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Research Article

On 4-domination and 4-rainbow domination of cylindrical graphs

Janez Žerovnik[1,2]

[1]FME, University of Ljubljana, Aškerčeva 6, Ljubljana, 1000, Slovenia

[2]Rudolfovo - Science and Technology Centre Novo Mesto,
Podbreznik 15, Novo mesto, 8000, Slovenia
*janez.zerovnik@fs.uni-lj.si

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Abstract: Cylindrical graphs and torus grid graphs are naturally constructed from subgraphs of the infinite grid by certain identifications of boundary vertices. Considering various domination type problems, it is usually possible to find an optimal solution on the infinite grid. To the contrary, exact values of invariants for the cylindrical and torus grid graphs are typically only known for special subfamilies, and are in general hard to compute. The 4-domination and 4-rainbow domination of cylindrical graphs are studied, and some new formulae and improved bounds are reported, generalizing recent results for the case $k = 2$ in [Computational and Applied Mathematics 44(5), 293 (2025)]. We also consider weak 4-domination and singleton 4-rainbow domination.

Keywords: 4-domination, weak 4-domination, singleton 4-rainbow domination, cylindrical graphs.

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1. Introduction

Domination is among the most studied topics in graph theory [15, 16]. There are many variations extensively studied in the literature, here we are interested in four of them including k-domination, k-rainbow domination, singleton k-rainbow domination, and weak k-domination. Two among them are well-known and are amog the most popular variations of graph domination. The first, k-domination has been studied in the literature for a long time [7]. More recently, k-rainbow domination problem was put forward in [2], also see [1]. Singleton k-rainbow domination [6] and weak k-domination are less popular, but as they are strongly related to the first two they are considered here as auxiliary notions that may help to clarify some interesting relations. To

avoid possible confusion note that weak k-domination is a natural generalization of 2-domination as defined in [2], and it is not the weak domination as defined in [21].
Cylindrical graphs, or cylindrical grid graphs, are Cartesian products of a path and a cycle. Domination number of cylindrical graphs has been studied in [5, 19] and [13]. The best so far lower bound for the 2-domination number of cylindrical graphs is proven in [18]. The computation of the 2-domination number of cylindrical graphs for arbitrary paths and small cylinders was addressed in [12], and for cylinders with small paths and arbitrary cycles in [11]. Domination numbers for 2-domination, 2-rainbow domination, singleton 2-rainbow domination, and weak 2-domination of cylindrical graphs were studid in [23]. For a recent brief outline of related previous work we refer to [23]. The work cited in [23] mainly refers to 2-domination and 2-rainbow domination, here we intend to add only a reference regarding the 4-rainbow domination. It seems that the only paper that reports a result on 4-rainbow domination of cylindrical graphs is [10], where it is shown that $\gamma_{r4}(P_m \Box C_3) = 2m$ and $\gamma_{r4}(P_3 \Box C_n) = 2n$.
As vertices in grid graphs are of degree at most four, the interesting k are 2,3, and 4. In this paper, we look at the case $k = 4$, the motivating question being whether the results known for $k = 2$ can be proven for $k = 4$ as well. In short, the answer is positive only in part. In particular there is no case in which the exact values of all the invariants would be equal. On the positive side, one of the constructions are general in the sense that they can be essentially used for more than one invariant and similarly holds for the proof of one lower bound. In some cases exact values are found, however, there is no case in which the domination numbers for all the variations would be equal.
The rest of the paper is organized as follows. In the next section, we recall some basic definitions and facts. Section 3 summarizes the results that are proven in Section 4, where some technical definitions and proofs are given. In the last section, we conclude with some challenges for future work.

