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Research Article

On Hermite-Hadamard type inequalities in stochastic fractional calculus

Israa Najm Abood[1], Rawa Khalil Ibrahim[2,*]

[1]Department of Mathematics, Faculty of Sciences, Diyala University, Diyala, Iraq
Israanajim@uodiyala.edu.iq

[2]Department of Mathematics, College of Dentistry, University of Diyala, Diyala, Iraq
*rawaa.kalil@uodiyala.edu.iq

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Abstract: This paper extends Hermite-Hadamard type inequalities within the framework of stochastic fractional calculus. We investigate how fractional integrals, which account for memory effects, interact with random processes. Our work presents three main contributions. First, we provide an error bound for approximating a standard integral of a smooth, deterministic function using stochastic fractional integrals. Second, we extend the well-known Hermite-Hadamard inequality, which applies to convex functions, to the setting of convex stochastic processes, showing how their expected values are bounded by these integrals. Finally, we derive specific mean-square error bounds when approximating a standard Brownian motion using its stochastic fractional integrals. These results enhance our understanding of stochastic fractional inequalities, offering new tools for analyzing complex systems influenced by both memory and randomness.

Keywords: fractional calculus, stochastic calculus, Hermite-Hadamard inequality, Riemann-Liouville fractional integral, Brownian motion, error bounds, convex stochastic process.

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1. Introduction

Fractional calculus, a generalization of classical calculus, extends the concepts of differentiation and integration to arbitrary non-integer orders. This field has witnessed a remarkable surge in interest over recent decades due to its profound utility

* *Corresponding Author*

in modeling complex phenomena across various scientific and engineering disciplines, including anomalous diffusion, viscoelasticity, control theory, and image processing [6, 10]. Its ability to describe systems with memory and hereditary properties offers a more accurate representation compared to traditional integer-order models.

Concurrently, stochastic calculus provides a robust framework for analyzing systems influenced by random fluctuations, with applications ranging from financial markets to biological systems. The interplay between fractional calculus and stochastic processes has given rise to the emerging field of stochastic fractional calculus, which seeks to integrate the memory effects of fractional operators with the inherent randomness of stochastic processes. This new paradigm allows for the development of more sophisticated models that capture both long-range dependence and stochastic variability.

A cornerstone in the theory of convex functions is the classical Hermite-Hadamard inequality, which provides bounds for the integral mean of a convex function [4, 8].

Definition 1 (Convex Function). A function $f : I \to \mathbb{R}$ is said to be convex on an interval I if for all $x, y \in I$ and for all $\lambda \in [0, 1]$, the following inequality holds:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

Geometrically, this means that the line segment connecting any two points on the graph of the function lies above or on the graph itself.

The classical Hermite-Hadamard inequality states that for a convex function $f : [a, b] \to \mathbb{R}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}$$

This inequality has proven to be a powerful tool in various branches of mathematics, including numerical analysis, optimization, and approximation theory. Also, when f' is a convex function, the following inequality holds:

Proposition 1. *Let $f \in C^1([a, b])$ and f' be convex, then for each $t \in [a, b]$*

$$|f(t)| \leq \max(|f(a)|, |f(b)|) + (b-a)(|f'(a)| + |f'(b)|).$$

Proof. For any $t \in [a, b]$, by the Mean Value Theorem, there exists $\xi \in (a, t)$ such that

$$f(t) - f(a) = f'(\xi)(t-a).$$

Thus

$$|f(t)| \leq |f(a)| + |f(t) - f(a)| \leq |f(a)| + (t-a) \sup_{\xi \in [a,t]} |f'(\xi)|. \tag{1.1}$$

Since f' is convex on $[a, b]$, there exists $\lambda \in (a, b)$ such that for any $\xi \in (a, b)$:

$$|f'(\xi)| = |f'(\lambda a + (1-\lambda)b)| \leq |\lambda f'(a) + (1-\lambda)f'(b)| \leq |f'(a)| + |f'(b)|. \tag{1.2}$$

since $\xi \in [a, b]$. Thus, the inequalities (1.1) and (1.2) imply that:

$$|f(t)| \leq |f(a)| + (b-a)(|f'(a)| + |f'(b)|).$$

Symmetrically, applying the same argument from t to b,

$$|f(t)| \leq |f(b)| + (b-t)(|f'(a)| + |f'(b)|) \leq |f(b)| + (b-a)(|f'(a)| + |f'(b)|).$$

