Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-22
https://doi.org/10.22049/cco.2025.31021.2714

CCO
Commun. Comb. Optim.

Research Article

New bounds on atom-bond sum-connectivity index in graphs

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Received: 22 September 2025; Accepted: 10 November 2025
Published Online: 16 December 2025

Abstract: The atom-bond sum-connectivity index (ABS) has recently been introduced as a variation of the classical atom-bond connectivity index, in which the product of vertex degrees in the denominator of each term is replaced by their sum. In this paper, we first establish a connection between the ABS index and the spectral radius of a graph. We then examine the relationships between the ABS index and several well-known degree-based topological indices. Furthermore, we improve several previous bounds presented in the literature.

Keywords: atom-bond sum-connectivity index, topological indices, Randić index, atom-bond-connectivity index, inverse sum indeg index, Sombor index, spectral radius.

AMS Subject classification: 05C09, 05C92

1. Introduction

Topological molecular descriptors strike a practical balance between chemical interpretability and computational efficiency, making them particularly useful for analyzing large molecular databases. The motivations behind introducing new topological indices are diverse-ranging from refining the predictive power of existing descriptors

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to addressing specific structural or mathematical characteristics of molecules. Occasionally, such developments are also driven by more theoretical or even abstract motivations.

Among these descriptors, degree-based topological indices form a substantial and influential category. Many of these indices have demonstrated significant potential in modeling physicochemical properties, biological activity, and reactivity of molecular structures (see, [10, 11, 18, 19]).

In chemical graph theory, numerical graph invariants are commonly referred to as *topological indices*, and their main purpose is to predict physico-chemical properties of compounds. One of the earliest examples is the Platt index, introduced in the late 1940s [16, 17]. Most of the terminology and notation of graph theory and chemical graph theory used in the present paper follow the standard references [5, 23, 26].

By merging elements from the definitions of the Randić, atom-bond connectivity (ABC), and sum-connectivity indices, a new degree-based topological index-called the *atom-bond sum-connectivity index* (ABS) was recently proposed in [1–4, 15]. For a graph G, this index is defined as

$$\mathrm{ABS}(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} = \sum_{uv\in E(G)} \sqrt{1 - \frac{2}{d_u + d_v}},$$

where d_u and d_v are the degrees of the end vertices u and v of the edge uv, and the summation runs over all edges in G.

Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. We denote the degree of a vertex v_i by $\deg(v_i) = d_{v_i}$, and the degree sequence of G by $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n > 0$. A vertex v_i is called *isolated* if $d_{v_i} = 0$ and *pendent* if $d_{v_i} = 1$. The *edge-connectivity* of a graph is the largest integer k such that the graph remains connected after the removal of any $k-1$ edges. We denote by S_n, P_n, C_n, and K_n to refer to the star, path, cycle, and complete graphs on n vertices, respectively.

The *adjacency matrix* $A(G)$ of the graph G is defined by entries $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. The eigenvalues of $A(G)$ are denoted by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

where λ_1 is called the *spectral radius* of G.

Throughout this paper, we also make use of several fundamental topological indices. Table 1 summarizes their definitions and references, which will be frequently used in our derivations.

The organization of the paper is as follows. We first establish a connection between the atom-bond sum-connectivity index (ABS) and the spectral radius of a graph in Section 2. We then derive new upper bounds for the ABS index in terms of the graph's order, size, minimum and maximum degrees in Section 3. In Section 4, we

Index	Formulation	Reference
ABS Index	$\mathrm{ABS}(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}}$	[3, 4, 15]
Randić Index	$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u d_v}}$	[20]
Sombor Index	$SO(G) = \sum_{uv\in E(G)} \sqrt{d_u^2+d_v^2}$	[25]
Modified Sombor Index	$SO^m(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u^2+d_v^2}}$	[21]
Harmonic Index	$H(G) = \sum_{uv\in E(G)} \frac{2}{d_u+d_v}$	[8]
Inverse sum indeg Index	$\mathrm{ISI}(G) = \sum_{uv\in E(G)} \frac{d_u d_v}{d_u+d_v}$	[24]
ABC Index	$ABC(G) = \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u d_v}}$	[7]
First Zagreb Index	$M_1(G) = \sum_{u\in V(G)} d_u^2$	[18]

Table 1. **Topological indices and their formulations**

investigate relationships between the ABS index and various well-known degree-based topological indices, including the Zagreb, Randić, Sombor, Modified Sombor, inverse sum indeg, ABC, and harmonic indices. Finally, we conclude the paper in Section 5. To prove our results, we will employ the following key tools and inequalities.

Lemma 1 (Cauchy–Schwarz Inequality). *For all sequences of real numbers a_i and b_i, $i = 1, 2, \cdots, n$, we have*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right).$$

Equality holds if and only if there exists a nonzero constant $k \in \mathbb{R}$ such that $a_i = kb_i$ for all i.

Theorem 1 (Turán, 1941). *Let G be a simple graph on n vertices that contains no complete subgraph K_r. Then the number of edges m in G satisfies:*

$$m \leq \left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}.$$

The extremal graph attaining this bound is the Turán graph $T_{n,r-1}$, the complete $(r-1)$-partite graph with parts as equal in size as possible.

2. Relationship between the *ABS* Index and the Spectral Radius in Graphs

This section is devoted to exploring the connection between the *ABS* index and the spectral radius of a graph. We aim to establish bounds and relations that highlight how these two graph invariants are correlated.

Theorem 2. *Let G be a connected graph on $n \geq 3$ vertices with m edges, minimum degree δ, maximum degree Δ and spectral radius λ_1. Then*

$$\frac{m\sqrt{\delta}}{\Delta\sqrt{\delta+2}}\,\lambda_1 \leq ABS(G) \leq \frac{m\sqrt{\Delta-1}}{\delta\sqrt{\Delta}}\,\lambda_1.$$

Equality holds in the upper bound if and only if G is regular, and equality holds in the lower bound if and only if $G \cong C_n$.

