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Research Article

On the global Italian domination of graphs

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The authors would like to dedicate this paper to Dr. Odile Favaron, in recognition of her outstanding career in graph theory.

Abstract: Let H be a graph with vertex set V. An Italian dominating function (IDF) on H is a function from V to the set $\{0, 1, 2\}$ having the property that any vertex assigned 0 is adjacent to two vertices assigned 1 or one vertex assigned 2. The value $\sum_{x\in V} h(x)$ is called the weight of an IDF h on H. A global Italian dominating function (GIDF) on H is an IDF on H and its complement. The minimum weight of an IDF (resp., GIDF) on H is the Italian (resp., global Italian) domination number of H. In this paper, we establish several relations between the global Italian domination and Italian domination numbers. In particular, we determine the difference between these two parameters of cubic graphs.

Keywords: Italian domination, global Italian domination, cubic graph.

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1. Introduction

Let $H = (V(H), E(H))$ be a graph with vertex set $V(H)$ and edge set $E(H)$. The *open neighborhood* $N_H(u)$ (briefly $N(u)$) of a vertex $u \in V(H)$ is the set of its neighbors,

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while its *closed neighborhood* $N_H[u]$ (briefly $N[u]$) is the set $\{u\} \cup N_H(u)$. We denote by $d(u) = d_H(u) = |N_H(u)|$ the *degree* of a vertex u in H and by $\Delta(H)$ (briefly Δ) the maximum degree among all vertices in H. For a vertex subset U of H, let $N_H(U)$ (briefly $N(U)$), $N_H[U]$ (briefly $N[U]$) and $H[U]$ be the sets $\bigcup_{x\in U} N_H(x)$, $\bigcup_{x\in U} N_H[x]$ and the subgraph induced by U, respectively. For subsets $U_1, U_2 \subseteq V(H)$, let $[U_1, U_2] = \{u_1u_2 \in E(H) : u_1 \in U_1 \text{ and } u_2 \in U_2\}$. For any vertices $u_1, u_2 \in V(H)$, the *distance* $d(u_1, u_2)$ between u_1 and u_2 is the length of a shortest (u_1, u_2)-path in H. The *diameter* $\mathrm{diam}(H)$ of H is defined as $\mathrm{diam}(H) = \max\{d(u_1, u_2) : u_1, u_2 \in V(H)\}$. The *complement* of H is denoted by $\overline{H}$, where $V(\overline{H}) = V(H)$ and $u_1u_2 \in E(\overline{H})$ if and only if $u_1u_2 \notin E(H)$. A *cubic graph* is a graph in which each vertex has degree three. For two subsets $U_1, U_2 \subseteq V(H)$, if $U_2 \subseteq N[U_1]$, then we say that U_1 dominates U_2. For a vertex $u \in V(H)$ and a set $U \subseteq V(H)$, we say that u dominates U if $\{u\}$ dominates U.

A vertex subset S of a graph H with $N[S] = V(H)$ is called a *dominating set* (D-*set*) of H. The *domination number*, $\gamma(H)$ of H is the minimum cardinality of a D-set of H. In [5], a variant of the domination parameters, namely Italian domination, was introduced where the authors called it Roman $\{2\}$-domination. A function h from the vertex set of a graph H to the set $\{0, 1, 2\}$ is called an *Italian dominating function* (IDF) on H if any vertex assigned 0 under h has two neighbors assigned 1 or one neighbor assigned 2. A *global Italian dominating function* (GIDF) on H, which is introduced in [6], is an IDF on both H and $\overline{H}$. Let $\omega(h)$ denote the *weight* $\sum_{x\in V(H)} h(x)$ of an IDF h on H. The minimum weight of an IDF (resp., GIDF) on H is called the *Italian domination number* (ID-number) $\gamma_I(H)$ (resp., *global Italian domination number* (GID-number) $\gamma_{gI}(H)$) of H. An IDF (resp., GIDF) on H having weight $\gamma_I(H)$ (resp., $\gamma_{gI}(H)$) is a $\gamma_I(H)$-*function* (resp., $\gamma_{gI}(H)$-*function*). For any $\gamma_I(H)$-function h, we use W_i^h to denote the set $\{x \in V(H) : h(x) = i\}$, where $i \in \{0, 1, 2\}$, based on which we may use the notation (W_0^h, W_1^h, W_2^h) to denote the function h. For a sake of simplicity, we write (W_0, W_1, W_2) rather than (W_0^h, W_1^h, W_2^h) when the $\gamma_I(H)$-function h is clear from the context. The concept on Italian domination was studied further in, for example, [1–4, 7–11].

In order to derive relations between GID-number and ID-number, we introduce a notation that is the key point for our following discussion. For arbitrary $\gamma_I(H)$-function $h = (W_0, W_1, W_2)$, let

$$X = \{w \in W_0 : |W_1 \backslash N(w)| \le 1 \text{ and } W_2 \subseteq N(w)\}.$$

We will utilize the following two well-known results.

Theorem 1 ([5]). *For any graph H, $\gamma_I(H) \le 2\gamma(H)$.*

Theorem 2 ([5]). *If H is a graph on n vertices, then $\gamma_I(H) \ge 2n/(\Delta + 2)$.*

2. Results for general graphs

Our aim in this section is to give some relations between GID-number and ID-number of general graphs.

It is known [6] that each graph H with diameter three or four satisfies $\gamma_{gI}(H) \leq \gamma_I(H)+4$. We now make a slight improvement on this bound for graphs with diameter four.