Taking the tighter of the two bounds, we have

$$|f(t)| \leq \max(|f(a)|, |f(b)|) + (b-a)(|f'(a)| + |f'(b)|). \qquad \square$$

The primary motivation for this paper is to bridge the gap between fractional integral inequalities for deterministic functions and their stochastic counterparts. While Hermite-Hadamard type inequalities have been well-established for deterministic fractional integrals, their extension to the stochastic setting, particularly involving Itô integrals, presents significant challenges and opens new avenues for research. Understanding these inequalities is crucial for developing robust bounds and approximations for stochastic systems with memory.

In this paper, we establish novel Hermite-Hadamard type inequalities for stochastic processes involving stochastic Riemann-Liouville fractional integrals. Specifically, we investigate the bounds for convex stochastic processes, extending classical results to this more general framework. We also analyze the mean-square error terms associated with approximations of these stochastic fractional integrals, providing quantitative measures of their accuracy.

The structure of this paper is as follows: Section 2 provides preliminary definitions and known results from Riemann-Liouvill fractional integral essential for our analysis. Section 3 presents the preliminary definitions from stochastic Riemann-Liouvill fractional integral. Moreover, in this section, we present main results in the stochastic Hermite-Hadamard type inequalities. Finally, Section 4 concludes the paper with a summary of our findings and potential directions for future research.

2. Hermite-Hadamard inequalities to Riemann-Liouville fractional integral

The advent of fractional calculus led to natural extensions of the Hermite-Hadamard inequality to fractional integral operators. The Riemann-Liouville (RL) fractional integral generalizes classical n-fold integration to non-integer orders $\alpha > 0$. It comes in two forms: left-sided and right-sided, depending on the direction of integration. For a function $f \in L^1([a,b])$ and $\alpha > 0$, the left-sided RL integral of order α is defined as [2, 9]

$$\left(I_{a+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t \in [a,b]. \tag{2.1}$$

Moreover, the right-sided RL integral of order α is defined as

$$\left(I_{b-}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\, d\tau, \quad t \in [a,b]. \tag{2.2}$$

These deterministic fractional integral inequalities have been extensively studied, yielding a rich body of literature on various generalizations and applications.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x)\, dx \le \frac{f(a)+f(b)}{2}.$$

Theorem 2. *Let $f \in C^1([a,b])$ be convex and $\alpha > 0$. Then*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} \left[I_{a+}^{\alpha} f(b) + I_{b-}^{\alpha} f(a)\right] \le \frac{f(a)+f(b)}{2}. \tag{2.3}$$

Proof. Adapts the classical proof using weighted integrals and convexity (see [1, 7]).

Remark 1. The Riemann–Liouville (RL) fractional integral is adopted in this paper because it provides a natural extension of the classical integral operator and is well-suited to stochastic settings. In contrast, Caputo-type derivatives require differentiability of the sample paths, which Brownian motion lacks. The RL integral, defined through a convolution kernel $(t-\tau)^{\alpha-1}$, can be directly combined with stochastic integrators, allowing us to apply Itô isometry and Fubini-type arguments. Possible extensions of the present results to Caputo or other fractional operators will be addressed in future work.

3. Stochastic Riemann-Liouville Fractional Integrals

The realm of stochastic fractional integrals introduces new complexities and opportunities.

Definition 2. Let $f\colon [a,b] \times \Omega \to \mathbb{R}$ be an $\mathcal{F}_t$-adapted stochastic process, and let $B(t)$ be a standard Brownian motion. For $\alpha > 0$, the left-sided stochastic Riemann-Liouville fractional integral of order α is defined as [5]

$$\left({}^{S}I_{a+}^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, dB(\tau), \quad t \in [a,b].$$

Moreover, the right-sided stochastic Riemann-Liouville fractional integral of order α is defined as:

$$\left({}^{S}I_{b-}^{\alpha}f\right)(t)=\frac{1}{\Gamma(\alpha)}\int_{t}^{b}(\tau-t)^{\alpha-1}f(\tau)\,dB(\tau),\quad t\in[a,b],$$

where the integral is interpreted in the Itô sense.