Proof. Since $n \geq 3$, we observe that for every edge $uv \in E(G)$, we have

$$2 + \delta \leq d_u + d_v \leq 2\Delta.$$

Recall that $\delta \leq \lambda_1 \leq \Delta$, with equality if and only if G is regular [22]. Hence,

$$\begin{aligned}
\frac{ABS(G)}{\lambda_1} &= \frac{\sum\limits_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}}}{\lambda_1} \\
&= \frac{\sum\limits_{uv\in E(G)} \sqrt{1-\frac{2}{d_u+d_v}}}{\lambda_1} \\
&\leq \frac{\sum\limits_{uv\in E(G)} \sqrt{1-\frac{2}{2\Delta}}}{\lambda_1} \\
&= \frac{m\sqrt{\Delta-1}}{\lambda_1\sqrt{\Delta}} \\
&\leq \frac{m\sqrt{\Delta-1}}{\delta\sqrt{\Delta}}.
\end{aligned}$$

Morevere,

$$\begin{aligned}
\frac{ABS(G)}{\lambda_1} &= \frac{\sum\limits_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}}}{\lambda_1} \\
&= \frac{\sum\limits_{uv\in E(G)} \sqrt{1-\frac{2}{d_u+d_v}}}{\lambda_1} \\
&\geq \frac{\sum\limits_{uv\in E(G)} \sqrt{1-\frac{2}{\delta+2}}}{\lambda_1} \\
&= \frac{m\sqrt{\delta}}{\lambda_1\sqrt{\delta+2}} \\
&\geq \frac{m\sqrt{\delta}}{\Delta\sqrt{\delta+2}}.
\end{aligned}$$

Hence, we deduce that,

$$\frac{m\sqrt{\delta}}{\Delta\sqrt{\delta+2}}\,\lambda_1 \leq ABS(G) \leq \frac{m\sqrt{\Delta-1}}{\delta\sqrt{\Delta}}\,\lambda_1.$$

We can see that equality holds in the upper bound if and only if, for every edge $uv \in E(G)$, $d_u + d_v = 2\Delta$ and $\delta = \lambda_1$, i.e., G is regular. Similarly, equality holds in the lower bound if and only if, for every edge $uv \in E(G)$, $d_u + d_v = 2 + \delta$ and $\Delta = \lambda_1$, i.e., G is regular and $\delta = 2$, implying $G \cong C_n$. □

Theorem 3. *Let G be a connected graph on $n \geq 3$ vertices with m edges, minimum degree $\delta \geq 2$, maximum degree Δ and spectral radius λ_1. Then*

$$\frac{m\sqrt{\delta-1}}{\Delta\sqrt{\delta}}\,\lambda_1 \leq ABS(G).$$

Equality holds if and only if G is regular.

Proof. Since $n \geq 3$, we observe that for every edge $uv \in E(G)$, we have $2\delta \leq d_u + d_v$, Recall that $\delta \leq \lambda_1 \leq \Delta$, with equality if and only if G is regular [22]. Hence,

$$\begin{aligned}
\frac{ABS(G)}{\lambda_1} &= \frac{\sum\limits_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}}}{\lambda_1} \\
&= \frac{\sum\limits_{uv\in E(G)} \sqrt{1-\frac{2}{d_u+d_v}}}{\lambda_1} \\
&\geq \frac{\sum\limits_{uv\in E(G)} \sqrt{1-\frac{2}{2\delta}}}{\lambda_1} \\
&= \frac{m\sqrt{\delta-1}}{\lambda_1\sqrt{\delta}} \\
&\geq \frac{m\sqrt{\delta-1}}{\Delta\sqrt{\delta}}.
\end{aligned}$$

Hence, we deduce that,

$$\frac{m\sqrt{\delta-1}}{\Delta\sqrt{\delta}}\lambda_1 \leq ABS(G).$$

We can see that equality holds if and only if, for every edge $uv \in E(G)$, $d_u + d_v = 2\delta$ and $\delta = \lambda_1$, i.e., G is regular. □

Theorem 4. *Let G be a connected graph on $n \geq 3$ vertices with spectral radius λ_1. Then*

$$\frac{ABS(G)}{\lambda_1} \leq \min\left\{\frac{n}{2}\sqrt{\frac{\delta(G)}{\delta(G)+2}}, \quad \frac{n}{2}\sqrt{\frac{\delta(G)-1}{\delta(G)}}\right\},$$

with equality if and only if G is regular.

Proof. Since $n \geq 3$, we observe that for every edge $uv \in E(G)$, we have

$$d_u + d_v \geq \max\{\delta(G)+2, 2\delta(G)\},$$

which implies

$$\sqrt{\frac{d_u+d_v-2}{d_u+d_v}} = \sqrt{1-\frac{2}{d_u+d_v}} \leq \alpha := \min\left\{\sqrt{1-\frac{2}{\delta(G)+2}}, \sqrt{1-\frac{1}{\delta(G)}}\right\}. \quad (2.1)$$

Recall that $\lambda_1 \geq \frac{2m}{n}$, where m is the number of edges in G, with equality if and only if G is regular [6]. Hence,

$$\frac{ABS(G)}{\lambda_1} = \frac{\sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}}}{\lambda_1} \leq \frac{m\cdot\alpha}{\lambda_1} \leq \frac{m\cdot\alpha}{\frac{2m}{n}} = \frac{n\alpha}{2}. \quad (2.2)$$

Combining two equations, (2.1) and (2.2), we deduce that

$$\frac{ABS(G)}{\lambda_1} \leq \min\left\{\frac{n}{2\sqrt{\delta(G)+2}}, \quad \frac{n}{2}\sqrt{1-\frac{1}{\delta(G)}}\right\}.$$

Now, suppose that equality holds. Then every inequality used in the proof must in fact be an equality. In particular, equality in (2.2) yields $\lambda_1 = \frac{2m}{n}$, which occurs if and only if G is regular.
Conversely, if G is r-regular, then for each edge $uv \in E(G)$,

$$\sqrt{\frac{d_u+d_v-2}{d_u+d_v}} = \sqrt{\frac{2r-2}{2r}} = \sqrt{1-\frac{1}{r}}.$$

Since $m = \frac{nr}{2}$, it follows that

$$ABS(G) = m\cdot\sqrt{1-\frac{1}{r}} = \frac{nr}{2}\cdot\sqrt{1-\frac{1}{r}}.$$

Moreover, since G is regular, the spectral radius is $\lambda_1 = r$, so

$$\frac{ABS(G)}{\lambda_1} = \frac{nr}{2}\cdot\sqrt{1-\frac{1}{r}}\cdot\frac{1}{r} = \frac{n}{2}\cdot\sqrt{1-\frac{1}{r}} = \frac{n}{2}\cdot\sqrt{1-\frac{1}{\delta(G)}},$$

which shows equality is attained if and only if G is regular. □

Corollary 1. *Let G be a connected graph on $n \geq 3$ vertices with spectral radius λ_1 and Randić index $R(G)$. Then*

$$ABS(G) \leq \alpha\, \lambda_1 R(G),$$

where

$$\alpha = \min\left\{\sqrt{1 - \frac{2}{\delta(G)+2}},\quad \sqrt{1 - \frac{1}{\delta(G)}}\right\}.$$

