Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-11
https://doi.org/10.22049/cco.2026.30339.2424

CCO
Commun. Comb. Optim.

Research Article

Generalized MacWilliams identity for λ-ply joint weight enumerators

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Received: 6 February 2025; Accepted: 26 December 2025
Published Online: 5 January 2026

Abstract: The MacWilliams identity provides a fundamental link between the weight enumerator of a linear code and its dual. While generalizations exist for joint weight enumerators and λ-ply weight enumerators, a unifying framework encompassing these extensions has remained elusive. In this paper, we introduce the λ-ply joint weight enumerators and obtain a novel generalization of the MacWilliams identity that subsumes previously known results as special cases.

Keywords: code, support, weight, weight enumerator, MacWilliams identity.

AMS Subject classification: 94B05, 94A24, 05A19

1. Introduction

The MacWilliams identities are cornerstone results in coding theory, providing a powerful relationship between the weight enumerator of a linear code and that of its dual code [4]. These identities are central to understanding the structure of codes with applications across several areas such as design theory [1, 2], quantum codes [3, 8], association schemes [11] and perfect codes [5]. By linking the weight enumerators of a code and its dual, MacWilliams identities allow for the optimization of coding schemes, helping to balance trade-offs between code rate and error-correcting capabilities and providing insights into the behavior of codes under various conditions [4, 11]. Moreover, the MacWilliams identities play a vital role in error-correction and

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optimizing code design, enabling the construction of codes that perform well under noisy transmission conditions.

Over the years, researchers have extended the classical MacWilliams identities in several directions to accommodate more complex coding structures [7, 9, 10, 14, 15, 17]. Recent works in this area has continued to explore and refine the generalization of MacWilliams identities. In particular, works from 2020 onward have proposed new formulations that expand the identities reach to more diverse coding structures. For instance, the MacWilliams-type identities for codes defined over several rings are introduced in [6, 12, 16, 18], thus extending the classical identities beyond vector spaces over finite fields. In [8], a new MacWilliams identity was derived for classical and quantum convolutional codes, offering insights into the behavior of codes when code length and structure are interdependent. Notably, two significant generalizations are the joint MacWilliams identity and the λ-ply MacWilliams identity [11, 13]. The joint identity generalizes the MacWilliams framework to scenarios involving two codes and their duals simultaneously, while the λ-ply identity examines the relationship between a code and its dual through a specific power of the weight enumerator. Both of these generalizations have provided deeper insights into the properties of codes, particularly in higher-order interactions and joint coding schemes.

In this paper, unified approach that encompasses both the λ-ply and joint MacWilliams identities are developed, providing a comprehensive framework for analyzing codes with more intricate dual relationships. In fact, a new generalization of the MacWilliams identities that subsumes both the λ-ply and joint MacWilliams identities as special cases is presented. Our approach introduces a unified framework that captures higher-order interactions and complex joint coding structures, extending the applicability of MacWilliams identities to a broader class of codes. This new identity not only simplifies and unifies several existing results but also provides a powerful tool for analyzing the structure of linear codes, particularly in settings that involve multiple codes and their duals. The remainder of this paper is organized as follows. In Section 2, we review some classical concepts and propositions from coding theory. Section 3 introduces the classical MacWilliams identities and some generalizations. Finally, Section 4 explores our new generalization and shows how both the λ-ply and joint identities emerge as special cases.

2. Preliminaries

Let $\mathbb{F}_q$ be the finite field of order q and let $\mathbb{F}_q^n$ be the vector space of all n-tuples over $\mathbb{F}_q$. A linear code $\mathcal{C}$ of length n over $\mathbb{F}_q$ is a subspace of $\mathbb{F}_q^n$. The elements of $\mathcal{C}$ are called codewords. The code $\mathcal{C}$ is described as a binary and ternary code if $q = 2$ and $q = 3$, respectively. The distance between any two words $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ in $\mathbb{F}_q^n$ is $d(u, v) = \#\{1 \leq i \leq n \mid u_i \neq v_i\}$. Moreover, the inner product of u and v is defined as $\langle u, v\rangle := \sum_{i=1}^{n} u_i v_i$. The support and the weight of $u = (u_1, u_2, \ldots, u_n) \in \mathbb{F}_q^n$ are $\mathrm{supp}(u) = \{1 \leq i \leq n | u_i \neq 0\}$ and $\mathrm{wt}(u) = \#\{1 \leq i \leq n | u_i \neq 0\}$, respectively. Clearly, $\mathrm{wt}(u) = |\mathrm{supp}(u)|$ and $d(x, y) = \mathrm{wt}(x - y)$ for any two words $x, y \in \mathbb{F}_q^n$. The minimum distance of the linear code $\mathcal{C}$ is defined as $d(\mathcal{C}) = \min\{d(c, c') \mid c, c' \in \mathcal{C},\ c \neq c'\}$. Clearly, $d(\mathcal{C}) = \min\{\mathrm{wt}(c) \mid c \in \mathcal{C},\ c \neq 0\}$. The linear code $\mathcal{C}$ of length n over $\mathbb{F}_q$ is called a q-ary $[n, k, d]$ code (or when d is unknown, an $[n, k]_q$ code) if $\dim_{\mathbb{F}_q}(\mathcal{C}) = k$ and $d(\mathcal{C}) = d$. The linear code $\mathcal{C}$ over $\mathbb{F}_q$ is an $[n, k, d]_q$ code if its length, dimension and minimum distance are n, k and d, respectively. The code $\mathcal{C}$ is called trivial if it is the singleton $\{0\}$, the whole space $\mathbb{F}_q^n$, a subspace $\langle v \rangle$ of dimension 1 or a subspace of dimension $n - 1$. The $[n, k, d]_q$ code $\mathcal{C}$ is optimal if it has the largest possible minimum distance among all the $[n, k]_q$ codes.