Some properties of stochastic Riemann-Liouville fractional integrals are as follows:

(a) Adaptedness: Both ${}^{S}I_{a+}^{\alpha}f$ and ${}^{S}I_{b-}^{\alpha}f$ are $\mathcal{F}_t$-adapted processes.

(b) Itô Isometry: For deterministic f,

$$\mathbb{E}\left[\left({}^{S}I_{a+}^{\alpha}f(t)\right)^{2}\right]=\frac{1}{\Gamma(\alpha)^{2}}\int_{a}^{t}(t-\tau)^{2\alpha-2}f(\tau)^{2}\,d\tau.$$

(c) Semimartingale Decomposition: If f is deterministic, ${}^{S}I_{a+}^{\alpha}f(t)$ is a square-integrable martingale.

Remark 2. When $\alpha=1$, the left-sided integral reduces to the standard Itô integral $\int_a^t f(\tau)\,dB(\tau)$. For $\alpha\neq 1$, the kernel $(t-\tau)^{\alpha-1}$ introduces memory effects, making these integrals suitable for modeling systems with non-Markovian noise.

Proposition 2. *Let $I_{a+}^{\alpha}f$ and ${}^{S}I_{a+}^{\alpha}f$ be defined as above. Then,*

$$\mathbb{E}\left[\left(I_{a+}^{\alpha}f(b)-{}^{S}I_{a+}^{\alpha}f(b)\right)^{2}\right]=Var({}^{S}I_{a+}^{\alpha}f(b))+(I_{a+}^{\alpha}f(b))^{2}.\tag{3.1}$$

Proof. For simplicity, put $J=I_{a+}^{\alpha}f(b)$ as a deterministic integral and ${}^{S}I={}^{S}I_{a+}^{\alpha}f(b)$ as a stochastic integral. We have

$$\mathbb{E}[(J-{}^{S}I)^{2}]=\mathbb{E}[J^{2}-2J\cdot{}^{S}I+({}^{S}I)^{2}]=J^{2}-2J\mathbb{E}[{}^{S}I]+\mathbb{E}[({}^{S}I)^{2}].$$

Since ${}^{S}I$ is an Itô integral of a deterministic integrand (as f is deterministic here):

$$\mathbb{E}[{}^{S}I]=0.$$

The second moment is the variance:

$$\mathbb{E}[({}^{S}I)^{2}]=\mathrm{Var}({}^{S}I).$$

Now, by substitute back:

$$\mathbb{E}[(J-{}^{S}I)^{2}]=J^{2}-2J\cdot 0+\mathrm{Var}({}^{S}I)=J^{2}+\mathrm{Var}({}^{S}I).\qquad\square$$

Assumptions. Throughout this section we assume that the stochastic process $f:[a,b]\times\Omega\to\mathbb{R}$ satisfies:

- f is $\mathcal{F}_t$-adapted and progressively measurable;
- $\mathbb{E}\left[\int_a^b |f(t)|^2\,dt\right] < \infty$;
- The fractional kernel $(b-t)^{\alpha-1}$ belongs to $L^2([a,b])$, which holds for $\alpha > \frac{1}{2}$.

Under these conditions, the stochastic fractional integral $SI^{\alpha}_{a+}f(b)$ is well-defined, and Itô isometry applies.

Theorem 3. *Let $f \in C^1([a,b])$ with f' convex, $\alpha \in (1/2,1)$, and $B(t)$ a Brownian motion. Then:*

$$\mathbb{E}\left[\left|\tfrac{f(a)+f(b)}{2} - \tfrac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left({}^{S}I^{\alpha}_{a+}f(b) + {}^{S}I^{\alpha}_{b-}f(a)\right)\right|\right] \le \tfrac{b-a}{4}(|f'(a)|+|f'(b)|) + \tfrac{2}{\Gamma(\alpha)}\sqrt{\tfrac{F(b-a)^{2\alpha-1}}{2\alpha-1}}(b-a+2),$$

where

$$F = \max(f(a)^2, f(b)^2) + (b-a)^2(|f'(a)|+|f'(b)|)^2.$$

Proof. Let

$$D = \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left(I^{\alpha}_{a+}f(b) + I^{\alpha}_{b-}f(a)\right),$$

and

$$S = \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left({}^{S}I^{\alpha}_{a+}f(b) + {}^{S}I^{\alpha}_{b-}f(a)\right).$$