2. Preliminaries

A finite, simple and undirected graph $G = (V(G), E(G))$ is given by a set of vertices $V(G)$ and a set of edges $E(G)$. Edges are pairs of vertices. As usual, the edge $\{i, j\} \in E(G)$ is shortly denoted by ij.
The Cartesian product of two graphs, $G \Box H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. The Cartesian product of graphs is one of the standard graph products [14]. The Cartesian product is commutative. In other words: $G \Box H$ is isomorphic to $H \Box G$. In Cartesian product, copies of each factor appear as subgraphs and are called layers. In $G \Box H$, there are $|V(G)|$ isomorphic copies of H and $|V(H)|$ isomorphic copies of G.
The path on m vertices P_m is usually defined as the graph on vertex set $V(P_m) = \{i \mid 0 \leq i < m\}$ and the set of edges $E(P_m) = \{ij \mid 0 \leq i < m-1, j = i+1\}$. The cycle on n vertices C_n is usually defined as the graph on vertex set $V(C_n) = \{i \mid 0 \leq i < n\}$

and the set of edges $E(C_n) = \{ij \mid 0 \leq i < n, j \equiv i+1 \pmod{n}\}$. Here we are mainly interested in *cylindrical graphs* that are Cartesian products of a path and a cycle, $P_m \Box C_n$.

A set of vertices S is a *dominating set* of G if every vertex in the complement $V(G) \setminus S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set of G is called the *domination number*, $\gamma(G)$.

There are many variations of the basic domination, see e.g. [16]. Here we are interested in 4-domination, 4-rainbow domination, singleton 4-rainbow domination, and weak 4-domination.

A *k-dominating set* D is a set of vertices such that each vertex not in the set has at least k neighbors in the set. The *k-domination number* $\gamma_k(G)$ is the number of vertices in a smallest k-dominating set for G. Equivalently, a dominating set D can be given in terms of a *k-domination function* (kDF), that is with a function $f : V(G) \to \{0,1\}$ for which $f(v) = 1$ iff $v \in D$ and $f(v) = 0$ iff $v \notin D$.

A *k-rainbow dominating function* (kRDF) f of G assigns subsets of $\{1,2,\ldots,k\}$ to vertices, such that for vertex v with $f(v) = \emptyset$, $\bigcup_{u \in N(v)} f(u) = \{1,2,\ldots,k\}$. The *weight* $w(f)$ of kRDF f is $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is the *k-rainbow domination number* denoted by $\gamma_{rk}(G)$.

An interesting special case are kRD functions that assign only singletons and empty sets. Such functions are called *singleton kRD functions* (SkRDF) and the minimal weight obtained when considering only SkRD functions is the *singleton k-rainbow domination number*, $\tilde{\gamma}_{rk}$. Clearly, $\gamma_{rk} \leq \tilde{\gamma}_{rk}$ [6].

Weak 2-domination was introduced in [2] as an auxiliary notion in the study of 2-rainbow domination on trees. Straightforward generalization to arbitrary k is given here (the original definition is obtained by taking $k = 2$). A function $f : V(G) \to \{0,1,2,\ldots,k\}$ is called a *weak k-domination* function (WkDF) if it has the following property: for any vertex with $f(v) = 0$ it holds $\sum_{u \in N(v)} f(u) \geq k$. The *weight* of f is $w(f) = \sum_{v \in V} f(v)$ and the *weak k-domination number* of G, $\gamma_{wk}(G)$, is the minimum weight over all (WkDF) of G. Obviously, any k-rainbow domination function f gives rise to a weak k-domination function w defined as $w(v) = |f(v)|$. In words, $w(v)$ is the cardinality of the set of colors assigned by f. Thus clearly $\gamma_{wk} \leq \gamma_{rk}$. For convenience, a weak k-domination function with minimum weight over all WkDF of G is called a $\gamma_{wk}(G)$-function.

Furthermore, it is straightforward to see that weak k-domination is also a relaxation of k-domination as any k-domination function is also a weak k-domination function. On the other hand, a weak k-domination function is a k-domination function only if $f(v) \leq 1$ for all $v \in V$. Consequently, $\gamma_{wk} \leq \gamma_k$.

Similarly, any singleton k-rainbow domination function is a k-domination function, while the opposite is not true in general, so we have $\gamma_k \leq \tilde{\gamma}_{rk}$ [6].