Theorem 3. *For any graph H with diameter four, $\gamma_{gI}(H) \leq \gamma_I(H) + 2$.*

Proof. Let $g = (W_0, W_1, W_2)$ be a $\gamma_I(H)$-function and let x_1 and x_2 be vertices of H with $d(x_1, x_2) = 4$. If $x_1 \in W_2$ (the case $x_2 \in W_2$ is similar), then the function η given by $\eta(x_2) = 2$ and $\eta(z) = g(z)$ for any $z \in V(H)\backslash\{x_2\}$, is a GIDF on H with $\omega(\eta) \leq \omega(g) + 2$. If $x_1 \in W_0$ (the case $x_2 \in W_0$ is similar), then x_1 has a neighbor w_1 in W_2 or two neighbors w_2 and w_3 in W_1. Clearly $d(x_2, w_i) \geq 3$ and so each vertex z of H is not adjacent to both w_i and x_2 for each i. Therefore, the function η given by $\eta(x_2) = 2$ and $\eta(z) = g(z)$ for each $z \in V(H)\backslash\{x_2\}$, is a GIDF on H with $\omega(\eta) \leq \omega(g) + 2$. Finally, assume that $x_1, x_2 \in W_1$. Then the function η given by $\eta(x_1) = \eta(x_2) = 2$ and $\eta(z) = g(z)$ for each $z \in V(H)\backslash\{x_1, x_2\}$, is a GIDF on H with $\omega(\eta) = \omega(g)+2$. All in all, we deduce that $\gamma_{gI}(H) \leq \omega(\eta) \leq \omega(g)+2 = \gamma_I(H)+2$. □

Theorem 4. *Let H be a graph. Then $\gamma_{gI}(H) \leq \gamma_I(H) + \gamma_I(\overline{H})$.*

Proof. Let $g = (W_0^g, W_1^g, W_2^g)$ be a $\gamma_I(H)$-function and let $h = (W_0^h, W_1^h, W_2^h)$ be a $\gamma_I(\overline{H})$-function. One can easily verify that the function η given by $\eta(x) = 0$ for any $x \in W_0^g \cap W_0^h$, $\eta(x) = 1$ for any $x \in (W_0^g \cap W_1^h) \cup (W_1^g \cap W_0^h) \cup (W_1^g \cap W_1^h)$ and $\eta(x) = 2$ for any $x \in W_2^g \cup W_2^h$, is a GIDF on H. This forces

$$\begin{aligned}
\gamma_{gI}(H) &\leq \omega(\eta) \\
&= |(W_0^g \cap W_1^h) \cup (W_1^g \cap W_0^h) \cup (W_1^g \cap W_1^h)| + 2|W_2^g \cup W_2^h| \\
&\leq |W_0^g \cap W_1^h| + (|W_1^g \cap W_0^h| + |W_1^g \cap W_1^h|) + 2(|W_2^g| + |W_2^h|) \\
&\leq |W_1^h| + |W_1^g| + 2(|W_2^g| + |W_2^h|) \\
&= (|W_1^g| + 2|W_2^g|) + (|W_1^h| + 2|W_2^h|) \\
&= \gamma_I(H) + \gamma_I(\overline{H}),
\end{aligned}$$

as desired. □

Recall that $X = \{w \in W_0 : |W_1\backslash N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\}$, where $h = (W_0, W_1, W_2)$ is a $\gamma_I(H)$-function.

Lemma 1. *For any graph H with a $\gamma_I(H)$-function $\eta = (W_0, W_1, W_2)$, if $\gamma_{gI}(H) \neq \gamma_I(H)$, then $X \neq \emptyset$.*

Proof. Note that η is $\gamma_I(H)$-function. Thus if η is an IDF on $\overline{H}$, then η is a GIDF on H with $\gamma_{gI}(H) \leq \omega(\eta) = \gamma_I(H)$. Also, since $\gamma_{gI}(H) \geq \gamma_I(H)$, it follows that $\gamma_{gI}(H) = \gamma_I(H)$, a contradiction. Therefore η is not an IDF on $\overline{H}$, leading that there must exist $w \in W_0$ with $|W_1 \cap N_{\overline{H}}(w)| \leq 1$ and $W_2 \cap N_{\overline{H}}(w) = \emptyset$. This forces $|W_1 \backslash N_H(w)| \leq 1$ and $W_2 \subseteq N_H(w)$. Hence $w \in X$, that is, $X \neq \emptyset$. ☐

Theorem 5. *For any graph H with a $\gamma_I(H)$-function $h = (W_0, W_1, W_2)$, if $\gamma_{gI}(H) = \gamma_I(H) + k$ $(k \geq 3)$, then*

(*a*) *Every subset of $V(H)$ that dominates X has cardinality at least $\lceil k/2 \rceil$ in $\overline{H}$.*

(*b*) *X is a D-set of H and $|X| \geq \gamma_I(H)/2$.*

Proof. Note that $\gamma_{gI}(H) = \gamma_I(H) + k$ $(k \geq 3)$. Thus by Lemma 1, $X = \{w \in W_0 : |W_1 \backslash N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let S be an arbitrary subset of $V(H)$ such that $X \subseteq N_{\overline{H}}[S]$. One can easily observe that the function η given by $\eta(z) = 2$ for every $z \in S$ and $\eta(z) = h(z)$ for each $z \in V(H) \backslash S$, is a GIDF on H and thus

$$\gamma_I(H) + k = \gamma_{gI}(H) \leq \omega(\eta) = (\omega(h) - \sum_{z \in S} h(z)) + \sum_{z \in S} \eta(z)$$

$$\leq \omega(h) + 2|S| = \gamma_I(H) + 2|S|,$$

implying that $2|S| \geq k$. Moreover, since $|S|$ is an integer, we have $|S| \geq \lceil k/2 \rceil$, implying that (a) is true. Further, since $k \geq 3$, we have that the set S has at least $\lceil k/2 \rceil \geq 2$ vertices and so every vertex in $V(\overline{H}) \backslash X$ is not adjacent to all vertices of X in $\overline{H}$. As a result, $V(H) = N_H[X]$. This forces that X is a D-set of H and so by Theorem 1, $|X| \geq \gamma(H) \geq \gamma_I(H)/2$, that is, (b) holds. ☐

Theorem 6. *For any graph H with $\gamma_I(H) \geq 3$,*

$$\gamma_{gI}(H) \leq \gamma_I(H) + 2\left\lceil \frac{2\Delta - \gamma_I(H) + 2}{\gamma_I(H) - 2} \right\rceil + 2.$$

Proof. Since $\gamma_I(H) \geq 3$, we have

$$\lceil (2\Delta - \gamma_I(H) + 2)/(\gamma_I(H) - 2) \rceil = \lceil 2\Delta/(\gamma_I(H) - 2) \rceil - 1 \geq -1.$$