By triangular inequality:

$$\left|\frac{f(a)+f(b)}{2} - S\right| \le \underbrace{\left|\frac{f(a)+f(b)}{2} - D\right|}_{(A)} + \underbrace{|D-S|}_{(B)}.$$

In view of (2.3):

$$(A) \le \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right) \le \frac{b-a}{4}(|f'(a)|+|f'(b)|).$$

The last inequality is proved in [3]. On ther hand, the stochastic error term (B) can be written by

$$(B) = \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left|\left(I^{\alpha}_{a+}f(b) - {}^{S}I^{\alpha}_{a+}f(b)\right) + \left(I^{\alpha}_{b-}f(a) - {}^{S}I^{\alpha}_{b-}f(a)\right)\right|.$$

By Proposition 2, we have:

$$\mathbb{E}\left[\left(I^{\alpha}_{a+}f(b) - {}^{S}I^{\alpha}_{a+}f(b)\right)^2\right] = \mathrm{Var}({}^{S}I) + J^2, \tag{3.2}$$

where $J = I^{\alpha}_{a+}f(b)$ and ${}^{S}I = {}^{S}I^{\alpha}_{a+}f(b)$. Now, by Itô Isometry,

$$\mathrm{Var}({}^{S}I) = \frac{1}{\Gamma(\alpha)^2}\int_a^b (b-t)^{2\alpha-2}\mathbb{E}[f(t)^2]dt. \tag{3.3}$$

It is readily seen that the deterministic term can be written by

$$\begin{aligned} J^2 &= \left(I^{\alpha}_{a+}f(b)\right)^2 = \left(\tfrac{1}{\Gamma(\alpha)}\int_a^b (b-\tau)^{\alpha-1}f(\tau)\,d\tau\right)^2 \\ &\leq \tfrac{1}{\Gamma^2(\alpha)}\left(\int_a^b (b-\tau)^{\alpha-1}f(\tau)\,d\tau\right)^2 \\ &\leq \tfrac{1}{\Gamma^2(\alpha)}\max_{t\in[a,b]} f^2(t)\left(\int_a^b (b-\tau)^{\alpha-1}\,d\tau\right)^2 \\ &= \tfrac{(b-a)^{2\alpha}}{\Gamma(\alpha)^2\alpha^2}\max_{t\in[a,b]} f^2(t). \end{aligned} \tag{3.4}$$

Since f' is convex, by Proposition 1 we have

$$|f(t)| \leq \max(|f(a)|, |f(b)|) + (b-a)(|f'(a)| + |f'(b)|).$$

Thus

$$(f(t))^2 \leq 2\left(\max(f(a)^2, f(b)^2) + (b-a)^2(|f'(a)| + |f'(b)|)^2\right). \tag{3.5}$$

Now, combining (3.2) through (3.5), we have

$$\mathbb{E}\left[\left(I^{\alpha}_{a+}f(b) - {}^{S}I^{\alpha}_{a+}f(b)\right)^2\right] \leq \frac{2F(b-a)^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)} + \frac{F(b-a)^{2\alpha}}{\Gamma(\alpha)^2\alpha^2}. \tag{3.6}$$

Since $\alpha^2 \geq 2\alpha - 1$ and $\alpha \in (1/2, 1)$, the inequality (3.6) implies that

$$\mathbb{E}\left[\left(I^{\alpha}_{a+}f(b) - {}^{S}I^{\alpha}_{a+}f(b)\right)^2\right] \leq \frac{2F(b-a)^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)}(b-a+2), \tag{3.7}$$

Thus

$$\mathbb{E}[|D-S|] \leq \frac{2}{\Gamma(\alpha)}\sqrt{\frac{F(b-a)^{2\alpha-1}}{2\alpha-1}(b-a+2)}.$$

Combining (A) and (B):

$$\mathbb{E}\left[\left|\frac{f(a)+f(b)}{2} - S\right|\right] \leq \frac{b-a}{4}(|f'(a)|+|f'(b)|) + \frac{2}{\Gamma(\alpha)}\sqrt{\frac{F(b-a)^{2\alpha-1}}{2\alpha-1}(b-a+2)}. \qquad \square$$

Remark 3. Let $f : [a,b] \times \Omega \to \mathbb{R}$ be an $\mathcal{F}_t$-adapted process. The interchange of the expectation and stochastic fractional integral operators is justified under the following conditions:

1. f is predictable (or progressively measurable) on $[a, b] \times \Omega$;
2. $f \in L^2([a, b] \times \Omega)$, that is, $\int_a^b \mathbb{E}[f(t)^2]\, dt < \infty$;
3. The kernel $(b-t)^{\alpha-1}$ is square integrable, which requires $\alpha > \frac{1}{2}$.