Proof. It was shown in [9] that for any graph with m edges and Randić index $R(G)$, the following inequality holds:

$$\frac{m}{R(G)} \leq \lambda_1.$$

Moreover, in the proof of Theorem 4, we established that

$$ABS(G) \leq \alpha\, m.$$

Combining these inequalities yields

$$ABS(G) \leq \alpha\, m \leq \alpha\, \lambda_1 R(G),$$

as desired. □

Theorem 5. *Let G be a connected graph on $n \geq 3$ vertices with minimum degree $\delta(G) \geq 1$ and spectral radius λ_1. Then:*

i) If $\delta(G) = 1$, then

$$\mathrm{ABS}(G) \geq \lambda_1 \sqrt{\frac{n-1}{3}},$$

with equality if and only if $G \cong S_3$.

ii) If $\delta(G) \geq 2$, then

$$\mathrm{ABS}(G) \geq \lambda_1 \frac{n\sqrt{\delta - 1}}{\sqrt{\delta(n+1)}},$$

with equality if and only if $G \cong C_n$.

Proof. Let $xy \in E(G)$ be an arbitrary edge. Without loss of generality, assume $\delta \leq d_x \leq d_y \leq n-1$. Since G is connected and $n \geq 3$, we have $d_y \geq 2$.
Consider the function

$$g(r, s) = \sqrt{1 - \frac{2}{r+s}},$$

defined for $\delta \leq r \leq s \leq \Delta$. Since $g(r, s)$ is increasing in both arguments, the minimum occurs at Thus,

$$\mathrm{ABS}(G) = \sum_{uv \in E(G)} \sqrt{1 - \frac{2}{d_u + d_v}} = \sum_{uv \in E(G)} g(d_u, d_v) \geq \tag{2.3}$$

It was shown in [12] that

$$\lambda_1 \leq \sqrt{2m-n+1}, \tag{2.4}$$

with equality if and only if G is either a star S_n or a complete graph K_n.
Combining (2.3) and (2.4), we obtain: Now define the function

$$f(x) = \frac{x}{\sqrt{2x-n+1}},$$

which is increasing for $x \geq n-1$. Since $m \geq n-1$ when $\delta = 1$ and $m \geq n$ when $\delta \geq 2$, the minimum of $f(x)$ occurs at $x = n-1$ for $\delta = 1$ and at $x = n$ for $\delta \geq 2$. Hence, Equality holds if and only if equality occurs in both (2.3) and (2.4), which implies $G \cong S_3$ in the case $\delta(G) = 1$, and $G \cong C_n$ in the case $\delta(G) \geq 2$. □

3. Upper Bounds for the *ABS* Index of Graphs

In this section, we establish several upper bounds for the atom–bond sum-connectivity index (ABS) of connected simple graphs. These inequalities are expressed in terms of various structural parameters such as the clique number, the maximum and minimum degrees, and the number of pendent vertices. We begin with two results involving the clique number, the maximum and minimum degrees.

Theorem 6 ([13]). *Let G be a connected graph having n vertices, m edges, maximum degree Δ, minimum degree δ, and clique number α. Then*

$$\mathrm{ABS}(G) \leq \frac{n^2(\alpha-1)}{2\alpha}\sqrt{\frac{\Delta-1}{\delta}}. \tag{11}$$

Theorem 7 ([14]). *Let G be a connected graph with m edges, minimum degree $\delta \geq 1$, and maximum degree Δ. Then,*

$$m\sqrt{\frac{\delta-1}{\delta}} \leq \mathrm{ABS}(G) \leq m\sqrt{\frac{\Delta-1}{\Delta}},$$

Equality holds in both bounds if and only if G is regular.

Theorem 8. *Let G be a connected graph having n vertices, maximum degree Δ, and clique number α. Then*

$$\mathrm{ABS}(G) \leq \frac{n^2(\alpha-1)}{2\alpha}\sqrt{\frac{\Delta-1}{\Delta}}.$$

Proof. Since G is $K_{\alpha+1}$-free, Turán's theorem implies that

$$m \leq \frac{n^2(\alpha-1)}{2\alpha}.$$

Also, by Theorem 7, we have

$$\mathrm{ABS}(G) \leq m\sqrt{\frac{\Delta-1}{\Delta}}.$$

Combining these two inequalities, we deduce that

$$\mathrm{ABS}(G) \leq \frac{n^2(\alpha-1)}{2\alpha}\sqrt{\frac{\Delta-1}{\Delta}}.$$

□

Remark 1. The bound obtained in Theorem 8 is sharper than that in Theorem 6 for all connected graphs with $n \geq 3$.

In the following theorem, we provide an upper bound for the ABS index of a connected graph in terms of its number of pendent vertices, the maximum degree, and the minimum degree among its non-pendent vertices.

Theorem 9. *Let G be a connected graph with n vertices, p pendent vertices, maximum degree Δ, and minimum degree among non-pendent vertices δ_1. Then*

$$\mathrm{ABS}(G) \leq p\sqrt{\frac{\Delta-1}{\Delta+1}} + \sqrt{\big(M_1(G) - 2m - p(\delta_1-1)\big)\left(\frac{H(G)}{2} - \frac{p}{\Delta+1}\right)}.$$

Proof. For each edge $ij \in E(G)$ such that both vertices v_i and v_j are not pendent (i.e., $d_i, d_j > 1$), define

$$a_{ij} = \sqrt{d_i + d_j - 2}, \quad b_{ij} = \frac{1}{\sqrt{d_i + d_j}}.$$

Applying the Cauchy–Schwarz inequality yields

$$\left(\sum_{\substack{ij\in E(G)\\ d_i,d_j>1}} \frac{\sqrt{d_i+d_j-2}}{\sqrt{d_i+d_j}}\right)^2 \leq \left(\sum_{\substack{ij\in E(G)\\ d_i,d_j>1}} (d_i+d_j-2)\right)\left(\sum_{\substack{ij\in E(G)\\ d_i,d_j>1}} \frac{1}{d_i+d_j}\right).$$

We now estimate these two sums. We have

$$\sum_{\substack{ij\in E(G)\\ d_i,d_j>1}} (d_i+d_j-2) = \sum_{ij\in E(G)} (d_i+d_j-2) - \sum_{\substack{ij\in E(G)\\ d_i=1}} (d_j-1),$$

$$\sum_{\substack{ij\in E(G)\\ d_i,d_j>1}} \frac{1}{d_i+d_j} = \sum_{ij\in E(G)} \frac{1}{d_i+d_j} - \sum_{\substack{ij\in E(G)\\ d_i=1}} \frac{1}{1+d_j}.$$