Summarizing, we have the inequalities

$$\gamma_{wk}(G) \leq \gamma_{rk}(G) \leq \tilde{\gamma}_{rk}(G) \qquad \text{and} \qquad \gamma_{wk}(G) \leq \gamma_k(G) \leq \tilde{\gamma}_{rk}(G)\,. \tag{2.1}$$

3. The results

In this paper we first prove lower bounds for weak 4-domination (Proposition 1) and for 4-domination (Proposition 2). For the upper bounds, we provide constructive proofs for singleton 4-rainbow domination (Lemma 8) and for weak 4-domination (Lemma 9). Summarizing and recalling the relations (2.1) we have the following theorems.

Theorem 1. *For $m, n \geq 3$ it holds*

$$\frac{(m+1)n}{2} \leq \gamma_{w4}(P_m \Box C_n)) \leq \begin{cases} \frac{(m+1)n}{2}, & n \equiv 0 \pmod 4 \\ \frac{(m+1)(n+1)}{2} + 1, & n \equiv 1 \pmod 4 \\ \frac{(m+1)n}{2} + 1, & n \equiv 2 \pmod 4 \\ \frac{(m+1)(n+1)}{2}, & n \equiv 3 \pmod 4 \end{cases} \tag{3.1}$$

Proof. Follows from Proposition 1 and Lemma 9. □

Corollary 1. *For $m \geq 3$ and $n \equiv 0 \pmod 4$,*

$$\gamma_{w4}(P_m \Box C_n)) = \frac{(m+1)n}{2}. \tag{3.2}$$

Theorem 2. *For $m \geq 3$, $n \geq 4$ it holds*

$$\frac{(m+2)n}{2} \leq \gamma_4(P_m \Box C_n)) \leq \tilde{\gamma}_{r4}(P_m \Box C_n)) \leq \begin{cases} \frac{(m+2)n}{2}, & n \equiv 0 \pmod 4 \\ \frac{(m+2)(n+1)}{2}, & n \equiv 1,3 \pmod 4 \\ \frac{(m+2)(n+2)}{2}, & n \equiv 2 \pmod 4 \end{cases} \tag{3.3}$$

Proof. Follows from Proposition 2 and Lemma 8, using the inequality $\gamma_4(G) \leq \tilde{\gamma}_{r4}(G)$ from (2.1). □

Corollary 2. *For $m \geq 3$ and $n \equiv 0 \pmod 4$,*

$$\tilde{\gamma}_{r4}(P_m \Box C_n)) = \gamma_4(P_m \Box C_n)) = \frac{(m+2)n}{2}. \tag{3.4}$$

Theorem 3. *For $m, n \geq 3$ it holds*

$$\frac{(m+1)n}{2} \leq \gamma_{r4}(P_m \Box C_n)) \leq \begin{cases} \frac{(m+2)n}{2}, & n \equiv 0 \pmod 4 \\ \frac{(m+2)(n+1)}{2}, & n \equiv 1,3 \pmod 4 \\ \frac{(m+2)(n+2)}{2}, & n \equiv 2 \pmod 4 \end{cases} \tag{3.5}$$

Proof. Follows from Proposition 1 and Corollary 3 to Lemma 8, using the inequality $\gamma_{w4}(G) \leq \gamma_{r4}(G)$. □

Remark. Note that there ia s significant gap between the two bounds for the 4-rainbow domination number γ_{r4}. However, the example provided in [10], namely $\gamma_{r4}(P_m \Box C_3)) = 2m$ shows that the lower bound is tight indicating that the lower bound may give exact values in some other cases as well. Unfortunately, we were not able to find a general construction for $\gamma_{r4}(P_m \Box C_n)$ that would improve our construction in Lemma 8.

4. Proofs

We start with some obsevations for general k in the first subsection, and then restrict to special case $k = 4$.

4.1. Lower bound for weak k-domination

The observations in this section are in part generalizations of the work in [23].
For simplicity, we introduce some more notation. For $0 \leq i \leq m-1$, the set of vertices $\mathcal{C}^i = \{(i,0),(i,1),(i,2),\cdots,(i,n-1)\}$ is called the i-th column of $P_m \Box C_n$.
Recall that a weak k-domination function is a function $f : V \to \{0,1,2,\ldots,k\}$, where for vertices with $f(v) = 0$ we have $\sum_{u \in N(v)} f(u) \geq k$. In words, a vertex may be assigned up to k colors, and the uncolored vertices must have at least k colors in the neighborhood. (We use term colors although we do not distinguish among them, just count.)
Let f be a kRDF of $P_m \Box C_n$ and $s_i = \sum_{x \in \mathcal{C}^i} f(x)$. (We write briefly $f(v)$ instead of $f(i,j)$ where no confusion is possible.) Note that s_i is a sum of contributions of vertices with $f(v) > 0$. The sequence $(s_0, s_1, \cdots, s_{m-1})$, is called the kRDF sequence corresponding to f.