The dual code of C is the orthogonal complement of C, i.e., $C^\perp := \{u \in \mathbb{F}_q^n \mid \forall c \in \mathcal{C},\ \langle u, c \rangle = 0\}$. The hull of $\mathcal{C}$ is $\mathcal{C} \cap \mathcal{C}^\perp$. If $\mathcal{C} \subseteq \mathcal{C}^\perp$ then $\mathcal{C}$ is self-orthogonal and also, if $\mathcal{C} = \mathcal{C}^\perp$ then $\mathcal{C}$ is self-dual. The weight enumerator of $\mathcal{C}$ is the homogeneous polynomial

$$W_{\mathcal{C}}(x, y) := \sum_{c \in \mathcal{C}} x^{n-\mathrm{wt}(c)} y^{\mathrm{wt}(c)} = \sum_{i=0}^{n} A_i x^{n-i} y^i,$$

where $A_i = \#\{c \in C \mid \mathrm{wt}(c) = i\}$. The sequence $A_0, A_1, \ldots, A_n$ is called the weight distribution of C. Two linear codes $\mathcal{C}$ and $\mathcal{C}'$ of length n over $\mathbb{F}_q$ are called equivalent if we can obtain $\mathcal{C}$ from $\mathcal{C}'$ by permuting the coordinate positions and possibly multiplying each coordinate position by a non-zero element of $\mathbb{F}_q$. Two codes $\mathcal{C}$ and $\mathcal{C}'$ are called isomorphic if we obtain $\mathcal{C}'$ from $\mathcal{C}$ by permuting the coordinate positions. In this case, we write $\mathcal{C} \cong \mathcal{C}'$. Each permutation of the coordinate positions of $\mathcal{C}$ that maps codewords of $\mathcal{C}$ to themselves is called an automorphism. The automorphism group of $\mathcal{C}$, denoted by $\mathrm{Aut}(\mathcal{C})$, consists of all the automorphisms of $\mathcal{C}$. We refer the reader to [4, 11] for further reading.

3. The MacWilliams identity

As it is known, the MacWilliams identity is an important tool for relating the minimum distance of a code to that of it's dual. In fact, this identity shows that the weight enumerator of $\mathcal{C}^\perp$ is completely determined by the weight enumerator of $\mathcal{C}$:

Theorem 1. *[11] (MacWilliams identity) Let $\mathcal{C}$ be a linear $[n, k, d]$ code over the finite field $\mathbb{F}_q$. Then,*

$$W_{\mathcal{C}^\perp}(x, y) = \frac{1}{|\mathcal{C}|} W_{\mathcal{C}}\big(x + (q-1)y, x - y\big).$$

There are some generalizations of the the weight enumerator $W_{\mathcal{C}}(x, y)$. One of them is as follows:

Definition 1. Let λ be an arbitary natural number and let $\mathcal{C}$ be a linear $[n, k, d]$ code over $\mathbb{F}_q$. The λ-ply weight enumerator of $\mathcal{C}$ is

$$W_{\mathcal{C}}^{(\lambda)}(x, y) = \sum_{u_1, \ldots, u_\lambda \in \mathcal{C}} x^{n-s(u_1, \ldots, u_\lambda)} y^{s(u_1, \ldots, u_\lambda)},$$

where $s(u_1, \ldots, u_\lambda) = |\text{supp}(u_1) \cup \cdots \cup \text{supp}(u_\lambda)|$.

It is clear that $W_{\mathcal{C}}^{(1)}(x, y) = W_{\mathcal{C}}(x, y)$.