Since there are exactly p pendent vertices, and each pendent vertex is adjacent to a vertex of degree at least δ_1 and at most Δ, we deduce that

$$\sum_{\substack{ij\in E(G)\\ d_i=1}} (d_j-1) \geq p(\delta_1-1), \qquad \sum_{\substack{ij\in E(G)\\ d_i=1}} \frac{1}{1+d_j} \geq \frac{p}{\Delta+1}.$$

Substituting into the bound yields

$$\left(\sum_{\substack{ij\in E(G)\\ d_i,d_j>1}} \sqrt{\frac{d_i+d_j-2}{d_i+d_j}}\right)^2 \leq \big(M_1(G)-2m-p(\delta_1-1)\big)\left(\frac{H(G)}{2}-\frac{p}{\Delta+1}\right).$$

Hence,

$$\sum_{\substack{ij\in E(G)\\ d_i,d_j>1}} \sqrt{\frac{d_i+d_j-2}{d_i+d_j}} \leq \sqrt{\big(M_1(G)-2m-p(\delta_1-1)\big)\left(\frac{H(G)}{2}-\frac{p}{\Delta+1}\right)}.$$

Now consider the contribution of edges incident to pendent vertices. For each such edge, we have

$$\sqrt{\frac{d_i+d_j-2}{d_i+d_j}} = \sqrt{\frac{d_j-1}{d_j+1}} \leq \sqrt{\frac{\Delta-1}{\Delta+1}},$$

and since there are p such edges,

$$\sum_{\substack{ij\in E(G)\\ d_i=1}} \sqrt{\frac{d_i+d_j-2}{d_i+d_j}} \leq p\sqrt{\frac{\Delta-1}{\Delta+1}}.$$

Combining both parts, we obtain

$$\begin{aligned}\mathrm{ABS}(G) &= \sum_{\substack{ij\in E(G)\\ d_i=1}} \sqrt{\frac{d_i+d_j-2}{d_i+d_j}} + \sum_{\substack{ij\in E(G)\\ d_i,d_j>1}} \sqrt{\frac{d_i+d_j-2}{d_i+d_j}} \\ &\leq p\sqrt{\frac{\Delta-1}{\Delta+1}} + \sqrt{\big(M_1(G)-2m-p(\delta_1-1)\big)\left(\frac{H(G)}{2}-\frac{p}{\Delta+1}\right)}.\end{aligned}$$

□

4. On relations of atom-bond sum-connectivity index with other degree-based topological indices

In this section, we provide several relationships between the atom-bond sum-connectivity (ABS) index and other degree-based topological indices.
Here, we present inequalities relating the atom-bond sum-connectivity index with the somber index. In [21], the authors established the following bounds for the ABS index of a graph in terms of sombor index.

Theorem 10 ([21]). *Let G be a connected graph of order at least 3 with minimum degree δ and maximum degree Δ. Then*

$$\frac{1}{2\Delta\sqrt{\Delta}}SO(G) \leq ABS(G) < \frac{1}{\sqrt{2\delta}}SO(G).$$

Theorem 11 ([21]). *Let G be a connected graph of order at least 3 with minimum degree $\delta \geq 2$ and maximum degree Δ. Then*

$$\frac{\sqrt{\delta - 1}}{\sqrt{2}\Delta^{3/2}}SO(G) \leq \mathrm{ABS}(G) \leq \frac{\sqrt{\Delta - 1}}{\delta\sqrt{2\Delta}}SO(G).$$

Theorem 12. *Let G be a connected graph of order at least three with minimum degree δ and maximum degree Δ. Then*

$$\frac{\sqrt{\delta}}{2\Delta^{3/2}}\,\mathrm{SO}(G) \leq ABS(G) < \frac{\sqrt{\Delta - 1}}{\delta\sqrt{2\Delta}}\,\mathrm{SO}(G).$$

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$ with $1 \leq d_u \leq d_v \leq \Delta$. Then we can see that $d_u \geq \delta$ and $d_v \geq 2$ and so $d_u + d_v - 2 \geq \delta + 2 - 2 = \delta$. Thus,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{\mathrm{SO}(G)} &= \frac{1}{\mathrm{SO}(G)} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\
&\geq \frac{1}{\sqrt{2\Delta}} \cdot \frac{1}{\mathrm{SO}(G)} \sum_{uv \in E(G)} \sqrt{d_u + d_v - 2} \\
&= \frac{1}{\sqrt{2\Delta}} \cdot \frac{1}{\mathrm{SO}(G)} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u^2 + d_v^2}} \cdot \sqrt{d_u^2 + d_v^2} \\
&\geq \frac{\sqrt{\delta}}{\sqrt{2\Delta} \cdot \sqrt{2\Delta}} \cdot \frac{1}{\mathrm{SO}(G)} \sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2} \\
&= \frac{\sqrt{\delta}}{2\Delta^{3/2}} \cdot \left(\frac{\sum_{uv \in E(G)} \sqrt{d_u^2 + d_v^2}}{\mathrm{SO}(G)} \right) \\
&= \frac{\sqrt{\delta}}{2\Delta^{3/2}}.
\end{aligned}$$

Hence,

$$ABS(G) \geq \frac{\sqrt{\delta}}{2\Delta^{3/2}}\,\mathrm{SO}(G).$$

Now, we turn to the proof of the lower bound. For each edge $uv \in E(G)$, let $x = d_u + d_v$. Then, clearly $2 \leq x \leq 2\Delta$, and the function

$$f(x) = \sqrt{\frac{x-2}{x}}$$

is increasing. Thus,

$$f(x) \leq f(2\Delta) = \sqrt{\frac{2\Delta - 2}{2\Delta}} = \sqrt{\frac{\Delta - 1}{\Delta}},$$

and so, for each edge $uv \in E(G)$, we have

$$\sqrt{\frac{d_u + d_v}{d_u + d_v - 2}} \leq \sqrt{\frac{\Delta - 1}{\Delta}}.$$

Therefore,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{\mathrm{SO}(G)} &= \frac{1}{\mathrm{SO}(G)} \sum_{uv\in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\
&\leq \sqrt{\frac{\Delta - 1}{\Delta}} \cdot \frac{1}{\mathrm{SO}(G)} \sum_{uv\in E(G)} \sqrt{\frac{d_u^2 + d_v^2}{d_u^2 + d_v^2}} \\
&\leq \sqrt{\frac{\Delta - 1}{\Delta}} \cdot \frac{1}{\sqrt{2\delta}} \cdot \frac{1}{\mathrm{SO}(G)} \sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2} \\
&= \frac{\sqrt{\Delta - 1}}{\delta\sqrt{2\Delta}} \left(\frac{\sum_{uv\in E(G)} \sqrt{d_u^2 + d_v^2}}{\mathrm{SO}(G)} \right) \\
&= \frac{\sqrt{\Delta - 1}}{\delta\sqrt{2\Delta}}.
\end{aligned}$$

Hence,

$$ABS(G) \leq \frac{\sqrt{\Delta - 1}}{\delta\sqrt{2\Delta}}\,\mathrm{SO}(G).$$

Combining the two inequalities gives the desired result. □

Remark 2. Clearly, the bounds in Theorem 12 are sharper than those in Theorem 10 for all connected graphs with $n \geq 3$.