It is clear to observe that $\gamma_{gI}(H) \leq \gamma_I(H) + 2\lceil (2\Delta - \gamma_I(H) + 2)/(\gamma_I(H) - 2) \rceil + 2$ if $\gamma_{gI}(H) = \gamma_I(H)$. Next, suppose that $\gamma_{gI}(H) \geq \gamma_I(H) + 1$. Let $h = (W_0, W_1, W_2)$ be a $\gamma_I(H)$-function. By Lemma 1, $X = \{w \in W_0 : |W_1 \backslash N_H(w)| \leq 1$ and $W_2 \subseteq$

$N_H(w)\} \neq \emptyset$. Let w be a vertex of X and $A = N_H(w) \cap X$. Note that $X \subseteq W_0$, $|W_1 \backslash N_H(w)| \leq 1$ and $W_2 \subseteq N_H(w)$. Thus

$$\begin{aligned}
|A| &= |N_H(w) \cap X| \\
&\leq |N_H(w) \cap W_0| \\
&= |N_H(w)| - |N_H(w) \cap W_1| - |N_H(w) \cap W_2| \\
&= d_H(w) - (|W_1| - |W_1 \backslash N_H(w)|) - |W_2| \\
&\leq \Delta - (|W_1| - 1) - |W_2| \\
&= \Delta - (|W_1| + |W_2|) + 1.
\end{aligned}$$

Moreover, since $W_2 \cup W_1$ is a D-set of H, this forces $\gamma(H) \leq |W_1| + |W_2|$ and hence

$$|A| \leq \Delta - (|W_1| + |W_2|) + 1 \leq \Delta - \gamma(H) + 1. \tag{2.1}$$

Now suppose that $A = N_H(w) \cap X = \emptyset$. Then the function η given by $\eta(w) = 2$ and $\eta(z) = h(z)$ for each $z \in V(H) \backslash \{w\}$, is a GIDF on H, implying that

$$\gamma_{gI}(H) \leq \omega(\eta) = (\omega(h) - h(w)) + \eta(w) = \gamma_I(H) + 2. \tag{2.2}$$

In addition, since $\gamma_I(H) \geq 3$ and $\Delta - \gamma(H) + 1 \geq |A| = 0$ by (2.1), we get $\lceil (\Delta - \gamma(H) + 1)/(\gamma_I(H) - 2) \rceil \geq 0$. Thus by Theorem 1 and (2.2),

$$\begin{aligned}
\gamma_{gI}(H) &\leq \gamma_I(H) + 2 \\
&\leq \gamma_I(H) + 2\lceil (\Delta - \gamma(H) + 1)/(\gamma_I(H) - 2) \rceil + 2 \\
&\leq \gamma_I(H) + 2\lceil (2\Delta - 2\gamma(H) + 2)/(\gamma_I(H) - 2) \rceil + 2 \\
&\leq \gamma_I(H) + 2\lceil (2\Delta - \gamma_I(H) + 2)/(\gamma_I(H) - 2) \rceil + 2.
\end{aligned}$$

Next suppose that $A = N_H(w) \cap X \neq \emptyset$. By Theorem 1, $\gamma(H) \geq \gamma_I(H)/2 \geq 3/2$. This forces $\gamma(H) \geq 2$. We now choose k disjoint subsets $A_1, A_2, \ldots, A_k$ of the set A as follows:

(a) If $|A| \leq \gamma(H) - 1$, then set $k = 1$ and $A_k = A$, and if $|A| > \gamma(H) - 1$, then let $B_1 = A$.

(b) If $|B_i| > \gamma(H) - 1$, then let $A_i \subseteq B_i$ with $|A_i| = \gamma(H) - 1$ and let $B_{i+1} = B_i \backslash A_i$.

(c) If $|B_{i+1}| \leq \gamma(H) - 1$, then set $k = i + 1$ and $A_k = B_{i+1}$. Otherwise, increment i and return to Step (b).

It is not complicated to check that $A = A_1 \cup A_2 \cup \cdots \cup A_k$. Since $|A_j| \leq \gamma(H) - 1$ for each $j \in \{1, 2, \ldots, k\}$, we have that A_j is not a D-set of H and thus there must exist some $a_j \in V(H) \setminus A_j$ with $A_j \cap N_H(a_j) = \emptyset$, implying that $A_j \subseteq N_{\overline{H}}(a_j)$. Let $S = \bigcup_{j=1}^{k} \{a_j\}$. Then

$$|S| \leq k = \lceil |A|/(\gamma(H) - 1) \rceil. \tag{2.3}$$

Observe that the function η given by $\eta(z) = 2$ for each $z \in \{w\} \cup S$ and $\eta(z) = h(z)$ for all other vertices z of H, is a GIDF on H. Thus by (2.1), (2.3) and Theorem 1,

$$\begin{aligned}
\gamma_{gI}(H) &\leq \omega(\eta) \\
&= \left(\omega(h) - \sum_{z \in \{w\} \cup S} h(z) \right) + \sum_{z \in \{w\} \cup S} \eta(z) \\
&\leq \omega(h) + 2|S| + 2 \\
&\leq \gamma_I(H) + 2\lceil |A|/(\gamma(H) - 1) \rceil + 2 \\
&= \gamma_I(H) + 2\lceil 2|A|/(\gamma_I(H) - 2) \rceil + 2 \\
&\leq \gamma_I(H) + 2\lceil 2(\Delta - \gamma(H) + 1)/(\gamma_I(H) - 2) \rceil + 2 \\
&\leq \gamma_I(H) + 2\lceil (2\Delta - \gamma_I(H) + 2)/(\gamma_I(H) - 2) \rceil + 2,
\end{aligned}$$

as desired. □

Theorem 7. *If H is a graph on $n \geq \Delta^2 + 3$ vertices, then $\gamma_{gI}(H) = \gamma_I(H)$.*

Proof. By contradiction, suppose that $\gamma_{gI}(H) \neq \gamma_I(H)$. Let $h = (W_0, W_1, W_2)$ be an IDF on H with weight $\gamma_I(H)$. By Lemma 1, $X = \{w \in W_0 : |W_1 \setminus N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let $w \in X$. Then $|W_1 \setminus N(w)| \leq 1$ and $W_2 \subseteq N(w)$.