Theorem 2. *[13]*

$$W_{\mathcal{C}^\perp}^{(\lambda)}(x, y) = \frac{1}{|C|^\lambda} W_C^{(\lambda)}\big(x + (q^\lambda - 1)y, x - y\big).$$

For any $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ of $\mathbb{F}_q^n$, define

$$\begin{aligned} i(u, v) &= \#\{i \in N | u_i = 0, v_i = 0\}, \\ j(u, v) &= \#\{i \in N | u_i = 0, v_i \neq 0\}, \\ k(u, v) &= \#\{i \in N | u_i \neq 0, v_i = 0\}, \\ l(u, v) &= \#\{i \in N | u_i \neq 0, v_i \neq 0\}, \end{aligned}$$

Clearly, $n = i(u, v) + j(u, v) + k(u, v) + l(u, v)$, $\text{wt}(u) = k(u, v) + l(u, v)$ and $\text{wt}(v) = j(u, v) + l(u, v)$.

Definition 2. Let $\mathcal{C}$ and $\mathcal{D}$ be two linear code over the finite field $\mathbb{F}_q$. The joint weight enumerator of $\mathcal{C}$ and $\mathcal{D}$ is

$$\begin{aligned} W_{\mathcal{C},\mathcal{D}}(x, y, z, t) &= \sum_{u \in \mathcal{C}} \sum_{v \in \mathcal{D}} x^{i(u,v)} y^{j(u,v)} z^{k(u,v)} t^{l(u,v)} \\ &= \sum_{i,j,k,l} A_{i,j,k,l} x^i y^j z^k t^l, \end{aligned}$$

where $A_{i,j,k,l} = \#\{(u, v) \in \mathcal{C} \times \mathcal{D} \mid i(u, v) = i, j(u, v) = j, k(u, v) = k, l(u, v) = l\}$.

By definition, we can say that

$$\begin{aligned} W_{\mathcal{D},\mathcal{C}}(x, y, z, t) &= W_{\mathcal{C},\mathcal{D}}(x, z, y, t) \\ W_{\mathcal{C}}(x, y) &= W_{\mathcal{C},\{0\}}(x, z, y, t), \\ W_{\mathcal{D}}(x, y) &= W_{\{0\},\mathcal{D}}(x, y, z, t), \\ W_{\mathcal{C}}(x, y) &= \frac{1}{|\mathcal{D}|} W_{\mathcal{C},\mathcal{D}}(x, x, y, y), \\ W_{\mathcal{D}}(x, y) &= \frac{1}{|\mathcal{C}|} W_{\mathcal{C},\mathcal{D}}(x, y, x, y), \\ W_{\mathcal{C}}(x, y) &= W_{\mathcal{C},\mathcal{D}}(x, 0, y, 0) \quad if\ 0 \in \mathcal{D}, \\ W_{\mathcal{D}}(x, y) &= W_{\mathcal{D},\mathcal{D}}(x, y, 0, 0) \quad if\ 0 \in \mathcal{C}. \end{aligned}$$

Theorem 3. *[10, 11] Let $\mathcal{C}$ and $\mathcal{D}$ be two linear codes over $\mathbb{F}_q$. Then,*
(i) $W_{\mathcal{C}^\perp,\mathcal{D}}(x,y,z,t) = (1/|\mathcal{C}|)W_{\mathcal{C},\mathcal{D}}\big(x+(q-1)z, y+(q-1)t, x-z, y-t\big)$.
(ii) $W_{\mathcal{C},\mathcal{D}^\perp}(x,y,z,t) = (1/|\mathcal{D}|)W_{\mathcal{C},\mathcal{D}}\big(x+(q-1)y, x-y, z+(q-1)t, z-t\big)$.
(iii) $W_{\mathcal{C}^\perp,\mathcal{D}^\perp}(x,y,z,t) = (1/|(\mathcal{C}||\mathcal{D}|))\, W_{\mathcal{C},\mathcal{D}}\big(x+(q-1)(y+z)+(q-1)^2t, x-y+(q-1)(z-t), x-z+(q-1)(y-t), x-y-z+t\big)$.

Theorems 2 and 3 generalize the MacWilliams identity for the λ-ply weight enumerators and the joint weight enumerators, respectively. These generalizations are into different directions.

4. Results

In this Section, the λ-ply joint weight enumerator is introduced and a version of the MacWilliams identity which generalizes the Theorems 2 and 3 is proved.