Theorem 13. *Let G be a connected graph of order $n \geq 3$, with minimum degree $\delta \geq 2$, maximum degree Δ, and Sombor index $SO(G)$. Then,*

$$\frac{\sqrt{\delta-1}}{\Delta\sqrt{2\delta}}SO(G) \leq \mathrm{ABS}(G) \leq \frac{\sqrt{\delta-1}}{\sqrt{2}\delta^{3/2}}SO(G).$$

Equality holds in both bounds if and only if G is a regular graph.

Proof. Define,

$$h(d_u, d_v) = \frac{\sqrt{\frac{d_u+d_v-2}{d_u+d_v}}}{\sqrt{d_u^2+d_v^2}}.$$

Thus,

$$\mathrm{ABS}(G) = \sum_{uv} h(d_u, d_v)\sqrt{d_u^2+d_v^2}.$$

We first proof upper boun. Since $\sqrt{d_u^2+d_v^2} \geq \frac{d_u+d_v}{\sqrt{2}}$, we have

$$h(d_u, d_v) = \frac{\sqrt{\frac{d_u+d_v-2}{d_u+d_v}}}{\sqrt{d_u^2+d_v^2}} \leq \sqrt{2}\,\frac{\sqrt{d_u+d_v-2}}{(d_u+d_v)^{3/2}}.$$

Let $x = d_u + d_v$. Then,

$$h(d_u, d_v) \leq \sqrt{2}\,\frac{\sqrt{x-2}}{x^{3/2}}.$$

The function $g(x) = \frac{\sqrt{x-2}}{x^{3/2}}$ is decreasing for $x \geq 2\delta \geq 4$, with maximum at $x = 2\delta$ and so

$$h(d_u, d_v) \leq \sqrt{2}\,\frac{\sqrt{x-2}}{x^{3/2}} \leq \sqrt{2}\,g(x) = \sqrt{2}\,g(2\delta) = \frac{\sqrt{\delta-1}}{\sqrt{2}\delta^{3/2}}.$$

Hence,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{\mathrm{SO}(G)} &= \frac{1}{\mathrm{SO}(G)}\sum_{uv\in E(G)}\sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \frac{1}{\mathrm{SO}(G)}\sum_{uv} h(d_u, d_v)\sqrt{d_u^2+d_v^2} \\
&\leq \frac{1}{\mathrm{SO}(G)}\sum_{uv}\frac{\sqrt{\delta-1}}{\sqrt{2}\delta^{3/2}}\sqrt{d_u^2+d_v^2} \\
&= \left(\frac{\sqrt{\delta-1}}{\sqrt{2}\delta^{3/2}}\right)\left(\frac{\sum_{uv}\sqrt{d_u^2+d_v^2}}{\mathrm{SO}(G)}\right) \\
&= \frac{\sqrt{\delta-1}}{\sqrt{2}\delta^{3/2}}.
\end{aligned}$$

Next we prove lower bound. Since $\sqrt{d_u^2+d_v^2} \le \sqrt{2}\Delta$, we have

$$h(d_u,d_v) \ge \frac{\sqrt{\frac{x-2}{x}}}{\sqrt{2}\Delta} = \frac{\sqrt{x-2}}{\sqrt{2}\Delta\sqrt{x}}.$$

The function $f(x)=\sqrt{\frac{x-2}{x}}$ is increasing, with minimum at $x=2\delta$ and so

$$h(d_u,d_v) \ge \frac{\sqrt{\frac{x-2}{x}}}{\sqrt{2}\Delta} \ge \frac{\sqrt{\delta-1}}{\Delta\sqrt{2\delta}},$$

Hence,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{\mathrm{SO}(G)} &= \frac{1}{\mathrm{SO}(G)}\sum_{uv\in E(G)}\sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \frac{1}{\mathrm{SO}(G)}\sum_{uv} h(d_u,d_v)\sqrt{d_u^2+d_v^2} \\
&\ge \frac{1}{\mathrm{SO}(G)}\sum_{uv} \frac{\sqrt{\delta-1}}{\Delta\sqrt{2\delta}}\sqrt{d_u^2+d_v^2} \\
&= \left(\frac{\sqrt{\delta-1}}{\Delta\sqrt{2\delta}}\right)\left(\frac{\sum_{uv}\sqrt{d_u^2+d_v^2}}{\mathrm{SO}(G)}\right) \\
&= \frac{\sqrt{\delta-1}}{\Delta\sqrt{2\delta}}.
\end{aligned}$$

Equality holds for two bounds when in two bounds G is regular. □

Remark 3. Clearly, the bounds in Theorem 13 are sharper than those in Theorems 11 and 12 for all connected graphs with $n \ge 3$.

In [21], the authors established the following bounds for the ABS index of a graph in terms of modified somber index.

Theorem 14 ([21]). *Let G be a connected graph of order at least three with minimum degree δ and maximum degree Δ. Then*

$$\frac{\delta}{\sqrt{\Delta}}SO^m(G) \le ABS(G) < \sqrt{2}\Delta SO^m(G).$$

Theorem 15 ([21]). *Let G be a connected graph with minimum degree $\delta \ge 2$ and maximum degree Δ. Then*

$$\frac{\delta\sqrt{2\delta-2}}{\sqrt{\Delta}}SO^m(G) \le ABS(G) \le \frac{\Delta\sqrt{2\Delta-2}}{\sqrt{\delta}}SO^m(G).$$

Equality holds for both if and only if G is regular.

Theorem 16. *Let G be a connected graph of order at least three with minimum degree δ and maximum degree Δ. Then*

$$\frac{\delta^{3/2}}{\sqrt{\Delta}}SO^m(G) \leq ABS(G) \leq \Delta\sqrt{\frac{2\delta}{\delta+2}}SO^m(G).$$

Equality holds in both bounds if and only if $G \cong C_n$.