First, assume that $|W_1 \setminus N(w)| = 0$. One can check that $W_2 \cup W_1 \subseteq N(w)$. Thus $|W_2 \cup W_1| \leq \Delta$ and any vertex in $W_2 \cup W_1$ has at most $\Delta - 1$ neighbors in $W_0 \setminus N[w]$. Moreover, since $W_2 \cup W_1$ is a D-set of H, this forces that $W_2 \cup W_1$ dominates $W_0 \setminus N[w]$, implying that $|W_0 \setminus N[w]| \leq (\Delta - 1)|W_2 \cup W_1| \leq (\Delta - 1)\Delta$. Note that $|(W_2 \cup W_1) \setminus N[w]| = 0$ since $W_2 \cup W_1 \subseteq N(w)$. Therefore,

$$\begin{aligned}
n &= |N[w]| + |V(H) \setminus N[w]| \\
&= |N[w]| + (|(W_2 \cup W_1) \setminus N[w]| + |W_0 \setminus N[w]|) \\
&= |N[w]| + |W_0 \setminus N[w]| \\
&\leq (\Delta + 1) + (\Delta - 1)\Delta \\
&= \Delta^2 + 1,
\end{aligned}$$

a contradiction.

Second, assume that $|W_1 \setminus N(w)| = 1$. Note that $W_2 \subseteq N(w)$. Thus

$$\begin{aligned}
|W_1| + |W_2| &= |W_1 \cup W_2| \\
&= |(W_1 \cup W_2) \cap N(w)| + |(W_1 \cup W_2) \setminus N(w)|
\end{aligned}$$

$$\begin{aligned} &= |(W_1 \cup W_2) \cap N(w)| + |W_1 \backslash N(w)| \\ &\leq |N(w)| + 1 \\ &\leq \Delta + 1. \end{aligned} \tag{2.4}$$

Now let $W_1 \backslash N(w) = \{v\}$. Moreover, since $w \in X$, this forces that w is adjacent to any vertex belonging to $W_2 \cup (W_1 \backslash \{v\})$ in H and so each vertex of $W_2 \cup (W_1 \backslash \{v\})$ has at most $\Delta - 1$ neighbors in $W_0 \backslash N[w]$. Thus by (2.4),

$$\begin{aligned} |(W_0 \backslash N[w]) \cap N((W_1 \backslash \{v\}) \cup W_2)| &\leq (\Delta - 1)|(W_1 \backslash \{v\}) \cup W_2| \\ &= (\Delta - 1)(|W_2| + |W_1| - 1) \\ &\leq \Delta(\Delta - 1). \end{aligned} \tag{2.5}$$

Since h is an IDF on H, every vertex of W_0 is adjacent to one vertex in W_2 or two vertices in W_1. Hence every neighbor of v in W_0 is adjacent to some vertex in $(W_1 \backslash \{v\}) \cup W_2$, implying that $N(v) \cap W_0 \subseteq N\big((W_1 \backslash \{v\}) \cup W_2\big) \cap W_0$. Thus $(W_0 \backslash N[w]) \cap N\big(W_1 \cup W_2\big) = (W_0 \backslash N[w]) \cap N\big((W_1 \backslash \{v\}) \cup W_2\big)$. Furthermore, since $W_1 \cup W_2$ is a D-set of H, this forces $W_0 \backslash N[w] \subseteq N(W_1 \cup W_2)$. Therefore, by (2.5),

$$\begin{aligned} |W_0 \backslash N[w]| &= |(W_0 \backslash N[w]) \cap N(W_1 \cup W_2)| \\ &= |(W_0 \backslash N[w]) \cap N\big((W_1 \backslash \{v\}) \cup W_2\big)| \\ &\leq \Delta(\Delta - 1) \\ &= \Delta^2 - \Delta. \end{aligned} \tag{2.6}$$

Since $W_2 \subseteq N(w)$ and $W_1 \backslash N(w) = \{v\}$, it follows from (2.6) that

$$\begin{aligned} n &= |N[w]| + |V(H) \backslash N[w]| \\ &= |N[w]| + \big(|W_2 \backslash N[w]| + |W_1 \backslash N[w]| + |W_0 \backslash N[w]|\big) \\ &= |N[w]| + |W_1 \backslash N[w]| + |W_0 \backslash N[w]| \\ &\leq (\Delta + 1) + 1 + (\Delta^2 - \Delta) \\ &= \Delta^2 + 2, \end{aligned}$$

a contradiction, and this complete our proof.

□

Next we demonstrate that the condition $n \geq \Delta^2 + 3$ in Theorem 7 is optimal. To show the optimality, we consider a graph H obtained from $\Delta \geq 4$ copies of stars $K_{1,\Delta-1}$, say $S_1, S_2, \ldots, S_\Delta$ centred at $x_1, x_2, \ldots, x_\Delta$ respectively, by adding a new vertex x and the edge xx_i for each $i \in \{1, 2, \ldots, \Delta\}$, and attaching a pendant edge at a unique leaf in S_1. One can easily see that H has $\Delta^2 + 2$ vertices and has a unique $\gamma_I(H)$-function which is not a GIDF on H. Thus $\gamma_{gI}(H) \neq \gamma_I(H)$. In fact, we have $\gamma_{gI}(H) = \gamma_I(H) + 1$.

3. Results for cubic graphs

Our aim in the section is to derive the difference between GID-number and ID-number for cubic graphs.

Lemma 2. *If H is a cubic graph on n vertices with $\gamma_{gI}(H) \geq \gamma_I(H)+1$, then $n \leq 10$. In particular, $n \in \{4,6,8,10\}$.*

Proof. By Theorem 7, $n \leq \Delta^2+2 = 11$. Moreover, since H is cubic, it follows from Euler's handshaking lemma that $2|E(H)| = \sum_{z\in V(H)} d(z) = 3n$. This forces that n is even and so $n \in \{4,6,8,10\}$. □

Lemma 3. *Let H be a graph with $\gamma_{gI}(H) \geq \gamma_I(H)+1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H. Then $\gamma_I(H) \leq d(w) + |W_2| + 1$, where $w \in X$.*

Proof. Since $w \in X$, we have $|W_1 \backslash N(w)| \leq 1$ and $W_2 \subseteq N(w)$. Therefore

$$\begin{aligned} d(w) &= |N(w)\cap W_0| + |N(w)\cap W_1| + |N(w)\cap W_2| \\ &\geq |N(w)\cap W_1| + |N(w)\cap W_2| \\ &= (|W_1| - |W_1\backslash N(w)|) + |W_2| \\ &\geq (|W_1|-1) + |W_2| \\ &= \gamma_I(H) - |W_2| - 1, \end{aligned}$$

as desired. □

Lemma 4. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H)+1$. Then $\gamma_I(H) \leq 7$.*