Under these assumptions, the stochastic Fubini theorem allows the exchange

$$\mathbb{E}[SI_{a+}^{\alpha} f(b)] = \frac{1}{\Gamma(\alpha)} \mathbb{E}\left[\int_a^b (b-t)^{\alpha-1} f(t)\, dB(t)\right] = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} \mathbb{E}[f(t)]\, dt.$$

This equality holds because $\mathbb{E}\left[\int_a^b (b-t)^{\alpha-1} f(t)\, dB(t)\right] = 0$ for deterministic kernels and the integrability condition $\int_a^b (b-t)^{\alpha-1} \mathbb{E}|f(t)|\, dt < \infty$ is satisfied.

Lemma 1 (Expectation preserves convexity). *Let $g : [a, b] \times \Omega \to \mathbb{R}$ satisfy:*

1. *For almost every $\omega \in \Omega$, the mapping $t \mapsto g(t, \omega)$ is convex on $[a, b]$;*
2. *$\mathbb{E}[|g(t, \cdot)|] < \infty$ for every $t \in [a, b]$.*

Then $\bar{g}(t) := \mathbb{E}[g(t, \cdot)]$ is finite and convex on $[a, b]$.

Proof. For any $t_1, t_2 \in [a, b]$ and $\lambda \in [0, 1]$, convexity of g yields

$$g(\lambda t_1 + (1-\lambda) t_2, \omega) \leq \lambda g(t_1, \omega) + (1-\lambda) g(t_2, \omega)$$

for almost all ω. Taking expectations and using integrability gives the claim. □

We now explicitly cite this lemma in the proof of the following theorem when invoking the convexity of the expectation.

Theorem 4. *Let $f(t)$ be a stochastic process in $C^1([a, b]; L^2(\Omega))$ whose sample paths are convex almost surely. Then the following inequality holds:*

$$\mathbb{E}\left[f\left(\frac{a+b}{2}\right)\right] \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \cdot \mathbb{E}\left[{}^{S}I_{a+}^{\alpha} f(b)\right] \leq \mathbb{E}\left[\frac{f(a)+f(b)}{2}\right].$$

Proof. The proof of this theorem proceeds in two main steps.

1. We show that the problem can be reduced to proving the corresponding inequality for a deterministic convex function.
2. We prove the inequality for the deterministic case.

Step 1. Reduction to a Deterministic Problem
Since $f(t)$ is a stochastic process with convex sample paths almost surely, we can define a deterministic function $g(t) = \mathbb{E}[f(t)]$. A key property of convex functions is that the expectation preserves convexity. Therefore, $g(t)$ is a convex deterministic function.
Now, let's analyze the terms in the theorem. The left-hand side is $\mathbb{E}\left[f\left(\frac{a+b}{2}\right)\right] = g\left(\frac{a+b}{2}\right)$. For the middle term, we can swap the expectation and the integral by Fubini's theorem we have

$$\mathbb{E}\left[{}^{S}I^{\alpha}_{a+}f(b)\right] = \mathbb{E}\left[\frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}f(t)dt\right] = \frac{1}{\Gamma(\alpha)}\int_a^b (b-t)^{\alpha-1}\mathbb{E}[f(t)]dt = I^{\alpha}_{a+}g(b).$$