Definition 3. Let $\mathcal{C}$ and $\mathcal{D}$ be two linear codes of length n over the finite field $\mathbb{F}_q$. The λ-ply joint weight enumerator of $\mathcal{C}$ and $\mathcal{D}$ is the homogeneous polynomial

$$W^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x,y,z,t) = \sum_{\substack{u_1,\ldots,u_\lambda\in\mathcal{C}\\ v_1,\ldots,v_\lambda\in\mathcal{D}}} x^{|(N-S)\cap(N-T)|}y^{|(N-S)\cap T|}z^{|S\cap(N-T)|}t^{|S\cap T|},$$

where $S = S(u_1,u_2,\ldots,u_\lambda) = \cup_{i=1}^{\lambda}\operatorname{supp}(u_i)$ and $T = T(v_1,v_2,\ldots,v_\lambda) = \cup_{i=1}^{\lambda}\operatorname{supp}(v_i)$ for any $u_1,u_2,\ldots,u_\lambda \in \mathcal{C}$ and $v_1,v_2,\ldots,v_\lambda \in \mathcal{D}$.

Now, it is clear that

$$\begin{aligned}
W^{(1)}_{\mathcal{C},\mathcal{D}}(x,y,z,t) &= W_{\mathcal{C},\mathcal{D}}(x,y,z,t),\\
W^{(\lambda)}_{\mathcal{D},\mathcal{C}}(x,y,z,t) &= W^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x,z,y,t),\\
W^{(\lambda)}_{\mathcal{C}}(x,y) &= W^{(\lambda)}_{\mathcal{C},\{0\}}(x,z,y,t),\\
W^{(\lambda)}_{\mathcal{D}}(x,y) &= W^{(\lambda)}_{\{0\},\mathcal{D}}(x,y,z,t),\\
W^{(\lambda)}_{\mathcal{C}}(x,y) &= \frac{1}{|\mathcal{D}|^\lambda}W^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x,x,y,y),\\
W^{(\lambda)}_{\mathcal{D}}(x,y) &= \frac{1}{|\mathcal{C}|^\lambda}W^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x,y,x,y),\\
W^{(\lambda)}_{\mathcal{C}}(x,y) &= W^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x,0,y,0), \qquad if\ 0\in\mathcal{D},\\
W^{(\lambda)}_{\mathcal{D}}(x,y) &= W^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x,y,0,0), \qquad if\ 0\in\mathcal{C}.
\end{aligned}$$

Moreover, for our purpose, We introduce the weight enumerator

$$\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x,y,z,t) = \sum_{R,S\subseteq N} |\mathcal{C}(R)|^\lambda|\mathcal{D}(S)|^\lambda x^{|(N\setminus R)\cap(N\setminus S)|}y^{|(N\setminus R)\cap S|}z^{|R\cap(N\setminus S)|}t^{|R\cap S|}$$

for any two linear code $\mathcal{C}$ and $\mathcal{D}$, where $\mathcal{C}(R) = \{u \in \mathcal{C} \mid \mathrm{supp}(u) \subseteq R\}$ and $\mathcal{D}(S) = \{v \in \mathcal{D} \mid \mathrm{supp}(v) \subseteq S\}$.

Lemma 1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two linear codes of length n over $\mathbb{F}_q$. Then,*
(i) $\widetilde{W}^{(\lambda)}_{\mathcal{C}^\perp,\mathcal{D}}(x,y,z,t) = (1/|\mathcal{C}|^\lambda)\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}(q^\lambda z, q^\lambda t, x, y)$,
(ii) $\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}^\perp}(x,y,z,t) = (1/|\mathcal{D}|^\lambda)\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}(q^\lambda y, x, q^\lambda t, z)$,
(iii) $\widetilde{W}^{(\lambda)}_{\mathcal{C}^\perp,\mathcal{D}^\perp}(x,y,z,t) = \big(1/(|\mathcal{C}|^\lambda|\mathcal{D}|^\lambda)\big)\,\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}(q^{2\lambda} t, q^\lambda z, q^\lambda y, x)$.

Proof. (i) Consider two linear codes $\mathcal{C}$ and $\mathcal{D}$ of length n over $\mathbb{F}_q$. Let R be an arbitary subset of N. It is known that

$$0 \longrightarrow \mathcal{C}^\perp(R) \xrightarrow{\text{inc}} \mathbb{F}_q^n(R) \xrightarrow{f} \mathcal{C}^* \xrightarrow{\text{res}} \mathcal{C}(N \setminus R)^* \longrightarrow 0$$

is an exact sequence of vector spaces over $\mathbb{F}_q$, where $\mathcal{C}^* = \mathrm{Hom}_{\mathbb{F}_q}(\mathcal{C}, \mathbb{F}_q) \cong \mathbb{F}_q^n/\mathcal{C}^\perp$, inc is the inclusion function, $f(v) : \mathcal{C} \to \mathbb{F}_q$ is a function with rule $v \xmapsto{f} (u \overset{\hat{v}}{\mapsto} \langle u, v\rangle)$ and res is the restriction function (see [13, 14, 17]). Therefore,