Proof. Let $h = (W_0, W_1, W_2)$ be a minimum IDF on H. Note that $\gamma_{gI}(H) \geq \gamma_I(H)+1$. Thus by Lemma 1, $X = \{w \in W_0 : |W_1\backslash N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let w_0 be a vertex of X. Clearly $W_2 \subseteq N(w_0)$. Thus by Lemma 3, $\gamma_I(H) \leq d(w_0)+|W_2|+1 \leq 3+|N(w_0)|+1 = 7$. □

Lemma 5. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H)+1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H. If $\gamma_I(H) = 7$, then $|W_2| = 3$ and $|W_1| = 1$.*

Proof. By Lemma 1, $X = \{w \in W_0 : |W_1\backslash N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let w_0 be a vertex of X. By Lemma 3, $|W_2| \geq \gamma_I(H) - d(w_0) - 1 = 3$. Moreover, since $|W_2| = (\gamma_I(H) - |W_1|)/2 \leq 7/2$ and $|W_2|$ is an integer, we have $|W_2| = 3$ and so $|W_1| = \gamma_I(H) - 2|W_2| = 1$. □

Lemma 6. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H)+1$. Then $\gamma_I(H) \leq 6$.*

Proof. By Lemma 2, H has order $n \in \{4,6,8,10\}$. Let $v \in V(H)$ and $N_H(v) = \{v_1, v_2, v_3\}$. If $n \in \{4,6,8\}$, then the function η given by $\eta(v) = 2$, $\eta(v_i) = 0$ for each $i \in \{1,2,3\}$ and $\eta(z) = 1$ for all other vertices z of H, is an IDF on H, leading that $\gamma_I(H) \leq 2 + (n-4) \leq 6$. Now let $n = 10$ and let $Y = V(H) \backslash \{v, v_1, v_2, v_3\} = \{u_1, u_2, \ldots, u_6\}$. Noting that H is cubic, we have $|[Y, N_H(v)]| \leq 6$ and $|[\{v\}, N_H(v)]| = 3$, leading that

$$\sum_{i=1}^{6} d_{H[Y]}(u_i) \geq \sum_{z \in V(H)} d_H(z) - 2|[Y, N_H(v)]| - 2|[\{v\}, N_H(v)]| \geq 30 - 12 - 6 = 12.$$

This forces $\Delta(H[Y]) \geq 2$. Wlog, assume that u_1 is adjacent to u_2 and u_3 in H. One can check that the mapping η given by $\eta(u_1) = \eta(v) = 2$, $\eta(z) = 0$ for each $z \in \{v_1, v_2, v_3, u_2, u_3\}$ and $\eta(z) = 1$ for all other vertices z of H, is an IDF on H, leading that $\gamma_I(H) \leq 7$. If $\gamma_I(H) = 7$, then η is a $\gamma_I(H)$-function with $W_2^\eta = \{u_1, v\}$, a contradiction to Lemma 5. Thus $\gamma_I(H) \leq 6$. □

By applying a similar approach as described in the proof of Lemma 5, we can get the next two results.

Lemma 7. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H. If $\gamma_I(H) = 6$, then $|W_2| = 3$ and $|W_1| = 0$, or $|W_2| = |W_1| = 2$.*

Lemma 8. *Let H be a cubic graph with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H. If $\gamma_I(H) = 5$, then $|W_2| = 2$ and $|W_1| = 1$, or $|W_2| = 1$ and $|W_1| = 3$.*

Lemma 9. *Let H be a cubic graph on n vertices with $\gamma_{gI}(H) \geq \gamma_I(H) + 1$ and let $h = (W_0, W_1, W_2)$ be a minimum IDF on H. Then*

$$n \leq \begin{cases} 1 + 3|W_2| + |W_1|, & \text{if } |W_1| \leq 1, \\ 1 + 3|W_2| + 2|W_1|, & \text{if } |W_1| \geq 2. \end{cases}$$

Proof. By Lemma 1, $X = \{w \in W_0 : |W_1 \backslash N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let v_0 be a vertex of X, $W_{01} = \{z \in W_0 \backslash \{v_0\} : |N(z) \cap W_2| \geq 1\}$ and let $W_{02} = \{z \in W_0 \backslash \{v_0\} : |N(z) \cap W_1| \geq 2\}$. Clearly $W_2 \subseteq N(v_0)$. Moreover, since H is cubic, we have that any vertex of W_2 is adjacent to at most two vertices of $W_0 \backslash \{v_0\}$ in H and hence $|W_{01}| \leq 2|W_2|$. Furthermore, by the definition of $\gamma_I(H)$-function, $W_0 \backslash \{v_0\} = W_{01} \cup W_{02}$. Hence

$$|W_0| = 1 + |W_0 \backslash \{v_0\}| = 1 + |W_{01} \cup W_{02}| \leq 1 + |W_{01}| + |W_{02}| \leq 1 + 2|W_2| + |W_{02}|. \quad (3.1)$$

If $|W_1| \leq 1$, then clearly $|W_{02}| = 0$ and hence by (3.1),

$$n = |W_0| + |W_1| + |W_2| \leq (1 + 2|W_2| + |W_{02}|) + |W_1| + |W_2| = 1 + |W_1| + 3|W_2|,$$

as desired. We next assume that $|W_1| \geq 2$. Since $v_0 \in X$, we have $|W_1 \backslash N(v_0)| \leq 1$. Moreover, since $W_{02} = \{z \in W_0 \backslash \{v_0\} : |N(z) \cap W_1| \geq 2\}$ and H is cubic, we obtain

$$\begin{aligned} |W_{02}| &\leq \frac{1}{2}|[W_{02}, W_1]| \\ &\leq \frac{1}{2}\Big(\sum_{z \in W_1} d(z) - |W_1 \cap N(v_0)|\Big) \\ &= \frac{1}{2}\big(3|W_1| - (|W_1| - |W_1 \backslash N(v_0)|)\big) \\ &\leq \frac{1}{2}(2|W_1| + 1) = |W_1| + \frac{1}{2}. \end{aligned}$$

