv \in E(G)$, we say that the edge e is incident to both u and v. Moreover, u and v are referred to as the endpoints of the edge e. Additionally, if two edges e_1 and e_2 of G share a common endpoint, we say that e_1 is incident to e_2 (and vice versa). The degree of an edge $e \in E(G)$ is the number of edges incident to e.
In 2022, Kulli and Gutman [13] introduced a novel variant of the Sombor index, named the KG-Sombor index, which depends on the degrees of both vertices and edges. The KG-Sombor index is defined as

$$KG(G) = \sum_{\substack{u\in V(G)\\ e\in E(G)\\ u\sim e}} \sqrt{d_G^2(u) + d_G^2(e)}$$

where the summation is over pairs of vertices u and edges e such that e is incident to u. Since the edge $e = uv$ is incident to both the vertices u and v, we have $d_G(e) = d_G(u) + d_G(v) - 2$ and it follows that the KG-Sombor index can be expressed as

$$KG(G) = \sum_{uv\in E(G)} \left[\sqrt{d_G^2(u) + (d_G(u) + d_G(v) - 2)^2} + \sqrt{d_G^2(v) + (d_G(u) + d_G(v) - 2)^2}\right].$$

Recently, several articles related to the KG-Sombor index have been published. In particular, Kulli and his colleagues [11, 13] studied some basic properties of the KG-Sombor index and its relationships with other topological indices. Gutman et al. studied Sombor index [7] and KG-Sombor index [8] of Kragujevac trees and empirically determined the extremal Kragujevac trees with respect to SO and KG indices. Later, an analytical proof of these results was provided in [18]. Kosari et al. [10] investigated the KG-Sombor index within the class of trees and established lower bounds involving the order and maximum degree. The extremal graphs with respect to the KG-Sombor index for trees, molecular trees, and unicyclic graphs were characterized in [1].
The Sombor index, originally introduced in [5], and its various relations have been extensively explored in numerous subsequent articles. For more detailed information, see [2–4, 6, 9, 12, 14–17].

The primary research on topological indices of graphs focuses on determining their extremal values within specific classes of graphs. Initially, topological indices are studied for the class of graphs of order n, the class of trees of order n, and the class of unicyclic graphs of order n. Subsequently, attention is extended to the class of graphs of order n with k pendent vertices (or pendent edges), the class of graphs of order n with k cut edges, as well as the class of unicyclic graphs of order n with girth g. Within this framework, we conducted a study on the maximum values of the KG-Sombor index for the aforementioned classes of graphs.

This paper is organized as follows: In Section 2, we present some lemmas that are crucial for deriving the main results. In Section 3, we give sharp upper bounds on the KG-Sombor index for classes of graphs of order n with k pendent vertices, graphs of order n with k cut edges and for unicyclic graphs of order n with girth g. Furthermore, each of the extremal graphs attaining these bounds are characterized.

2. Preliminaries

In this section, we introduce some notations and definitions, and present several lemmas that will be instrumental in proving our main results.
Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be non-increasing sequences of real numbers. We say that A majorizes B if, for each $k = 1, 2, \ldots, n$,

$$\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k} b_i,$$

with equality holding when $k = n$.

Lemma 1. *(Karamata's inequality) Let $f : I \to \mathbb{R}$ be a strictly convex function. Let $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$ be non-increasing sequences on I. If A majorizes B then*

$$f(a_1) + f(a_2) + \ldots + f(a_n) \geq f(b_1) + f(b_2) + \ldots + f(b_n)$$

with equality if and only if $a_i = b_i$ for all $1 \leq i \leq n$.