$$|\mathcal{C}| \cdot |\mathcal{C}^\perp(R)| = |\mathbb{F}_q^n(R)| \cdot |\mathcal{C}(N \setminus R)|.$$

and we can write

$$\begin{aligned}
\widetilde{W}^{(\lambda)}_{\mathcal{C}^\perp,\mathcal{D}}(x,y,z,t) &= \sum_{R,S\subseteq N} |\mathcal{C}^\perp(R)|^\lambda |\mathcal{D}(S)|^\lambda x^{|(N\setminus R)\cap(N\setminus S)|} y^{|(N\setminus R)\cap S|} z^{|R\cap(N\setminus S)|} t^{|R\cap S|} \\
&= \frac{1}{|\mathcal{C}|^\lambda} \sum_{R,S\subseteq N} |\mathbb{F}_q^n(R)|^\lambda |\mathcal{C}(N\setminus R)|^\lambda |\mathcal{D}(S)|^\lambda x^{|(N\setminus R)\cap(N\setminus S)|} y^{|(N\setminus R)\cap S|} \\
&\qquad\qquad \times z^{|R\cap(N\setminus S)|} t^{|R\cap S|} \\
&= \frac{1}{|\mathcal{C}|^\lambda} \sum_{R,S\subseteq N} |\mathbb{F}_q^n(N\setminus R)|^\lambda |\mathcal{C}(R)|^\lambda |\mathcal{D}(S)|^\lambda x^{|R\cap(N\setminus S)|} y^{|R\cap S|} \\
&\qquad\qquad \times z^{|(N\setminus R)\cap(N\setminus S)|} t^{|(N\setminus R)\cap S|} \\
&= \frac{1}{|\mathcal{C}|^\lambda} \sum_{R,S\subseteq N} |\mathcal{C}(R)|^\lambda |\mathcal{D}(S)|^\lambda x^{|R\cap(N\setminus S)|} y^{|R\cap S|} (q^\lambda z)^{|(N\setminus R)\cap(N\setminus S)|} \\
&\qquad\qquad \times (q^\lambda t)^{|(N\setminus R)\cap S|} \\
&= \frac{1}{|\mathcal{C}|^\lambda} \widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}(q^\lambda z, q^\lambda t, x, y).
\end{aligned}$$

(ii) It is similar to (i).
(iii) By (i) and (ii),

$$\begin{aligned}
\widetilde{W}^{(\lambda)}_{\mathcal{C}^\perp,\mathcal{D}^\perp}(x,y,z,t) &= (1/|\mathcal{C}|^\lambda)\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}^\perp}(q^\lambda z, q^\lambda t, x, y) \\
&= (1/|\mathcal{C}|^\lambda)(1/|\mathcal{D}|^\lambda)\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}(q^{2\lambda} t, q^\lambda z, q^\lambda y, x).
\end{aligned}$$

□

Lemma 2. *If $\mathcal{C}$ and $\mathcal{D}$ are two linear codes of length n over $\mathbb{F}_q$ then*

$$\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x,y,z,t) = W^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x+y+z+t, y+t, z+t, t).$$

Proof.