Noting that $|W_{02}|$ and $|W_1|$ are integers, we obtain $|W_{02}| \leq |W_1|$. Thus by (3.1),

$$\begin{aligned} n = |W_0| + |W_1| + |W_2| &\leq (1 + 2|W_2| + |W_{02}|) + |W_1| + |W_2| \\ &\leq (1 + 2|W_2| + |W_1|) + |W_1| + |W_2| = 1 + 3|W_2| + 2|W_1|, \end{aligned}$$

which completes our proof. □

Proposition 1. *Let H be a cubic graph on 10 vertices. Then $\gamma_{gI}(H) = \gamma_I(H)$.*

Proof. Suppose that $\gamma_{gI}(H) \neq \gamma_I(H)$ and let $h = (W_0, W_1, W_2)$ be a $\gamma_I(H)$-function. By Lemma 1, $X = \{w \in W_0 : |W_1 \backslash N(w)| \leq 1 \text{ and } W_2 \subseteq N(w)\} \neq \emptyset$. Let v_0 be a vertex of X. Clearly $v_0 \in W_0$, $|W_1 \backslash N(v_0)| \leq 1$ and $W_2 \subseteq N(v_0)$. Since $n = 10$ and $\Delta = 3$, it follows from Theorem 2 that $\gamma_I(H) \geq 2n/(\Delta+2) = 4$. On the other hand, by Lemma 6, $\gamma_I(H) \leq 6$. Therefore $\gamma_I(H) \in \{4, 5, 6\}$. Let $V(H) \backslash \{v_0\} = \{v_i : 1 \leq i \leq 9\}$.

Case 1. $\gamma_I(H) = 4$.

Noting that $2|W_2| + |W_1| = \gamma_I(H) = 4$, we have that $|W_1| = 0$ and $|W_2| = 2$, or $|W_1| = 2$ and $|W_2| = 1$, or $|W_1| = 4$ and $|W_2| = 0$. By Lemma 9, if $|W_1| = 0$ and $|W_2| = 2$, then $n \leq 1 + 3|W_2| + |W_1| = 7$; if $|W_1| = 2$ and $|W_2| = 1$, then $n \leq 1+3|W_2|+2|W_1| = 8$ and if $|W_1| = 4$ and $|W_2| = 0$, then $n \leq 1+3|W_2|+2|W_1| = 9$. In each case, we have a contradiction to the assumption $n = 10$.

Case 2. $\gamma_I(H) = 5$.

By Lemma 8, we have two possibilities $|W_1| = 1$ and $|W_2| = 2$, or $|W_1| = 3$ and $|W_2| = 1$. If $|W_1| = 1$ and $|W_2| = 2$, then by Lemma 9, $n \leq 1 + 3|W_2| + |W_1| = 8$, a contradiction. Therefore $|W_1| = 3$ and $|W_2| = 1$. Let $W_2 = \{v_1\}$ and let $W_1 = \{v_2, v_3, v_4\}$. Then $W_0 \backslash \{v_0\} = \{v_i : 5 \leq i \leq 9\}$. Since $d(v_0) = 3$, $|W_1 \backslash N(v_0)| \leq 1$ and $\{v_1\} = W_2 \subseteq N(v_0)$, we have

$$|N(v_0) \cap W_1| = 2. \tag{3.2}$$

Therefore we can assume that $N(v_0) = \{v_1, v_2, v_3\}$. Furthermore, since $d(v_1) = 3$, we have $|N(v_1) \cap (W_0 \backslash \{v_0\})| \leq 2$.
First, suppose that $|N(v_1) \cap (W_0 \backslash \{v_0\})| \leq 1$. Let $v_6, v_7, v_8, v_9 \notin N(v_1) \cap (W_0 \backslash \{v_0\})$. It is evident from the definition of $\gamma_I(H)$-function that $|N(v_i) \cap W_1| \geq 2$ for each $i \in \{6, 7, 8, 9\}$. Thus by (3.2),

$$\begin{aligned} d(v_2) + d(v_3) + d(v_4) &\geq |[\{v_0\}, W_1]| + |[\{v_6, v_7, v_8, v_9\}, W_1]| \\ &\geq |N(v_0) \cap W_1| + 2|\{v_6, v_7, v_8, v_9\}| \\ &= 10, \end{aligned}$$

a contradiction.
Second, suppose that $|N(v_1) \cap (W_0 \backslash \{v_0\})| = 2$. Let $N(v_1) \cap (W_0 \backslash \{v_0\}) = \{v_5, v_6\}$. Note that $|N(v_i) \cap \{v_2, v_3, v_4\}| = |N(v_i) \cap W_1| \geq 2$ for each $i \in \{7, 8, 9\}$. Moreover, since H is cubic and $N(v_0) = \{v_1, v_2, v_3\}$, the function η given by $\eta(v_i) = 1$ for each $i \in \{2, 3, 4, 5, 6, \}$ and $\eta(v_i) = 0$ for each $i \in \{0, 1, 7, 8, 9, \}$ is a GIDF on H, leading that $\gamma_{gI}(H) \leq 5$. Moreover, since $\gamma_{gI}(H) \geq \gamma_I(H) = 5$, implying that $\gamma_{gI}(H) = 5 = \gamma_I(H)$, a contradiction.

Case 3. $\gamma_I(H) = 6$.

It follows from Lemma 7 that $|W_1| = 0$ and $|W_2| = 3$, or $|W_1| = |W_2| = 2$. First, assume that $|W_1| = 0$ and $|W_2| = 3$. Let $W_2 = \{v_1, v_2, v_3\}$. It is clear that $W_0 \backslash \{v_0\} = \{v_4, v_5, \ldots, v_9\}$.

Claim. $N(W_2) \cap (W_0 \backslash \{v_0\}) = W_0 \backslash \{v_0\}$ and any vertex in W_2 is adjacent to exactly two vertices of $W_0 \backslash \{v_0\}$.