Lemma 1. *Let f be a $\gamma_{wk}(P_m \Box C_n)$-function. Then we have, for $1 \leq i \leq m-2$,*

$$s_{i-1} + s_{i+1} \geq kn - (k+2)s_i, .$$

Remark. Statement of Lemma 1 holds for "inner" indices only. We wish to note that the same argument allows a proof for products of two cycles and even of graph bundles [20] of cycles over cycles. In these generalizations there is no restriction on indices, the statement holds for every i (taken modulo m).
We continue with the proof of Lemma 1.

Proof. Write

$$s_i = s_{i,1} + 2s_{i,2} + 3s_{i,3} + \cdots + ks_{i,k} \tag{4.1}$$

where $s_{i,j}$ is the number of vertices in $\mathcal{C}^i$ that are assigned j colors (i.e. $f(v) = j$).

For a later reference observe that

$$\begin{aligned}(k+2)s_i &= (k+2)(s_{i,1}+2s_{i,2}+\cdots+ks_{i,k}) \\ &= (k+2)s_{i,1}+(2k+4)s_{i,2}+\cdots+(k^2+2k)s_{i,k} \\ &\geq (k+2)s_{i,1}+(k+4)s_{i,2}+\cdots+(k+2k)s_{i,k} \\ &= \sum_{j=1}^{k}(k+2j)s_{i,j} \qquad (4.2)\end{aligned}$$

The demand of vertices in $\mathcal{C}^i$ is kn, and the colored vertices (of weight s_i) fulfill at most

$$(k+2)s_{i,1}+(k+4)s_{i,2}+\cdots+(k+2k)s_{i,k}=\sum_{j=1}^{k}(k+2j)s_{i,j} \qquad (4.3)$$

within $\mathcal{C}^i$ as colored vertex v is k-dominated if it is assigned a positive value $f(v)>0$, and this also contributes $f(v)$ to each of its neighbor(s) demand.
Hence, using (4.2), and assuming that f is a kRDF, i.e. $s_{i-1}+s_{i+1}+\sum_{j=1}^{k}(k+2j)s_{i,j}\geq kn$, we have

$$s_{i-1}+s_{i+1}+(k+2)s_i\geq s_{i-1}+s_{i+1}+\sum_{j=1}^{k}(k+2j)s_{i,j}\geq kn \qquad (4.4)$$

or, $s_{i-1}+s_{i+1}\geq kn-(k+2)s_i$, as needed. □

For a later reference, observe that in case $k=4$, Lemma 1 reads as

$$s_{i-1}+s_i+s_{i+1}\geq 4n-5s_i \qquad (4.5)$$

Intuitively, from (4.4) we observe that if all s_i are equal, then $s_i\geq\frac{k}{k+4}n$, hence we expect the lower bound to be $\simeq\frac{k}{k+4}nm$. However, such balanced assignment on cylindrical graphs obviously does not exist. Before analyzing how much larger than the average s_0 and s_{m-1} must be let us show that it is not possible to have three consecutive s_i smaller from the intuitive lower bound $\frac{k}{k+4}n$.
Assume that for some i, $1\leq i\leq m-2$ it holds $s_{i-1}<\frac{k}{k+4}n$, $s_i<\frac{k}{k+4}n$, and $s_{i+1}<\frac{k}{k+4}n$. Then $s_{i-1}+s_{i+1}+(k+2)s_i<(k+4)\frac{k}{k+4}n=kn$, contradiction to Lemma 4.5 (more precisely, to (4.4) in its proof). So we have

Lemma 2. *Let f be a $\gamma_{wk}(P_m\Box C_n)$-function. If for some i, $1\leq i\leq m-2$, we have $s_i<\frac{k}{k+4}n$, then $s_{i-1}>\frac{k}{k+4}n$ or $s_{i+1}>\frac{k}{k+4}n$.*

We now consider the border indices. Clearly, if $s_0=s_{0,1}=n$ then all vertices in $\mathcal{C}^0$ are colored, so we can assume that $s_0\leq n$. Similarly, $s_{m-1}\leq n$. For a later reference, let us write it as a lemma.