For simplicity, we will consider the following symmetric function and its properties:

$$\psi(x, y) = \sqrt{x^2 + (x + y - 2)^2} + \sqrt{y^2 + (x + y - 2)^2},$$

where $x, y \in [1, +\infty)$. Then we can define the KG-Sombor index of a graph G as follows:

$$KG(G) = \sum_{uv \in E(G)} \psi(d_G(u), d_G(v)).$$

Lemma 2. *For given constants $a, b, c \in [1, +\infty)$ and the function $\psi(x, y)$, we have*
(i) *$\psi(x, c)$ is an increasing function on $[1, +\infty)$*
(ii) *$\psi(a + 1, c) + \psi(b - 1, c) \geq \psi(a, c) + \psi(b, c)$ where $a \geq b$.*
(iii) *$\psi(a + 1, b - 1) \geq \psi(a, b)$ where $a \geq b$.*
(iv) *$\psi(a + b - 1, 1) \geq \psi(a, b)$.*

Proof. (i) Let us consider a function

$$f(x) = \psi(x, c) = \sqrt{x^2 + (x + c - 2)^2} + \sqrt{c^2 + (x + c - 2)^2}$$

for $y = c$. Then we have

$$f'(x) = \frac{2x + c - 2}{\sqrt{x^2 + (x + c - 2)^2}} + \frac{x + c - 2}{\sqrt{c^2 + (x + c - 2)^2}} > 0$$

and it follows that $f(x) = \psi(x, c)$ is an increasing function.
(ii) We also have

$$f''(x) = \frac{(c - 2)^2}{(x^2 + (x + c - 2)^2)^{3/2}} + \frac{c^2}{(c^2 + (x + c - 2)^2)^{3/2}} > 0$$

and it follows that $f(x) = \psi(x, c)$ is a strictly convex on $[1, +\infty)$. Now we must prove that

$$f(a + 1) + f(b - 1) \geq f(a) + f(b)$$

and we easily get the required inequality by the Karamata's inequality.
(iii) Consider a function $g(x) = \sqrt{1 + x^2}$. Then

$$\psi(a + 1, b - 1) = (a + b - 2)\left[g\left(\frac{a + 1}{a + b - 2}\right) + g\left(\frac{b - 1}{a + b - 2}\right)\right]$$
$$\psi(a, b) = (a + b - 2)\left[g\left(\frac{a}{a + b - 2}\right) + g\left(\frac{b}{a + b - 2}\right)\right]$$

and it is easy to see that the function $g(x)$ is strictly convex on an interval $[0, +\infty)$. From $a \geq b$, $A = \left(\frac{a+1}{a+b-2}, \frac{b-1}{a+b-2}\right)$ majorizes $B = \left(\frac{a}{a+b-2}, \frac{b}{a+b-2}\right)$. Hence, we have

$$g\left(\frac{a + 1}{a + b - 2}\right) + g\left(\frac{b - 1}{a + b - 2}\right) \geq g\left(\frac{a}{a + b - 2}\right) + g\left(\frac{b}{a + b - 2}\right)$$

and one can easily obtain the desired inequality (iii) in Lemma 2.
(iv) Since the function $\psi(x, y)$ is symmetric, we may assume without loss of generality that $a \geq b$. By repeatedly applying inequality (iii), we obtain

$$\psi(a + b - 1, 1) \geq \psi(a + b - 2, 2) \geq \psi(a + b - 3, 3) \geq \cdots \geq \psi(a, b).$$

This completes the proofs. □

Lemma 3. *Let G be a connected graph and uv be a non-pendent cut edge in G. Denote by G' the graph obtained by the contraction of uv onto the vertex u and attaching a pendent vertex v to u. Then $KG(G) < KG(G')$.*

Proof. Let $N_G(u) \setminus \{v\} = \{u_1, u_2, \ldots, u_s\}$ and $N_G(v) \setminus \{u\} = \{v_1, v_2, \ldots, v_t\}$. Then $s, t \geq 1$ because uv is non-pendent. Also we have $d_G(u) = s+1$, $d_G(v) = t+1$, $d_{G'}(u) = d_G(u) + d_G(v) - 1 = s+t+1$, $d_{G'}(v) = 1$ and $d_{G'}(w) = d_G(w)$ for $w \in V(G)$ which different from u, v. From (i) of Lemma 2, we have

$$\begin{aligned}\psi(d_{G'}(u), d_{G'}(u_i)) =& \psi(s+t+1, d_G(u_i)) > \psi(s+1, d_G(u_i)) = \psi(d_G(u), d_G(u_i)),\\ \psi(d_{G'}(u), d_{G'}(v_i)) =& \psi(s+t+1, d_G(v_i)) > \psi(t+1, d_G(v_i)) = \psi(d_G(v), d_G(v_i))\end{aligned}$$