Proof of Claim. By the definition of $\gamma_I(H)$-function, any vertex of $W_0 \backslash \{v_0\}$ has at least one neighbor in W_2, implying that $(W_0 \backslash \{v_0\}) \cap N(W_2) = W_0 \backslash \{v_0\}$ and $|[W_0 \backslash \{v_0\}, W_2]| \geq |W_0 \backslash \{v_0\}| = 6$. Also, since H is cubic and $W_2 \subseteq N(v_0)$, we have $|[W_2, W_0 \backslash \{v_0\}]| \leq 2|W_2| = 6$. Thus $|[W_2, W_0 \backslash \{v_0\}]| = 6$. This forces that any vertex in W_2 is adjacent to exactly two vertices in $W_0 \backslash \{v_0\}$. ■

By Claim, we let $N(v_1) \cap (W_0 \backslash \{v_0\}) = \{v_4, v_5\}$, $N(v_2) \cap (W_0 \backslash \{v_0\}) = \{v_6, v_7\}$ and let $N(v_3) \cap (W_0 \backslash \{v_0\}) = \{v_8, v_9\}$. One can verify that the function η given by $\eta(v_1) = 2$, $\eta(v_i) = 1$ for each $i \in \{6, 7, 8, 9\}$ and $\eta(v_i) = 0$ for each $i \in \{0, 2, 3, 4, 5\}$, is a GIDF on H, leading that $\gamma_{gI}(H) \leq \omega(\eta) = 6$. Moreover, since $\gamma_{gI}(H) \geq \gamma_I(H) = 6$, this forces $\gamma_{gI}(H) = 6 = \gamma_I(H)$, a contradiction.
Second, assume that $|W_1| = |W_2| = 2$. Let $W_2 = \{v_1, v_2\}$ and let $W_1 = \{v_3, v_4\}$. Let $U = (W_0 \backslash \{v_0\}) \cap N(v_3) \cap N(v_4)$. If $U = \emptyset$, then by the definition of $\gamma_I(H)$-function, each vertex of $W_0 \backslash \{v_0\}$ has a neighbor in W_2. Furthermore, since $W_2 \subseteq N(v_0)$, we obtain $d(v_1) + d(v_2) \geq 7$, a contradiction. Suppose next that $U \neq \emptyset$. Since $|W_1 \backslash N(v_0)| \leq 1$ and $W_2 \subseteq N(v_0)$, we have $|N(v_0) \cap W_1| = 1$ (noting that H is cubic). Let $N(v_0) \backslash \{v_1, v_2\} = \{v_3\}$. Thus v_3 has at most two neighbors in $W_0 \backslash \{v_0\}$, implying that $|U| \in \{1, 2\}$.

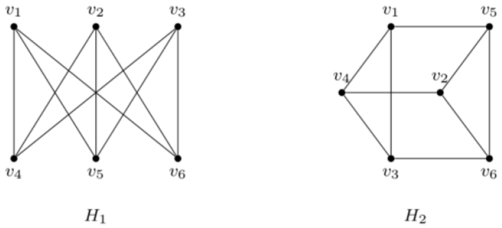

Figure 1. **Two non-isomorphic cubic graphs H_1 and H_2 of order 6.**

Now suppose that $|U| = 1$. Let $U = \{v_9\}$. Clearly $W_0 \backslash (\{v_0\} \cup U) = \{v_5, v_6, v_7, v_8\}$. We deduce from the method analogous to the proof of Claim that $N(W_2) \cap (W_0 \backslash (\{v_0\} \cup U)) = W_0 \backslash (\{v_0\} \cup U)$ and any vertex in W_2 is adjacent to exactly two vertices of $W_0 \backslash (\{v_0\} \cup U)$. We now let $N(v_1) \cap (W_0 \backslash (\{v_0\} \cup U)) = \{v_5, v_6\}$ and $N(v_2) \cap (W_0 \backslash (\{v_0\} \cup U)) = \{v_7, v_8\}$. Observe that the function η given by $\eta(v_1) = 2$, $\eta(v_i) = 1$ for each $i \in \{3, 4, 7, 8\}$ and $\eta(v_i) = 0$ for each $i \in \{0, 2, 5, 6, 9\}$, is a GIDF on H, leading that $\gamma_{gI}(H) \leq \omega(\eta) = 6$. Moreover, since $\gamma_{gI}(H) \geq \gamma_I(H) = 6$, this forces $\gamma_{gI}(H) = 6 = \gamma_I(H)$, a contradiction.

We next suppose that $|U| = 2$. Let $U = \{v_8, v_9\}$. Then $W_0 \backslash (\{v_0\} \cup U) = \{v_5, v_6, v_7\}$. We conclude from the method similar to the proof of Claim that $N(W_2) \cap (W_0 \backslash (\{v_0\} \cup U)) = W_0 \backslash (\{v_0\} \cup U)$ and one vertex in W_2 is adjacent to exactly two vertices in $W_0 \backslash (\{v_0\} \cup U)$ and the other is adjacent to exactly one or two vertices in $W_0 \backslash (\{v_0\} \cup U)$. We now let $N(v_1) \cap (W_0 \backslash (\{v_0\} \cup U)) = \{v_5, v_6\}$ and $v_7 \in N(v_2) \cap (W_0 \backslash (\{v_0\} \cup U))$. Observe that the function η given by $\eta(v_1) = 2$, $\eta(v_i) = 1$ for each $i \in \{2, 7, 8, 9\}$ and $\eta(v_i) = 0$ for each $i \in \{0, 3, 4, 5, 6\}$, is a GIDF on H, leading that $\gamma_{gI}(H) \leq \omega(\eta) = 6$. Moreover, since $\gamma_{gI}(H) \geq \gamma_I(H) = 6$, this forces $\gamma_{gI}(H) = 6 = \gamma_I(H)$, a contradiction. This concludes the proof. □

Theorem 8. *For any cubic graph H on n vertices,*

$$\gamma_{gI}(H) - \gamma_I(H) = \begin{cases} 2, & \text{if } n = 4, \\ 1, & \text{if } n = 6, \\ 0, & \text{if } n \notin \{4, 6\}. \end{cases}$$

Proof. Noting that H is cubic, we obtain n is even. If $n = 4$, then $H = K_4$ and clearly $\gamma_{gI}(H) - \gamma_I(H) = 4 - 2 = 2$. If $n = 6$, then $H \in \{H_1, H_2\}$ (see Figure 1). Then the mapping g given by $g(v_i) = 0$ for each $i \in \{1, 2, 3\}$ and $g(v_i) = 1$ for each $i \in \{4, 5, 6\}$ is a $\gamma_I(H)$-function. Furthermore, the mapping h given by $h(v_1) = h(v_4) = 0$ and $h(v_i) = 1$ for each $i \in \{2, 3, 5, 6\}$ is a $\gamma_{gI}(H)$-function. Thus $\gamma_{gI}(H) - \gamma_I(H) = \omega(h) - \omega(g) = 4 - 3 = 1$. If $n \geq 10$, then by Lemma 2 and Proposition 1, $\gamma_{gI}(H) = \gamma_I(H)$. Suppose next that $n = 8$. Suppose that $f = (W_0, W_1, W_2)$ be a $\gamma_I(H)$-function.