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$ with $1 \leq d_u \leq d_v \leq \Delta$. Then we can see that $d_u \geq \delta$ and $d_v \geq 2$ and so $d_u + d_v - 2 \geq \delta + 2 - 2 = \delta$. Thus,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{SO^m(G)} &= \frac{1}{SO^m(G)} \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&\geq \frac{1}{\sqrt{2\Delta}} \cdot \frac{1}{SO^m(G)} \sum_{uv\in E(G)} \sqrt{d_u+d_v-2} \\
&= \frac{1}{\sqrt{2\Delta}} \cdot \frac{1}{SO^m(G)} \sum_{uv\in E(G)} \sqrt{(d_u+d_v-2)(d_u^2+d_v^2)} \cdot \frac{1}{\sqrt{d_u^2+d_v^2}} \\
&\geq \frac{\sqrt{2\delta^3}}{\sqrt{2\Delta}} \cdot \frac{1}{SO^m(G)} \sum_{uv\in E(G)} \frac{1}{\sqrt{d_u^2+d_v^2}} \\
&= \frac{\delta^{3/2}}{\sqrt{\Delta}} \cdot \left(\frac{\sum_{uv\in E(G)} \frac{1}{\sqrt{d_u^2+d_v^2}}}{SO^m(G)} \right) \\
&= \frac{\delta^{3/2}}{\sqrt{\Delta}}.
\end{aligned}$$

Hence,

$$ABS(G) \geq \frac{\delta^{3/2}}{\sqrt{\Delta}}SO^m(G).$$

Moreover, the chain of inequalities shows that equality holds if and only if, for all $uv \in E(G)$, we have $d_u + d_v = \delta + 2$, $d_u^2 + d_v^2 = 2\delta^2$, and $d_u + d_v = 2\Delta$. Thus, $\delta = \Delta = 2$, and hence $G \cong C_n$.
Now we prove the upper bouns. We know that $d_u + d_v \geq \delta + 2$. Hence,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{SO^m(G)} &= \frac{1}{SO^m(G)} \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \frac{1}{SO^m(G)} \sum_{uv\in E(G)} \sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \sqrt{\frac{d_u^2+d_v^2}{d_u^2+d_v^2}} \\
&= \frac{1}{SO^m(G)} \sum_{uv\in E(G)} \sqrt{1-\frac{2}{d_u+d_v}} \sqrt{\frac{d_u^2+d_v^2}{d_u^2+d_v^2}}
\end{aligned}$$

$$
\begin{aligned}
&\leq \sqrt{1-\frac{2}{2+\delta}}\frac{1}{SO^m(G)}\sum_{uv\in E(G)}\sqrt{\frac{d_u^2+d_v^2}{d_u^2+d_v^2}}\\
&\leq \sqrt{1-\frac{2}{2+\delta}}\cdot\sqrt{2}\Delta\cdot\frac{1}{SO^m(G)}\sum_{uv\in E(G)}\sqrt{\frac{1}{d_u^2+d_v^2}}\\
&= \sqrt{1-\frac{2}{2+\delta}}\cdot\sqrt{2}\Delta\cdot\left(\frac{\sum_{uv\in E(G)}\frac{1}{\sqrt{d_u^2+d_v^2}}}{SO^m(G)}\right)\\
&= \Delta\sqrt{\frac{2\delta}{\delta+2}}.
\end{aligned}
$$

Hence,

$$ABS(G) \leq \Delta\sqrt{\frac{2\delta}{\delta+2}}SO^m(G).$$

Moreover, the chain of inequalities shows that equality holds if and only if, for all $uv \in E(G)$, we have $d_u + d_v = \delta + 2$ and $d_u^2 + d_v^2 = 2\Delta^2$. Hence, $\delta = \Delta = 2$, and therefore $G \cong C_n$. □

Remark 4. Clearly, the bounds in Theorem 16 are sharper than those in Theorem 14 for all connected graphs with $n \geq 3$.

Theorem 17. *Let G be a connected graph of order $n \geq 3$, with minimum degree $\delta \geq 2$, maximum degree Δ, and modified Sombor index $SO^m(G)$. Then:*

$$\sqrt{2(\delta-1)\delta}SO^m(G) \leq \mathrm{ABS}(G) \leq \sqrt{2\Delta(\Delta-1)}SO^m(G).$$

Moreover, equality holds in both bounds if and only if G is a regular graph.

Proof. Define:

$$h(d_u, d_v) = \sqrt{\frac{d_u+d_v-2}{d_u+d_v}}\cdot\sqrt{d_u^2+d_v^2}.$$

Thus:

$$\mathrm{ABS}(G) = \sum_{uv} h(d_u, d_v)\,\frac{1}{\sqrt{d_u^2+d_v^2}}.$$

Let $x = d_u + d_v$. Since $\sqrt{d_u^2+d_v^2} \leq \sqrt{2}\Delta$, we have

$$h(d_u, d_v) \leq \sqrt{\frac{x-2}{x}}\cdot\sqrt{2\Delta^2}.$$

The function $f(x) = \sqrt{\frac{x-2}{x}}$ is increasing, with maximum occurs at $x = 2\Delta$, with $d_u = d_v = \Delta$ and so

$$h(d_u, d_v) \leq \sqrt{\frac{\Delta - 1}{\Delta}} \cdot \sqrt{2\Delta^2} = \sqrt{2\Delta(\Delta - 1)}.$$

Hence,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{SO^m(G)} &= \frac{1}{\mathrm{SO^m}(G)} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\
&= \frac{1}{SO^m(G)} \sum_{uv \in E(G)} h(d_u, d_v) \frac{1}{\sqrt{d_u^2 + d_v^2}} \\
&\leq \frac{1}{SO^m(G)} \sum_{uv \in E(G)} \sqrt{2\Delta(\Delta - 1)} \frac{1}{\sqrt{d_u^2 + d_v^2}} \\
&= \sqrt{2\Delta(\Delta - 1)} \left(\frac{\sum_{uv \in E(G)} \frac{1}{\sqrt{d_u^2 + d_v^2}}}{SO^m(G)} \right) \\
&= \sqrt{2\Delta(\Delta - 1)}.
\end{aligned}$$

Next we prove lower bound. Since $\sqrt{d_u^2 + d_v^2} \geq \sqrt{2}\delta$, we have

$$h(d_u, d_v) \geq \sqrt{\frac{x - 2}{x}} \cdot \sqrt{2\delta^2}.$$

The function $f(x) = \sqrt{\frac{x-2}{x}}$ is increasing, with minimum occurs at $x = 2\delta$, with $d_u = d_v = \delta$ and so

$$h(d_u, d_v) \geq \sqrt{\frac{\delta - 1}{\delta}} \cdot \sqrt{2\delta^2} = \sqrt{2\delta(\delta - 1)}.$$

Hence,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{SO^m(G)} &= \frac{1}{\mathrm{SO^m}(G)} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\
&= \frac{1}{SO^m(G)} \sum_{uv \in E(G)} h(d_u, d_v) \frac{1}{\sqrt{d_u^2 + d_v^2}} \\
&\geq \frac{1}{SO^m(G)} \sum_{uv \in E(G)} \sqrt{2\delta(\delta - 1)} \frac{1}{\sqrt{d_u^2 + d_v^2}} \\
&= \sqrt{2\delta(\delta - 1)} \left(\frac{\sum_{uv \in E(G)} \frac{1}{\sqrt{d_u^2 + d_v^2}}}{SO^m(G)} \right) \\
&= \sqrt{2\delta(\delta - 1)}.
\end{aligned}$$

Clearly, equality in both bounds holds if and only if G is a regular graph. □

Remark 5. Clearly, the bounds in Theorem 17 are sharper than those in Theorem 15 for all connected graphs with $n \geq 3$.