Lemma 3. *For any $\gamma_{wk}(P_m \Box C_n)$-function we have $s_0 \leq n$, and $s_{m-1} \leq n$.*

Lemma 4. *Let f be a $\gamma_{wk}(P_m \Box C_n)$-function. Then $s_0 = n - \Delta \leq n$, then $s_1 \geq (k-2)\Delta$.*

Proof. If $s_0 = n - \Delta$, then this fulfills at most demand $ks_0 + 2(n - s_0)$ in $\mathcal{C}^0$ (out of kn). The remaining demand, $(k-2)(n - s_0) = (k-2)\Delta$ in $\mathcal{C}^0$ must be covered from $\mathcal{C}^1$ hence $s_1 \geq (k-2)\Delta$. □

4.2. Lower bound for weak 4-domination

We continue the discussion focusing on the special case, $k = 4$. The proof of the next proposition is based on idea of discharging. We follow the ideas of [22] and [4].
We define a discharging rule in which the rows with sufficiently large s_i give half of their overweight to one or both of the neighboring columns. More precisely, let f' be a new function on the vertex set of $P_m \Box C_n$ that assigns a positive real number to each vertex. Note that f' need not be a W4RDF because noninteger weights are allowed. Denote by $s'_i = \sum_{x \in \mathcal{P}^i} f'(x)$ and let $(s'_1, s'_2, \ldots, s'_m)$ be the sequence corresponding to f'.
We aim to obtain $s'_i \geq \frac{n}{2}$ for $0 < i < m-1$, $s'_0 \geq \frac{3n}{4}$ and $s'_{m-1} \geq \frac{3n}{4}$. Indeed, it turns out that this can be done by one round of discharging from the s_i corresponding to an arbitrary W4RDF f we start with. Roughly speaking, the overloaded columns give half of their overweights to the two neighbors whenever applicable. Formally, the rules are given below. Although in principle very simple, the rules are somewhat technical because there are obviously some exceptions needed around the first and the last row.
Formally, the discharging rules are as follows:

1. For $1 < i < m-2$: If $s_i > \frac{n}{2}$ then set $s'_i = \frac{n}{2}$. If $s_i \leq \frac{n}{2}$, then
   (a) if $s_{i-1} > \frac{n}{2}$ and $s_{i+1} > \frac{n}{2}$, then $s'_i = s_i + \frac{1}{2}(s_{i-1} - \frac{n}{2}) + \frac{1}{2}(s_{i+1} - \frac{n}{2})$,
   (b) if $s_{i-1} > \frac{n}{2}$ and $s_{i+1} < \frac{n}{2}$, then $s'_i = s_i + \frac{1}{2}(s_{i-1} - \frac{n}{2})$,
   (c) if $s_{i-1} < \frac{n}{2}$ and $s_{i+1} > \frac{n}{2}$, then $s'_i = s_i + \frac{1}{2}(s_{i+1} - \frac{n}{2})$.
2. For $i = 1$: If $s_1 > \frac{n}{2}$ then set $s'_1 = \frac{n}{2}$. If $s_1 \leq \frac{n}{2}$, then
   (a) if $s_0 > \frac{3n}{4}$ and $s_2 > \frac{n}{2}$, then $s'_1 = s_1 + (s_0 - \frac{3n}{4}) + \frac{1}{2}(s_2 - \frac{n}{2})$,
   (b) if $s_0 > \frac{3n}{4}$ and $s_2 < \frac{n}{2}$, then $s'_1 = s_1 + s_0 - \frac{3n}{4}$,
   (c) if $s_0 < \frac{3n}{4}$ and $s_2 > \frac{n}{2}$, then $s'_1 = s_1 + \frac{1}{2}(s_2 - \frac{n}{2})$.
3. For $i = m-2$: If $s_{m-2} > \frac{n}{2}$ then set $s'_{m-2} = \frac{n}{2}$. If $s_{m-2} \leq \frac{n}{2}$, then
   (a) if $s_{m-3} > \frac{n}{2}$ and $s_{m-1} > \frac{3n}{4}$, then $s'_{m-2} = s_{m-2} + (s_{m-1} - \frac{3n}{4}) + \frac{1}{2}(s_{m-3} - \frac{n}{2})$,
   (b) if $s_{m-3} > \frac{n}{2}$ and $s_{m-1} < \frac{3n}{4}$, then $s'_{m-2} = s_{m-2} + \frac{1}{2}(s_{m-3} - \frac{n}{2})$,
   (c) if $s_{m-3} < \frac{n}{2}$ and $s_{m-1} > \frac{3n}{4}$, then $s'_{m-2} = s_{m-2} + s_{m-1} - \frac{3n}{4}$.