Thus, the middle term of the inequality is $\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \cdot I^{\alpha}_{a+}g(b)$. The right-hand side is $\mathbb{E}\left[\frac{f(a)+f(b)}{2}\right] = \frac{\mathbb{E}[f(a)]+\mathbb{E}[f(b)]}{2} = \frac{g(a)+g(b)}{2}$.
Therefore, the stochastic theorem is equivalent to proving the following deterministic Hermite-Hadamard type inequality for a convex function $g(t)$:

$$g\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \cdot I^{\alpha}_{a+}g(b) \leq \frac{g(a)+g(b)}{2}, \tag{3.8}$$

that is true by Theorem 2. □

Corollary 1. *(Limit cases). When $\alpha \to 1$, the fractional kernel and the Gamma factor satisfy $\Gamma(\alpha+1) \to \Gamma(2) = 1$ and $(b-a)^{-\alpha} \to (b-a)^{-1}$. Hence, the stochastic fractional integral reduces to the classical one, and we obtain*

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(t)\,dt \leq \frac{g(a)+g(b)}{2},$$

which is the classical Hermite-Hadamard inequality.
Moreover, if the stochastic term vanishes in mean square (i.e., $\mathrm{Var}(SI^{\alpha}_{a+}f) = 0$), then $SI^{\alpha}_{a+}f$ coincides almost surely with the deterministic fractional integral, and the stochastic inequality reduces to its deterministic form.

Theorem 5. *Let $B(t)$ be a standard Brownian motion. For $\alpha \in (1/2, 1)$, the following inequality holds for the interval $[a, b]$:*

$$\mathbb{E}\left[\left|\frac{B(a)+B(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left({}^{S}I^{\alpha}_{a+}B(b) + {}^{S}I^{\alpha}_{b-}B(a)\right)\right|\right] \leq \sqrt{\mathbb{E}[X^2]}$$

where $\mathbb{E}[X^2]$ is the mean-square error given by

$$\mathbb{E}[X^2] = \frac{3a+b}{4} - 1 + \frac{\Gamma(\alpha+1)^2}{2(b-a)}\left(\frac{1}{\Gamma(\alpha)^2(2\alpha-1)} + \frac{1}{\Gamma(2\alpha)}\right).$$

Proof. Let the two terms be $M = \frac{B(a)+B(b)}{2}$ and $S = \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left({}^S I^\alpha_{a+} B(b) + {}^S I^\alpha_{b-} B(a)\right)$. Our goal is to bound the term $\mathbb{E}[|M-S|]$.
By Jensen's inequality, we know that $\mathbb{E}[|M-S|] \le \sqrt{\mathbb{E}[(M-S)^2]}$. The proof will therefore focus on computing the mean-square error $\mathbb{E}[(M-S)^2]$.
Using the expansion of a squared term, we have

$$\mathbb{E}[(M-S)^2] = \mathbb{E}[M^2] - 2\mathbb{E}[MS] + \mathbb{E}[S^2]. \tag{3.9}$$

Let's break down the stochastic integral term: $S = c(S_L + S_R)$, where $c = \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}$, $S_L = ({}^S I^\alpha_{a+} B)(b)$ and $S_R = ({}^S I^\alpha_{b-} B)(a)$. The integrals are defined as

$$S_L = \frac{1}{\Gamma(\alpha)} \int_a^b (b-\tau)^{\alpha-1} dB(\tau) \quad \text{and} \quad S_R = \frac{1}{\Gamma(\alpha)} \int_a^b (\tau-a)^{\alpha-1} dB(\tau).$$

In order to compute the first term of (3.9), i.e. $\mathbb{E}[M^2]$, using the properties of a standard Brownian motion where $\mathbb{E}[B(t)^2] = t$ and $\mathbb{E}[B(t)B(s)] = \min(t,s)$, we get

$$\begin{aligned}
\mathbb{E}[M^2] &= \mathbb{E}\left[\left(\frac{B(a)+B(b)}{2}\right)^2\right] \\
&= \frac{1}{4}\left(\mathbb{E}[B(a)^2] + 2\mathbb{E}[B(a)B(b)] + \mathbb{E}[B(b)^2]\right) \\
&= \frac{1}{4}(a + 2\min(a,b) + b).
\end{aligned}$$

Since $a < b$, we have $\min(a,b) = a$, which simplifies the expression to:

$$\mathbb{E}[M^2] = \frac{1}{4}(a + 2a + b) = \frac{3a+b}{4}.$$

In order to compute the second term of (3.9), i.e. $-2\mathbb{E}[MS]$, we note this is the cross-correlation term between the trapezoidal approximation and the total stochastic integral.