The next results give lower and upper bounds for the atom-bond sum-connectivity index involving the inverse sum indeg index ISI and the order n of a graph.

Theorem 18. *Let G be a connected graph of order $n \geq 3$, with minimum degree $\delta \geq 2$, maximum degree Δ, and inverse sum indeg index* $\mathrm{ISI}(G)$. *Then,*

$$\frac{2\sqrt{\delta(\delta-1)}}{\Delta^2}\ \mathrm{ISI}(G) \leq \mathrm{ABS}(G) \leq \frac{2\sqrt{\Delta(\Delta-1)}}{\delta^2}\ \mathrm{ISI}(G).$$

Moreover, equality holds in both bounds if and only if G is a regular graph.

Proof. Define the function

$$h(r,s) = \frac{\sqrt{\frac{r+s-2}{r+s}}}{\frac{rs}{r+s}} = \frac{\sqrt{(r+s)(r+s-2)}}{rs}.$$

Thus, we have

$$\mathrm{ABS}(G) = \sum_{uv \in E(G)} h(d_u, d_v) \cdot \frac{d_u d_v}{d_u + d_v}.$$

We first prove the upper bound. For $2\delta \leq r+s \leq 2\Delta$, we have
Therefore,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{ISI(G)} &= \frac{1}{ISI(G)} \sum_{uv \in E(G)} h(d_u, d_v) \cdot \frac{d_u d_v}{d_u + d_v} \\
&\leq \frac{1}{ISI(G)} \sum_{uv \in E(G)} \frac{2\sqrt{\Delta(\Delta-1)}}{\delta^2} \cdot \frac{d_u d_v}{d_u + d_v} \\
&= \frac{2\sqrt{\Delta(\Delta-1)}}{\delta^2} \left(\frac{1}{ISI(G)} \sum_{uv \in E(G)} \frac{d_u d_v}{d_u + d_v} \right) \\
&= \frac{2\sqrt{\Delta(\Delta-1)}}{\delta^2}.
\end{aligned}$$

We now prove the lower bound. For $2\delta \leq r+s \leq 2\Delta$, we have Therefore,

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{ISI(G)} &= \frac{1}{ISI(G)} \sum_{uv \in E(G)} h(d_u, d_v) \cdot \frac{d_u d_v}{d_u + d_v} \\
&\geq \frac{1}{ISI(G)} \sum_{uv \in E(G)} \frac{2\sqrt{\delta(\delta-1)}}{\Delta^2} \cdot \frac{d_u d_v}{d_u + d_v}
\end{aligned}$$

$$= \frac{2\sqrt{\delta(\delta-1)}}{\Delta^2}\left(\frac{1}{ISI(G)}\sum_{uv\in E(G)}\frac{d_ud_v}{d_u+d_v}\right)$$
$$= \frac{2\sqrt{\delta(\delta-1)}}{\Delta^2}.$$

Clearly, equality in both bounds holds if and only if G is a regular graph. □

In [21], the authors established the following bounds for the ABS index of a graph in terms of ABC index.

Theorem 19 ([21]). *Let G be a connected graph of order at least 3 with minimum degree δ and maximum degree Δ. Then*

$$\frac{\delta}{\sqrt{2\Delta}}ABC(G) \le ABS(G) \le \frac{\Delta}{\sqrt{2\delta}}ABC(G).$$

Equality holds for both if and only if G is regular.

Theorem 20. *Let G be a connected graph of order $n \ge 3$, with minimum degree $\delta \ge 1$, maximum degree Δ, and atom-bond connectivity index $\mathrm{ABC}(G)$. Then:*

$$\sqrt{\frac{\delta}{2}}\,\mathrm{ABC}(G) \le \mathrm{ABS}(G) \le \sqrt{\frac{\Delta}{2}}\,\mathrm{ABC}(G).$$

Equality holds in both bounds if and only if G is a regular graph.

Proof. Let $u, v \in V(G)$ such that $uv \in E(G)$ with $1 \le \delta \le d_u \le d_v \le \Delta$. Then,

$$\frac{2}{\Delta} \le \frac{1}{d_u} + \frac{1}{d_v} \le \frac{2}{\delta}.$$

We first prove the upper bound, we have

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{\mathrm{ABC}(G)} &= \frac{1}{\mathrm{ABC}(G)}\sum_{uv\in E(G)}\sqrt{\frac{d_u+d_v-2}{d_u+d_v}} \\
&= \frac{1}{\mathrm{ABC}(G)}\sum_{uv\in E(G)}\sqrt{\frac{d_u+d_v-2}{d_ud_v}}\sqrt{\frac{d_ud_v}{d_u+d_v}} \\
&= \frac{1}{\mathrm{ABC}(G)}\sum_{uv\in E(G)}\sqrt{\frac{d_u+d_v-2}{d_ud_v}}\sqrt{\frac{1}{\frac{1}{d_u}+\frac{1}{d_v}}} \\
&\le \sqrt{\frac{\Delta}{2}}\left(\frac{\sum_{uv\in E(G)}\sqrt{\frac{d_u+d_v-2}{d_ud_v}}}{\mathrm{ABC}(G)}\right) \\
&= \sqrt{\frac{\Delta}{2}}.
\end{aligned}$$

Now we prove the lower bound, we have

$$\begin{aligned}
\frac{\mathrm{ABS}(G)}{\mathrm{ABC}(G)} &= \frac{1}{\mathrm{ABC}(G)} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u + d_v}} \\
&= \frac{1}{\mathrm{ABC}(G)} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \sqrt{\frac{d_u d_v}{d_u + d_v}} \\
&= \frac{1}{\mathrm{ABC}(G)} \sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}} \sqrt{\frac{1}{\frac{1}{d_u} + \frac{1}{d_v}}} \\
&\geq \sqrt{\frac{\delta}{2}} \left(\frac{\sum_{uv \in E(G)} \sqrt{\frac{d_u + d_v - 2}{d_u d_v}}}{\mathrm{ABC}(G)} \right) \\
&= \sqrt{\frac{\delta}{2}}.
\end{aligned}$$

Clearly, equality in both bounds holds if and only if G is a regular graph. □

Remark 6. Clearly, the bounds in Theorem 20 are sharper than those in Theorem 19 for all connected graphs with $n \geq 3$.