$$\begin{aligned}
\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x,y,z,t) &= \sum_{R,S\subseteq N} |\mathcal{C}(R)|^{\lambda}|\mathcal{D}(S)|^{\lambda} x^{|(N\setminus R)\cap(N\setminus S)|} y^{|(N\setminus R)\cap S|} z^{|R\cap(N\setminus S)|} t^{|R\cap S|} \\
&= \sum_{R,S\subseteq N} \left(\sum_{T\subseteq R} |\{c\in\mathcal{C}|\operatorname{supp}(c)=T\}|\right)^{\lambda} \\
&\qquad \times \left(\sum_{U\subseteq S} |\{d\in\mathcal{C}|\operatorname{supp}(d)=U\}|\right)^{\lambda} \\
&\qquad \times x^{|(N\setminus R)\cap(N\setminus S)|} y^{|(N\setminus R)\cap S|} z^{|R\cap(N\setminus S)|} t^{|R\cap S|} \\
&= \sum_{R,S\subseteq N} \left(\sum_{T_1,\dots,T_\lambda\subseteq R} |\big\{(c_1,\dots,c_\lambda)|c_i\in\mathcal{C},\operatorname{supp}(c_i)=T_i\big\}|\right) \\
&\qquad \times \left(\sum_{U_1,\dots,U_\lambda\subseteq S} |\big\{(d_1,\dots,d_\lambda)|d_j\in\mathcal{C},\operatorname{supp}(d_j)=U_j\big\}|\right) \\
&\qquad \times x^{|(N\setminus R)\cap(N\setminus S)|} y^{|(N\setminus R)\cap S|} z^{|R\cap(N\setminus S)|} t^{|R\cap S|} \\
&= \sum_{\substack{T_1,\dots,T_\lambda\subseteq N\\ U_1,\dots,U_\lambda\subseteq N}} |\big\{(c_1,\dots,c_\lambda)|c_i\in\mathcal{C},\operatorname{supp}(c_i)=T_i\big\}| \\
&\qquad \times |\big\{(d_1,\dots,d_\lambda)|d_j\in\mathcal{C},\operatorname{supp}(d_j)=U_j\big\}| \\
&\qquad \times \sum_{\substack{T_1\cup\cdots\cup T_\lambda\subseteq R\subseteq N\\ U_1\cup\cdots\cup U_\lambda\subseteq S\subseteq N}} x^{|(N\setminus R)\cap(N\setminus S)|} y^{|(N\setminus R)\cap S|} z^{|R\cap(N\setminus S)|} t^{|R\cap S|} \\
&= \sum_{\substack{T_1,\dots,T_\lambda\subseteq N\\ U_1,\dots,U_\lambda\subseteq N}} |\big\{(c_1,\dots,c_\lambda)|c_i\in\mathcal{C},\operatorname{supp}(c_i)=T_i\big\}| \\
&\qquad \times |\big\{(d_1,\dots,d_\lambda)|d_j\in\mathcal{C},\operatorname{supp}(d_j)=U_j\big\}| \\
&\qquad \times \sum_{\substack{\emptyset\subseteq R'\subseteq N\setminus T\\ \emptyset\subseteq S'\subseteq N\setminus U}} x^{|((N\setminus T)\setminus R')\cap((N\setminus U)\setminus S')|} y^{|((N\setminus T)\setminus R')\cap S'|} \\
&\qquad \times y^{|((N\setminus T)\setminus R')\cap U|} z^{|((N\setminus U)\setminus S')\cap T|} z^{|((N\setminus U)\setminus S')\cap R'|} \\
&\qquad \times t^{|U\cap T|} t^{|S'\cap T|} t^{|R'\cap U|} t^{|R'\cap S'|} \\
&= \sum_{\substack{T_1,\dots,T_\lambda\subseteq N\\ U_1,\dots,U_\lambda\subseteq N}} |\big\{(c_1,\dots,c_\lambda)|c_i\in\mathcal{C},\operatorname{supp}(c_i)=T_i\big\}|
\end{aligned}$$

$$\times |\{(d_1,\ldots,d_\lambda)|d_j \in \mathcal{C}, \mathrm{supp}(d_j)=U_j\}|$$
$$\times\, t^{|T\cap U|}(y+t)^{|(N\setminus T)\cap U|}(z+t)^{|(N\setminus U)\cap T|}$$
$$\times\, (x+y+z+t)^{|(N\setminus T)\cap(N\setminus U)|}$$
$$= W^{(\lambda)}_{\mathcal{C},\mathcal{D}}(x+y+z+t, y+t, z+t, t),$$

where $T = T_1\cup\cdots\cup T_\lambda$ and $U = U_1\cup\cdots\cup U_\lambda$. □

Now, the main result is as follows:

Theorem 4. *Let $\mathcal{C}$ and $\mathcal{D}$ be two linear codes of length n over the finite field $\mathbb{F}_q$. Then,*
(i) $W^{(\lambda)}_{\mathcal{C}^\perp,\mathcal{D}}(x,y,z,t) = (1/|\mathcal{C}|^\lambda)W^{(\lambda)}_{\mathcal{C},\mathcal{D}}\big(x+(q^\lambda-1)z, y+(q^\lambda-1)t, x-z, y-t\big)$,
(ii) $W^{(\lambda)}_{\mathcal{C},\mathcal{D}^\perp}(x,y,z,t) = (1/|\mathcal{D}|^\lambda)W^{(\lambda)}_{\mathcal{C},\mathcal{D}}\big(x+(q^\lambda-1)y, x-y, z+(q^\lambda-1)t, z-t\big)$,
(iii) $W^{(\lambda)}_{\mathcal{C}^\perp,\mathcal{D}^\perp}(x,y,z,t) = (1/(|\mathcal{C}|^\lambda|\mathcal{D}|^\lambda))W^{(\lambda)}_{\mathcal{C},\mathcal{D}}\big(x+(q^\lambda-1)(y+z)+(q^\lambda-1)^2t, x-y+(q^\lambda-1)(z-t), x-z+(q^\lambda-1)(y-t), x-y-z+t\big)$.