$$-2\mathbb{E}[MS] = -\mathbb{E}\left[\frac{B(a)+B(b)}{2} \cdot c \cdot (S_L + S_R)\right].$$

Using the properties of stochastic integrals and the independence of increments:

$$\begin{aligned}
-2\mathbb{E}[MS] &= -c\left(\mathbb{E}[B(a)S_L] + \mathbb{E}[B(b)S_L] + \mathbb{E}[B(a)S_R] + \mathbb{E}[B(b)S_R]\right) \\
&= -c\left(0 + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)} + 0 + \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}\right) \\
&= -\left(\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\right)\left(\frac{2(b-a)^\alpha}{\Gamma(\alpha+1)}\right) = -1.
\end{aligned}$$

In order to compute the third term of (3.9), i.e $\mathbb{E}[S^2]$, one note this is the mean-square value of the total stochastic integral. Then,

$$\mathbb{E}[S^2] = c^2\mathbb{E}[(S_L + S_R)^2] = c^2(\mathbb{E}[S_L^2] + 2\mathbb{E}[S_L S_R] + \mathbb{E}[S_R^2]), \tag{3.10}$$

where each term is computed as follows:

- $\mathbb{E}[S_L^2]$: Using the Ito Isometry for the left integral we have

$$\mathbb{E}[S_L^2] = \frac{1}{\Gamma(\alpha)^2}\int_a^b (b-\tau)^{2\alpha-2}d\tau = \frac{(b-a)^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)}.$$

- $\mathbb{E}[S_R^2]$: By symmetry, this is identical to $\mathbb{E}[S_L^2]$.
- $2\mathbb{E}[S_L S_R]$: This is the cross-correlation between the two integrals.

$$2\mathbb{E}[S_L S_R] = \frac{2}{\Gamma(\alpha)^2}\int_a^b (b-\tau)^{\alpha-1}(\tau-a)^{\alpha-1}d\tau \tag{3.11}$$

  Using the Beta function integral identity, the equation (3.11) can be computed by

$$2\mathbb{E}[S_L S_R] = \frac{2}{\Gamma(\alpha)^2}(b-a)^{2\alpha-1}\frac{\Gamma(\alpha)^2}{\Gamma(2\alpha)} = \frac{2(b-a)^{2\alpha-1}}{\Gamma(2\alpha)}.$$

Combining these terms for $\mathbb{E}[S^2]$ in (3.10) we obtain

$$\begin{aligned}
\mathbb{E}[S^2] &= c^2\left(2\frac{(b-a)^{2\alpha-1}}{\Gamma(\alpha)^2(2\alpha-1)} + \frac{2(b-a)^{2\alpha-1}}{\Gamma(2\alpha)}\right)\\
&= \left(\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\right)^2 2(b-a)^{2\alpha-1}\left(\frac{1}{\Gamma(\alpha)^2(2\alpha-1)} + \frac{1}{\Gamma(2\alpha)}\right)\\
&= \frac{\Gamma(\alpha+1)^2}{2(b-a)}\left(\frac{1}{\Gamma(\alpha)^2(2\alpha-1)} + \frac{1}{\Gamma(2\alpha)}\right).
\end{aligned}$$

Summing the three terms gives the complete expression for the mean-square error:

$$\mathbb{E}[X^2] = \frac{3a+b}{4} - 1 + \frac{\Gamma(\alpha+1)^2}{2(b-a)}\left(\frac{1}{\Gamma(\alpha)^2(2\alpha-1)} + \frac{1}{\Gamma(2\alpha)}\right).$$

The final bound is the square root of this expression. □

4. Conclusion

This paper successfully extended a series of important mathematical inequalities into the field of stochastic fractional calculus. Our work provides new tools for understanding and working with systems that have both memory and random behavior. We first developed a way to measure the error when using stochastic fractional integrals to approximate regular integrals of smooth functions. This helps us understand the accuracy of these powerful new tools. We then proved that a well-known inequality for convex functions, the Hermite-Hadamard inequality, also holds for the expected values of convex random processes. This result provides a fundamental new rule for this field. Lastly, we gave a precise formula for the error when using our methods to approximate a standard Brownian motion.

In summary, this work provides fundamental new bounds for stochastic fractional integrals, extending the classical Hermite–Hadamard framework to random processes. Future work could apply these methods to other types of random processes or explore new applications.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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