5. Conclusion

In this paper, we investigated the atom-bond sum-connectivity index (ABS) and established several new bounds and relationships between ABS and other classical topological indices. We began by connecting the ABS index to the spectral radius of a graph and obtained sharp upper and lower bounds involving the graph's order, size, minimum degree, and spectral radius. In particular, we showed that the regular graphs attain equality in these bounds. Moreover, we derived new inequalities between the ABS index and other well-known indices such as the inverse sum indeg index (ISI), the Sombor index (SO), and the modified Sombor index (SO^m). These results refine and extend previously known bounds by incorporating parameters such as minimum degree δ, maximum degree Δ, and the structure of K_α-free graphs and graphs with pendent vertices.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] K. Aarthi, S. Elumalai, S. Balachandran, and S. Mondal, *On difference between atom-bond sum-connectivity index and Randić index of graphs*, J. Appl. Math. Comput. **71** (2025), no. 2, 2727–2748.
https://doi.org/10.1007/s12190-024-02339-2.

[2] A. Ali, T. Došlić, and Z. Raza, *On trees of a fixed maximum degree with extremal general atom-bond sum-connectivity index*, J. Appl. Math. Comput. **71** (2025), no. 1, 1035–1049.
https://doi.org/10.1007/s12190-024-02275-1.

[3] A. Ali, B. Furtula, I. Redžepović, and I. Gutman, *Atom-bond sum-connectivity index*, J. Math. Chem. **60** (2022), no. 10, 2081–2093.
https://doi.org/10.1007/s10910-022-01403-1.

[4] A. Ali, I. Gutman, B. Furtula, I. Redžepović, T. Došlić, and Z. Raza, *Extremal results and bounds for atom-bond sum-connectivity index*, MATCH Commun. Math. Comput. Chem. **92** (2024), no. 2, 271–314.
https://doi.org/10.46793/match.92–2.271A.

[5] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, north-Holland, 1979.

[6] D.M. Cvetković, P. Rowlinson, and S. Simić, *An Introduction to the Theory of Graph Spectra*, vol. 75, Cambridge university press Cambridge, 2010.

[7] E. Estrada, L. Torres, L. Rodriguez, and I. Gutman, *An atom-bond connectivity index: modelling the enthalpy of formation of alkanes*, Indian J. Chem., Sect. A **37** (1998), no. 10, 849–855.

[8] S. Fajtlowicz, *On conjectures of Graffiti-II*, Congr. Numer. **60** (1987), 187–197.

[9] O. Favaron, M. Mahéo, and J.F. Saclé, *Some eigenvalue properties in graphs (conjectures of Graffiti-II)*, Discrete Math. **111** (1993), no. 1-3, 197–220.
https://doi.org/10.1016/0012-365X(93)90156-N.

[10] B. Furtula and I. Gutman, *Degrees-Based Topological Indices*, University of Kragujevac, Faculty of Science, 2017.

[11] I. Gutman and O.E. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer Science & Business Media, 2012.

[12] Y. Hong, *Bounds of eigenvalues of graphs*, Discrete Math. **123** (1993), no. 1-3, 65–74.
https://doi.org/10.1016/0012-365X(93)90007-G.

[13] Z. Hussain, H. Liu, S. Zhang, and H. Hua, *Bounds for the atom-bond sum-connectivity index of graphs*, MATCH Commun. Math. Comput. Chem. **94** (2025), no. 3, 805–824.
https://doi.org/10.21203/rs.3.rs–3353933/v1.

[14] A. Jahanbani and I. Redžepović, *On the generalized abs index of graphs*, Filomat **37** (2023), no. 30, 10161–10169.

[15] P. Nithya, S. Elumalai, and S. Balachandran, *Minimum atom-bond sum-connectivity index of unicyclic graphs with maximum degree*, Discrete Math. Lett

13 (2024), 82–88.
https://doi.org/10.47443/dml.2023.191.

[16] J.R. Platt, *Influence of neighbor bonds on additive bond properties in paraffins*, J. Chem. Phys. **15** (1947), no. 6, 419–420.
https://doi.org/10.1063/1.1746554.

[17] ______, *Prediction of isomeric differences in paraffin properties*, J. Phys. Chem. **56** (1952), no. 3, 328–336.
https://doi.org/10.1021/j150495a009.

[18] H. Rahbani, H. Abdollahzadeh Ahangar, and M.A. Henning, *New bounds on Zagreb connection indices for trees with fixed domination number*, AKCE Int. J. Graphs Comb. (2025), 1–9.
https://doi.org/10.1080/09728600.2025.2523028.

[19] ______, *New upper bounds on Zagreb indices with given domination number*, Appl. Math. Comput. **514** (2026), 129815.
https://doi.org/10.1016/j.amc.2025.129815.

[20] M. Randić, *Characterization of molecular branching*, J. Am. Chem. Soc. **97** (1975), no. 23, 6609–6615.
https://doi.org/10.1021/ja00856a001.

[21] A. Rosalio Jr, R.U. Gobithaasan, H. Roslan, and N. Jafari Rad, *Relation between ABS index with some other topological indices*, Commun. Comb. Optim. (2025), In press.
https://doi.org/10.22049/cco.2025.30348.2429.

[22] J. Shu and Y. Wu, *Sharp upper bounds on the spectral radius of graphs*, Linear Algebra Appl. **377** (2004), 241–248.
https://doi.org/10.1016/j.laa.2003.07.015.

[23] N. Trinajstić, *Chemical Graph Theory*, CRC press, 2018.

[24] D. Vukičević and M. Gašperov, *Bond additive modeling 1. Adriatic indices*, Croat. Chem. Acta **83** (2010), no. 3, 243–260.

[25] Z. Wang, Y. Mao, Y. Li, and B. Furtula, *On relations between Sombor and other degree-based indices*, J. Appl. Math. Comput. **68** (2022), no. 1, 1–17.
https://doi.org/10.1007/s12190-021-01516-x.

[26] D.B. West, *Introduction to Graph Theory*, vol. 2, Prentice hall Upper Saddle River, 2001.