Proof. By Lemma 1 and Lemma 2,
(i)

$$\begin{aligned}
W^{(\lambda)}_{\mathcal{C}^\perp,\mathcal{D}}(x,y,z,t) &= \widetilde{W}^{(\lambda)}_{\mathcal{C}^\perp,\mathcal{D}}(x-y-z+t, y-t, z-t, t)\\
&= \frac{1}{|\mathcal{C}|^\lambda}\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}\big(q^\lambda(z-t), q^\lambda t, x-y-z+t, y-t\big)\\
&= \frac{1}{|\mathcal{C}|^\lambda}W^{(\lambda)}_{\mathcal{C},\mathcal{D}}\big(x+(q^\lambda-1)z, y+(q^\lambda-1)t, x-z, y-t\big),
\end{aligned}$$

(ii)

$$\begin{aligned}
W^{(\lambda)}_{\mathcal{C},\mathcal{D}^\perp}(x,y,z,t) &= \widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}^\perp}(x-y-z+t, y-t, z-t, t)\\
&= \frac{1}{|\mathcal{D}|^\lambda}\widetilde{W}^{(\lambda)}_{\mathcal{C},\mathcal{D}}\big(q^\lambda(y-t), x-y-z+t, q^\lambda t,, z-t\big)\\
&= \frac{1}{|\mathcal{D}|^\lambda}W^{(\lambda)}_{\mathcal{C},\mathcal{D}}\big(x+(q^\lambda-1)y, x-y, z+(q^\lambda-1)t, z-t\big),
\end{aligned}$$

(iii)

$$\begin{aligned}
W^{(\lambda)}_{\mathcal{C}^\perp,\mathcal{D}^\perp}(x,y,z,t) &= \frac{1}{|\mathcal{C}|^\lambda}W^{(\lambda)}_{\mathcal{C},\mathcal{D}^\perp}\big(x+(q^\lambda-1)z, y+(q^\lambda-1)t, x-z, y-t\big)\\
&= \frac{1}{|\mathcal{C}|^\lambda|\mathcal{D}|^\lambda}W^{(\lambda)}_{\mathcal{C},\mathcal{D}}\big(x+(q^\lambda-1)(y+z)+(q^\lambda-1)^2t,\\
&\qquad x-y+(q^\lambda-1)(z-t), x-z+(q^\lambda-1)(y-t),\\
&\qquad x-y-z+t\big).
\end{aligned}$$

□

Example 1. Let $\mathrm{Ham}(r,q)$ denote the q-ary Hamming code and let $S(r,q)$ denote its dual, namely the q-ary simplex code. It is well known that $\mathrm{Ham}(r,q)$ is a perfect code with parameters $[(q^r-1)/(q-1),(q^r-1)/(q-1)-r,3]$ and $S(r,q)$ is a $[(q^r-1)/(q-1),r,q^{r-1}]$ code which attains the Griesmer bound. Moreover, every nonzero codeword in $S(r,q)$ has constant weight q^{r-1}. See [11] for more details. Now, by Theorem 4,

$$\begin{aligned}
&W^{(\lambda)}_{\mathrm{Ham}(r,q),\mathrm{Ham}(r,q)}(x,y,z,t)\\
&= \frac{1}{|S(r,q)|^{2\lambda}} W^{(\lambda)}_{S(r,q),S(r,q)}\big(x+(q^\lambda-1)(y+z)+(q^\lambda-1)^2t, x-y\\
&\quad + (q^\lambda-1)(z-t), x-z+(q^\lambda-1)(y-t), x-y-z+t\big)\\
&= \frac{1}{q^{2\lambda r}} \sum_{\substack{u_1,\ldots,u_\lambda\in S(r,q)\\ v_1,\ldots,v_\lambda\in S(r,q)}} \Big(x+(q^\lambda-1)(y+z)+(q^\lambda-1)^2t\Big)^{|(N-S)\cap(N-T)|}\\
&\qquad\qquad \times \Big(x-y+(q^\lambda-1)(z-t)\Big)^{|(N-S)\cap T|}\\
&\qquad\qquad \times \Big(x-z+(q^\lambda-1)(y-t)\Big)^{|S\cap(N-T)|}(x-y-z+t)^{|S\cap T|},
\end{aligned}$$

where $S=S(u_1,u_2,\ldots,u_\lambda)=\cup_{i=1}^{\lambda}\mathrm{supp}(u_i)$ and $T=T(v_1,v_2,\ldots,v_\lambda)=\cup_{i=1}^{\lambda}\mathrm{supp}(v_i)$ for any $u_1,u_2,\ldots,u_\lambda,v_1,v_2,\ldots,v_\lambda\in S(r,q)$.