and

$$\begin{aligned}KG(G') - KG(G) =& \sum_{i=1}^{s} [\psi(d_{G'}(u), d_{G'}(u_i)) - \psi(d_G(u), d_G(u_i))]\\ &+ \sum_{i=1}^{t} [\psi(d_{G'}(u), d_{G'}(v_i)) - \psi(d_G(v), d_G(v_i))]\\ &+ \psi(d_{G'}(u), d_{G'}(v)) - \psi(d_G(u), d_G(v))\\ >& \psi(d_{G'}(u), d_{G'}(v)) - \psi(d_G(u), d_G(v))\\ =& \psi(s+t+1, 1) - \psi(s+1, t+1) > 0\end{aligned}$$

by Lemma 2 (iv). It follows that $KG(G') > KG(G)$ and the proof is finished. □

Lemma 4. *Let G be a connected graph of order n with k cut edges. If G attains the maximum value of KG-sombor index in the class of graphs of order n with k cut edges, then all k cut edges of G are pendent.*

Proof. If there is a non-pendent cut edge uv in G, then denoted by G' the graph obtained by the contraction of uv onto the vertex u and adding a pendent vertex v to u. Then by the Lemma 3, we have $KG(G') > KG(G)$. This contradicts the fact that G attains the maximum value of KG-sombor index. Because G' belongs to the class of graphs of order n with k cut edges. Therefore, there is no non-pendent cut edge in G. □

Lemma 5. [10] *Let G be a connected graph and u, v be non-adjacent vertices in G. Denote $G' = G + uv$. Then $KG(G') > KG(G)$.*

3. Main results

A complete graph is a graph in which every pair of distinct vertices is adjacent. The complete graph of order n is denoted by K_n. A cycle graph of order n, denoted by C_n, is a graph that consists of n vertices and n edges in which each vertex has degree 2, forming a single cycle. Let us denote by K_{n-k}^k the graph obtained by attaching k pendent edges to a vertex of K_{n-k}.

Theorem 1. *Let G be a connected graph of order n with k pendent edges. Then*

$$\begin{aligned} KG(G) \leq & k\sqrt{1^2+(n-2)^2}+k\sqrt{(n-1)^2+(n-2)^2} \\ & +(n-k-1)\left(\sqrt{(n-k-1)^2+(2n-k-4)^2}+\sqrt{(n-1)^2+(2n-k-4)^2}\right) \\ & +(n-k-1)(n-k-2)\sqrt{(n-k-1)^2+4(n-k-2)^2} \end{aligned} \tag{3.1}$$

with equality holds if and only if G is isomorphic to K_{n-k}^k.

Proof. If G is isomorphic to K_{n-k}^k then the equality holds in (3.1). Suppose that G is not isomorphic to this graph and $KG(G)$ is maximum among all connected graphs of order n with k pendent edges. Then by Lemma 4 and Lemma 5, all k cut edges in G are pendent and any two non-pendent vertices in G are adjacent. Let $u_1, u_2, \ldots, u_{n-k}$ be the non-pendent vertices in G. It will be convenient to denote $d_G(u_i) = d_i, i = 1, 2, \ldots, n-k$. Without loss of generality we may assume that $d_1 \geq d_2 \geq \ldots \geq d_{n-k} \geq n-k-1$. Let $k_i = d_i - (n-k-1)$ represents the number of pendent vertices adjacent to u_i in G.
Let $\{x_1, x_2, \ldots, x_{k_1}\}$ be the set of pendent vertices adjacent to u_1. From the fact that $G \ncong K_{n-k}^k$, it follows that $k_2 \geq 1$. Then we can assume that $\{y_1, y_2, \ldots, y_{k_2-1}, x\}$ is the set of pendent vertices adjacent to u_2. Consider $G' = G - u_2x + u_1x$ then we have