By Theorem 2 and the fact that $\gamma_I(H)$ is an integer, we obtain $\gamma_{gI}(H) \geq \gamma_I(H) \geq 4$. Thus it suffices to prove that $\gamma_{gI}(H) \leq 4$. Let $v_1v_2 \in E(H)$. Since $n = 8$ and $d(v_1) = d(v_2) = 3$, there must exist two vertices, say v_3 and v_4, in H with $v_3, v_4 \notin N(v_1) \cup N(v_2)$. Let $V(H)\backslash\{v_i \mid 1 \leq i \leq 4\} = \{v_i \mid 5 \leq i \leq 8\}$.

Case 1. $v_3v_4 \in E(H)$.

Note that $v_1v_2, v_3v_4 \in E(H)$ and $v_3, v_4 \notin N(v_1) \cup N(v_2)$. Moreover, since H is cubic, we have that each vertex of $\{v_1, v_2, v_3, v_4\}$ has exactly two neighbors in $\{v_5, v_6, v_7, v_8\}$. Thus the mapping h given by $h(v_i) = 0$ for each $i \in \{1, 2, 3, 4\}$ and $h(v_i) = 1$ for each $i \in \{5, 6, 7, 8\}$, is a GIDF on H, implying that $\gamma_{gI}(h) \leq 4$.

Case 2. $v_3v_4 \notin E(H)$.

First, suppose that $N(v_1) \cap N(v_2) \neq \emptyset$. Now let $v_8 \in N(v_1) \cap N(v_2)$. Since $v_1v_2 \in E(H)$, we obtain $\{v_1, v_2, v_8\} \subseteq N[v_1] \cap N[v_2]$. Moreover, since H is cubic, we obtain

$$|N[v_1] \cup N[v_2]| = |N[v_1]| + |N[v_2]| - |N[v_1] \cap N[v_2]| \leq 4 + 4 - |\{v_1, v_2, v_8\}| = 5.$$

Note that $n = 8$ and $v_3, v_4 \notin N[v_1] \cup N[v_2]$. Thus there must exist some vertex, say v_5, in $\{v_5, v_6, v_7\}$ with $v_5 \notin N[v_1] \cup N[v_2]$. If v_3, v_4 and v_5 are pairwise nonadjacent vertices in H, then since each of v_3, v_4 and v_5 has degree three, we have $v_3, v_4, v_5 \in \bigcap_{i=6}^{8} N(v_i)$. Further, since $v_8 \in N(v_1) \cap N(v_2)$, we have $\{v_i : 1 \leq i \leq 5\} \subseteq N(v_8)$ and hence $d(v_8) \geq 5$, a contradiction. Noting that $v_3v_4 \notin E(H)$, we have either $v_3v_5 \in E(H)$ or $v_4v_5 \in E(H)$. We may presume that $v_3v_5 \in E(H)$. Moreover, since H is cubic, we obtain:

(*i*) For any $i \in \{1, 2\}$, $v_{3-i}, v_8 \in N(v_i)$ and $|N(v_i) \cap \{v_6, v_7\}| = 1$.

(*ii*) $|N(v_3) \cap \{v_6, v_7, v_8\}| = 2$ and $|N(v_5) \cap \{v_4, v_6, v_7, v_8\}| = 2$.

Thus the mapping h given by $h(v_i) = 0$ for each $i \in \{1, 2, 3, 5\}$ and $h(v_i) = 1$ for each $i \in \{4, 6, 7, 8\}$, is a GIDF on H, leading that $\gamma_{gI}(H) \leq 4$.

Second, suppose that $N(v_1) \cap N(v_2) = \emptyset$. Since $d(v_1) = d(v_2) = 3$, $v_1v_2 \in E(H)$ and $v_3, v_4 \notin N(v_1) \cup N(v_2)$, we may presume that $N(v_1)\backslash\{v_2\} = \{v_5, v_6\}$ and $N(v_2)\backslash\{v_1\} = \{v_7, v_8\}$. Moreover, since $d(v_3) = 3$ and $v_1, v_2, v_4 \notin N(v_3)$, we may presume that $N(v_3) = \{v_5, v_6, v_7\}$. Noting that $v_1, v_2, v_3 \notin N(v_4)$, we have $N(v_4) \subseteq \{v_i : 5 \leq i \leq 8\}$. If $v_8 \notin N(v_4)$, then since $d(v_4) = 3$, we have $N(v_4) = \{v_5, v_6, v_7\}$ and so $d(v_8) = |N(v_8)| = |\{v_2\}| = 1$, a contradiction. Therefore $v_8 \in N(v_4)$, implying that $N(v_4)\backslash\{v_8\} \subseteq \{v_5, v_6, v_7\}$. Recall that H is cubic. If $N(v_4)\backslash\{v_8\} = \{v_i, v_7\}$ for some $i \in \{5, 6\}$, then clearly $v_{11-i}v_8 \in E(H)$ and so the mapping h given by $h(v_j) = 1$ for each $j \in \{2, 3, 8, i\}$ and $h(v_j) = 0$ for each $j \notin \{2, 3, 8, i\}$, is a GIDF on H, implying that $\gamma_{gI}(H) \leq 4$, and if $N(v_4)\backslash\{v_8\} = \{v_5, v_6\}$, then $v_7v_8 \in E(H)$ and hence the mapping h given by $h(v_j) = 1$ for each $j \in \{2, 3, 4, 6\}$ and $h(v_j) = 0$ for each $j \notin \{2, 3, 4, 6\}$, is a GIDF on H, leading that $\gamma_{gI}(H) \leq 4$, which completes our proof. □

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