4. if $s_0 > \frac{3n}{4}$ then set $s'_0 = \frac{3n}{4}$ and otherwise, if $s_0 < \frac{3n}{4}$ then $s'_0 = s_0 + \frac{1}{2}(s_1 - \frac{n}{2})$.
5. if $s_{m-1} > \frac{3n}{4}$ then set $s'_{m-1} = \frac{3n}{4}$ and otherwise, if $s_{m-1} < \frac{3n}{4}$ then $s'_{m-1} = s_{m-1} + \frac{1}{2}(s_{m-2} - \frac{n}{2})$.

Lemma 5. *Let $m \geq 4$, $n \geq 4$, and f be a $\gamma_{w4}(P_m \Box C_n)$-function. Then* $\sum_{i=0}^{m-1} s_i \geq \sum_{i=0}^{m-1} s'_i \geq \frac{(m+1)n}{2}$.

Proof. Clearly, if $s_0 \geq \frac{3n}{4}$, $s_{m-1} \geq \frac{3n}{4}$, and $s_i \geq \frac{n}{2}$ for $0 < i < m-1$, then $\sum_{i=0}^{m-1} s_i \geq \frac{(m+1)n}{2}$, and there is nothing to prove.
Observe that it is not possible to have three consecutive s_i with $s_i < \frac{n}{2}$. Indeed, from Lemma 1 we have $s_{i-1}+s_i+s_{i+1} \geq 4n-5s_i$ (recall eqn. (4.5)). Below we also assume $s_0 \leq n$ and $s_{m-1} \leq n$, recalling Lemma 3.
First consider the inner columns, $1 < i < m-2$. We claim that $s'_i \geq \frac{n}{2}$. If $s_i \geq \frac{n}{2}$ then $s'_i = \frac{n}{2}$ by discharging rule 1. If $s_i < \frac{n}{2}$ then by the Lemma 2, we can not have both $s_{i-1} < \frac{n}{2}$ and $s_{i+1} < \frac{n}{2}$. Let us distinguish the other three possibilities.

- If $s_{i-1} \geq \frac{n}{2}$ and $s_{i+1} \geq \frac{n}{2}$, then by Lemma 1, $s_{i-1} + s_{i+1} \geq 4n - 6s_i$. Write $s_i = \frac{n}{2} - \Delta$, $s_{i-1} = \frac{n}{2} + \Delta_1$, $s_{i+1} = \frac{n}{2} + \Delta_2$, $s_{i-1} + s_{i+1} = \frac{n}{2} + \Delta_1 + \frac{n}{2} + \Delta_2 \geq 4n - 6(\frac{n}{2} - \Delta) = n + 4\Delta$. so $\Delta_1 + \Delta_2 \geq 4\Delta$ and hence, by rule 1(a),
$$s'_i = s_i + \frac{1}{2}(s_{i-1} - \frac{n}{2}) + \frac{1}{2}(s_{i+1} - \frac{n}{2}) = \frac{n}{2} - \Delta + \frac{1}{2}\Delta_1 + \frac{1}{2}\Delta_2 > \frac{n}{2}.$$
- Next, assume $s_{i-1} \geq \frac{n}{2}$ and $s_{i+1} < \frac{n}{2}$. Then by Lemma 1, $s_{i-1} \geq 4n - 6s_i - s_{i+1}$. Write $s_i = \frac{n}{2} - \Delta$, hence $s_{i-1} \geq 4n - 6s_i - s_{i+1} = n + 6\Delta - s_{i+1} > \frac{n}{2} + 6\Delta$ and by rule 1(b),
$$s'_i = s_i + \frac{1}{2}(s_{i-1} - \frac{n}{2}) = \frac{n}{2} - \Delta + \frac{1}{2}(6\Delta) > \frac{n}{2}.$$
- Finaly, assume $s_{i-1} < \frac{n}{2}$ and $s_{i+1} \geq \frac{n}{2}$ and proceed analogously as in the previous case.