$$\begin{aligned} KG(G') - KG(G) = & \sum_{i=1}^{k_1}[\psi(d_{G'}(u_1), d_{G'}(x_i)) - \psi(d_G(u_1), d_G(x_i))] \\ & + \sum_{i=1}^{k_2-1}[\psi(d_{G'}(u_2), d_{G'}(y_i)) - \psi(d_G(u_2), d_G(y_i))] \\ & + \psi(d_{G'}(u_1), d_{G'}(x)) - \psi(d_G(u_2), d_G(x)) \\ & + \sum_{i=3}^{n-k}[\psi(d_{G'}(u_1), d_{G'}(u_i)) - \psi(d_G(u_1), d_G(u_i))] \\ & + \sum_{i=3}^{n-k}[\psi(d_{G'}(u_2), d_{G'}(u_i)) - \psi(d_G(u_2), d_G(u_i))] \\ & + [\psi(d_{G'}(u_1), d_{G'}(u_2)) - \psi(d_G(u_1), d_G(u_2))] \end{aligned}$$

$$\begin{aligned}
=&k_1[\psi(d_1+1,1)-\psi(d_1,1)]+(k_2-1)[\psi(d_2-1,1)-\psi(d_2,1)]\\
&+\psi(d_1+1,1)-\psi(d_2,1)\\
&+\sum_{i=3}^{n-k}[\psi(d_1+1,d_i)-\psi(d_1,d_i)+\psi(d_2-1,d_i)-\psi(d_2,d_i)]\\
&+[\psi(d_1+1,d_2-1)-\psi(d_1,d_2)].
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
A=&(k_1+1)\psi(d_1+1,1)-k_1\psi(d_1,1)+(k_2-1)\psi(d_2-1,1)-k_2\psi(d_2,1)\\
=&k_1[\psi(d_1+1,1)+\psi(d_2-1,1)-\psi(d_1,1)-\psi(d_2,1)]\\
&+(k_1-k_2)[\psi(d_2,1)-\psi(d_2-1,1)]+[\psi(d_1+1,1)-\psi(d_2-1,1)]>0
\end{aligned}$$

since $k_1 \geq k_2$. Because $\psi(d_1+1,1)+\psi(d_2-1,1)-\psi(d_1,1)-\psi(d_2,1) \geq 0$ by Lemma 2 (ii), and $\psi(d_2,1)-\psi(d_2-1,1) > 0, \psi(d_1+1,1)-\psi(d_2-1,1) > 0$ by Lemma 2 (i). Also, we have

$$\begin{aligned}
B=&\sum_{i=3}^{n-k}[\psi(d_1+1,d_i)-\psi(d_1,d_i)+\psi(d_2-1,d_i)-\psi(d_2,d_i)]\\
&+[\psi(d_1+1,d_2-1)-\psi(d_1,d_2)]\geq 0
\end{aligned}$$

where the inequality follows from applying Lemma 2 (ii) and (iii) to each term in the sum over $i=3$ to $n-k$. Therefore, we get

$$KG(G')-KG(G)=A+B>0$$

and it follows that $KG(G') > KG(G)$. Hence, we have a contradiction to the assumption that $KG(G)$ is maximum among all connected graphs of order n with k pendent edges. This completes the proof. □

Let G be a connected graph of order n with k cut edges. If G attains the maximum value of KG-sombor index in the class of graphs of order n with k cut edges, then all k cut edges of G are pendent by Lemma 4. Therefore, the following theorem follows directly from the preceding result.

Theorem 2. *Let G be a connected graph of order n with k cut edges. Then*

$$\begin{aligned}
KG(G)\leq&k\left(\sqrt{1^2+(n-2)^2}+\sqrt{(n-1)^2+(n-2)^2}\right)\\
&+(n-k-1)\left(\sqrt{(n-k-1)^2+(2n-k-4)^2}+\sqrt{(n-1)^2+(2n-k-4)^2}\right)\\
&+(n-k-1)(n-k-2)\sqrt{(n-k-1)^2+4(n-k-2)^2}
\end{aligned}$$

with equality holds if and only if G is isomorphic to K_{n-k}^k.