5. Conclusion

This paper has presented a significant advancement in the theory of weight enumerators by introducing a generalized MacWilliams identity that unifies and extends previously known results for joint weight enumerators and λ-ply weight enumerators. Our approach, based on generalizations of classical ideas and arguments, provides a powerful and elegant framework for analyzing the weight distributions of linear codes and their duals. The key contributions of this work include the MacWilliams identity for λ-ply joint weight enumerators. Future research directions include introducing λ-ply triple weight enumerators and presenting a MacWilliams-type identity for three codes and their dual. The generalized MacWilliams identity presented here offers a valuable tool for researchers in coding theory and related fields, providing a more comprehensive understanding of the structure and properties of linear codes.

Acknowledgements: The authors are partially supported by the University of Kashan under grant number 1073211. The authors would like to thank the referees for carefully reading the manuscript and their useful comments which improved the paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] E.F. Assmus and J.D. Key, *Designs and Their Codes*, no. 103, Cambridge University Press, 1994.

[2] C. Ding and C. Tang, *Designs from Linear Codes*, World Scientific, 2022.

[3] C. Du, Z. Ma, and M. Xiong, *On the complete weight distributions of quantum error-correcting codes*, Chinese Physics B **32** (2023), no. 5, 050307.

[4] W.C. Huffman, J.L. Kim, and P. Solé, *Concise Encyclopedia of Coding Theory*, Chapman and Hall/CRC, 2021.

[5] J.Y. Hyun, *Generalized MacWilliams identities and their applications to perfect binary codes*, Designs, Codes and Cryptography **50** (2009), no. 2, 173–185. https://doi.org/10.1007/s10623-008-9222–6.

[6] D.S. Irwansyah, *Linear codes over a general infinite family of rings and MacWilliams-type relations*, **11** (2022), 53–60. https://doi.org/10.47443/dml.2022.091.

[7] H.K. Kim and D.Y. Oh, *A classification of posets admitting the MacWilliams identity*, IEEE Trans. Info. Theory **51** (2005), no. 4, 1424–1431. https://doi.org/10.1109/TIT.2005.844067.

[8] C.Y. Lai, M.H. Hsieh, and H.F. Lu, *On the MacWilliams identity for classical and quantum convolutional codes*, IEEE Trans. Comm. **64** (2016), no. 8, 3148–3159. https://doi.org/10.1109/TCOMM.2016.2585641.

[9] T. Li, R. Wu, and J. Xu, *Linear codes over a non-chain ring and the MacWilliams identities*, IEEE Access **8** (2020), 87115–87120. https://doi.org/10.1109/ACCESS.2020.2992586.

[10] F. MacWilliams, C. Mallows, and N. Sloane, *Generalizations of Gleason's theorem on weight enumerators of self-dual codes*, IEEE Trans. Info. Theory **18** (1972), no. 6, 794–805. https://doi.org/10.1109/TIT.1972.1054898.

[11] F.J. MacWilliams and N.J. A. Sloane, *The Theory of Error-Correcting Codes*, vol. 16, Elsevier, 1977.

[12] V. Sagar, A. Yadav, and R. Sarma, *Constacyclic codes over $\mathbb{Z}_2[u]/\{u^2\} \times \mathbb{Z}_2[u]/\{u^3\}$ and the macwilliams identities*, Appl. Algebra Engin. Comm. Comput. **36** (2024), no. 6, 1295–1324. https://doi.org/10.1007/s00200-024-00662-6.

[13] K. Shiromoto, *A new MacWilliams type identity for linear codes*, Hokkaido Math. J. **25** (1996), 651–656.

[14] ———, *Singleton bounds for codes over finite rings*, J. Algebraic Combin. **12** (2000), no. 1, 95–99.

[15] J. Simonis, *MacWilliams identities and coordinate partitions*, Linear Algebra Appl. **216** (1995), 81–91. https://doi.org/10.1016/0024-3795(93)00106-A.

[16] M. Sridhar and M.L. Kaushik, *MacWilliams type identities on Sharma-Kaushik weights for linear codes over ring* $\mathbb{Z}_q$, Int. J. Info. Tec. **15** (2023), no. 6, 2995–3001.
https://doi.org/10.1007/s41870-023-01368-7.

[17] Y. Tomoyuki, *MacWilliams identities for linear codes with group action*, Kumamoto J. Math. **6** (1993), 29–45.

[18] H. Wu, X. Shen, and H. Wang, *MacWilliams identities for linear codes over the ring* $\mathbb{Z}_4 + u\mathbb{Z}_4$ *on the Hermitian inner product*, International Conference on Cryptography, Network Security, and Communication Technology (CNSCT 2023), vol. 12641, SPIE, 2023, pp. 218–223.
https://doi.org/10.1117/12.2678862.