The proofs of Theorem 1 and Lemma 4 depend only on the properties in Lemma 1. Hence, Theorem 1 and Theorem 2 extend to all topological indices of the form

$$TI_{\psi}(G) = \sum_{uv \in E(G)} \psi(d_G(u), d_G(v)), \tag{3.2}$$

where $\psi(x, y)$ satisfies properties (i)–(iii) of Lemma 1. Therefore, for functions $\psi(x, y)$ satisfying the properties (i)–(iii) of Lemma 1, the following extended results hold.

Theorem 3. *Let G be a connected graph of order n with k cut edges, and G' be a connected graph of order n with k pendent vertices. Then*

$$\begin{aligned}\max\{TI_{\psi}(G), TI_{\psi}(G')\} \leq & k \cdot \psi(n-1, 1) + (n-k-1) \cdot \psi(n-1, n-k-1) \\ & + \frac{(n-k-1)(n-k-2)}{2} \cdot \psi(n-k-1, n-k-1)\end{aligned}$$

with equality holds if and only if G is isomorphic to K_{n-k}^{k}.

Now we give the sharp upper bound on KG over the class of unicyclic graphs of order n with girth g. Denote by $U_{n,g}$ the graph obtained by attaching $n-g$ pendent edges to a vertex of C_g.

Theorem 4. *Let G be a unicyclic graph of order n with girth g. Then*

$$\begin{aligned}KG(G) \leq & (n-g)\left(\sqrt{1^2 + (n-g+1)^2} + \sqrt{(n-g+2)^2 + (n-g+1)^2}\right) \\ & + 2\sqrt{4 + (n-g+2)^2} + (2n + 2g - 4)\sqrt{2}\end{aligned}$$

with equality holds if and only if G is isomorphic to $U_{n,g}$.

Proof. Suppose that G is the graph which $KG(G)$ is the maximum among all unicyclic graphs of order n with girth g. Denote by C_g the unique cycle of G. Since all edges that do not belong to C_g are cut edges, G has $n-g$ pendent edges as stated in Lemma 3. Then each pendent edge of G is attached to C_g. Denote (in clockwise order) by $u_1, u_2, \ldots, u_g$ the vertices on C_g. It will be convenient to denote $d_G(u_i) = d_i, i = 1, 2, \ldots, g$. Then, we have

$$2 \leq d_i \leq n-g+2 \quad \text{and} \quad d_1 + d_2 + \ldots + d_g = n + g. \tag{3.3}$$

Therefore, by (3.3) we obtain

$$\begin{aligned}KG(G) &= \sum_{i=1}^{g} (d_i - 2)\psi(d_i, 1) + \sum_{i=1}^{g} \psi(d_i, d_{i+1}) \\ &\leq \sum_{i=1}^{g} (d_i - 2)\psi(n-g+2, 1) + \sum_{i=1}^{g} \psi(d_i, d_{i+1})\end{aligned}$$

$$
\begin{aligned}
=&(n-g)\left(\sqrt{1^2+(n-g+1)^2}+\sqrt{(n-g+2)^2+(n-g+1)^2}\right)\\
&+\sum_{i=1}^{g}\left(d_i\sqrt{1+\left(1+\frac{d_{i+1}-2}{d_i}\right)^2}+d_{i+1}\sqrt{1+\left(1+\frac{d_i-2}{d_{i+1}}\right)^2}\right),
\end{aligned}
$$

where $d_{g+1}=d_1$.
Consider a function $f(x)=\sqrt{1+(1+x)^2}$ and it is easy to see that $f(x)$ is strictly convex on an interval $[0,+\infty)$. The above inequality can be written in the form

$$
\begin{aligned}
KG(G)\leq&(n-g)\left(\sqrt{1^2+(n-g+1)^2}+\sqrt{(n-g+2)^2+(n-g+1)^2}\right)\\
&+d_1f\left(\frac{d_2-2}{d_1}\right)+d_2f\left(\frac{d_1-2}{d_2}\right)+d_2f\left(\frac{d_3-2}{d_2}\right)+d_3f\left(\frac{d_2-2}{d_3}\right)+\dots\\
&+d_gf\left(\frac{d_1-2}{d_g}\right)+d_1f\left(\frac{d_g-2}{d_1}\right).
\end{aligned}
$$

To complete the proof using Karamata's inequality, we define two non-increasing sequences, $A=\{a_i\}$ and $B=\{b_i\}$, of equal length $2(n+g)$. Specifically, A is a non-increasing permutation of the following numbers:

$$
\underbrace{\frac{d_2-2}{d_1},\dots,\frac{d_2-2}{d_1}}_{d_1},\underbrace{\frac{d_1-2}{d_2},\dots,\frac{d_1-2}{d_2}}_{d_2},\underbrace{\frac{d_3-2}{d_2},\dots,\frac{d_3-2}{d_2}}_{d_2},\underbrace{\frac{d_2-2}{d_3},\dots,\frac{d_2-2}{d_3}}_{d_3},\dots,
$$

$$
\underbrace{\frac{d_g-2}{d_{g-1}},\dots,\frac{d_g-2}{d_{g-1}}}_{d_{g-1}},\underbrace{\frac{d_{g-1}-2}{d_g},\dots,\frac{d_{g-1}-2}{d_g}}_{d_g},\underbrace{\frac{d_1-2}{d_g},\dots,\frac{d_1-2}{d_g}}_{d_g},\underbrace{\frac{d_g-2}{d_1},\dots,\frac{d_g-2}{d_1}}_{d_1}.
$$

The sequence B is defined as:

$$
B=\left(\frac{n-g}{2},\frac{n-g}{2},\frac{n-g}{2},\frac{n-g}{2},\underbrace{0,0,\dots,0}_{2n+2g-4}\right).
$$

It is easy to see that both the summations of all elements of A and B are equal to $2(n-g)$. Denote $A_i=a_1+a_2+\dots+a_i$ and $B_i=b_1+b_2+\dots+b_i$ for $1\leq i\leq 2(n+g)$. Since $a_i=\frac{d_p-2}{d_q}\leq\frac{n-g}{2}$, it follows that $A_i\leq B_i$ for all i. Hence, B majorizes A. Then, by the Karamata's inequality, we obtain

$$
\begin{aligned}
&d_1f\left(\frac{d_2-2}{d_1}\right)+d_2f\left(\frac{d_1-2}{d_2}\right)+d_2f\left(\frac{d_3-2}{d_2}\right)+d_3f\left(\frac{d_2-2}{d_3}\right)+\dots\\
&+d_gf\left(\frac{d_1-2}{d_g}\right)+d_1f\left(\frac{d_g-2}{d_1}\right)\leq 4f\left(\frac{n-g}{2}\right)+(2n+2g-4)f(0)
\end{aligned}
$$

and hence we get

$$\begin{aligned}KG(G) \leq &(n-g)\left(\sqrt{1^2+(n-g+1)^2}+\sqrt{(n-g+2)^2+(n-g+1)^2}\right)\\&+4f\left(\frac{n-g}{2}\right)+(2n+2g-4)f(0)\\=&(n-g)\left(\sqrt{1^2+(n-g+1)^2}+\sqrt{(n-g+2)^2+(n-g+1)^2}\right)\\&+\sqrt{4+(n-g+2)^2}+(2n+2g-4)\sqrt{2}\\=&KG(U_{n,g})\end{aligned}$$

and this is the required inequality. Th equality holds if and only if $d_1 = n-g+2, d_2 = 2, \ldots, d_g = 2$. Hence the equality condition holds if and only if $G \cong U_{n,g}$. □

4. Conclusion

In this paper, we studied extremal problems concerning the KG-Sombor index. By employing graph transformations and inequality techniques, we identified the graphs that maximize the KG-Sombor index within three classes: graphs of order n with k pendent edges, graphs of order n with k cut edges, and unicyclic graphs of order n with girth g. For these classes, extremal values and the corresponding extremal graphs have been established for almost all well-known topological indices. In the first two cases, the problem can be solved by analyzing certain properties of the function $\psi(x, y)$ (see Theorem 3). In this work, we focused only on maximal values and their corresponding extremal graphs; extending these results to determine minimal values and the associated extremal graphs remains an interesting direction for future research.

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