It remains to check the four rows $i = 0, 1, m-2$, and $m-1$.

- Consider first the column 0. Recall that by Lemma 4 $s_0 = n - \Delta$ implies $s_1 \geq 2\Delta$. Assume $s_0 < \frac{3n}{4}$. Then, since column 1 can give half of $s_1 - \frac{n}{2}$ to each of its two neighbors,
$$s'_0 = n - \Delta + \frac{1}{2}(s_1 - \frac{n}{2}) \geq n - \Delta + \frac{1}{2}(2\Delta - \frac{n}{2}) \geq \frac{3n}{4},$$
as needed.

- Now consider the column 1. First observe that $s_0 \le \frac{7n}{8}$ implies $s_1 \ge 2\Delta > \frac{n}{2}$. Therefore, if $s_1 < \frac{n}{2}$ we must have $s_0 \ge \frac{7n}{8}$. Write, as before $s_0 = \frac{n}{2} + \Delta_1$. The difference when considering $i = 1$ to the previous reasonning in case of general i is that one neighbor, column 0, must have $\frac{3n}{4}$ instead of $\frac{n}{2}$, but, on the other hand can give to column 1 all its weight over $\frac{3n}{4}$. We claim that this is possible, because it holds
$$\Delta_1 - \frac{n}{4} \ge \frac{\Delta_1}{4}\,.$$
This is true because, first, we have just observed that $s_0 - \frac{3n}{4} \ge \frac{n}{8}$, and on the other hand, $\Delta_1 \le \frac{n}{2}$ implies $\frac{\Delta_1}{4} \le \frac{n}{8}$. Now we proceed similarly as for $1 < i < m-2$. Recall that we have, by Lemma 1, $s_{i-1} + s_{i+1} \ge 4n - 6s_i$, and as before, write $s_i = \frac{n}{2} - \Delta$, $s_{i-1} = \frac{n}{2} + \Delta_1$, $s_{i+1} = \frac{n}{2} + \Delta_2$. Observe that $s_{i-1} + s_{i+1} = \frac{n}{2} + \Delta_1 + \frac{n}{2} + \Delta_2 \ge 4n - 6(\frac{n}{2} - \Delta) = n + 4\Delta$. So $\Delta_1 + \Delta_2 \ge 4\Delta$ and hence, if $s_0 \ge \frac{3n}{4}$ and $s_2 \ge \frac{n}{2}$, by rule 2(a),
$$s_1' = s_1 + (s_0 - \frac{3n}{4}) + \frac{1}{2}(s_2 - \frac{n}{2}) = \frac{n}{2} - \Delta + \frac{1}{4}\Delta_1 + \frac{1}{2}\Delta_2 > \frac{n}{2} - \Delta + \frac{1}{4}\Delta_1 + \frac{1}{4}\Delta_2 \ge \frac{n}{2}\,.$$
Similarly for the other two cases, applying rules 2(b) and 2(c).

The remaining cases, columns $m-1$ and $m-2$, can be handled analogously to columns 0 and 1, by obvious symmetry. □

Proposition 1. *For $m \ge 3$ and $n \ge 3$, $\gamma_{w4}(P_m \Box C_n) \ge \frac{(m+1)n}{2}$.*

Proof. The statement follows directly from Lemma 5. □

4.3. Lower bound for 4-domination

Recall that in definition of k-domination, each vertex is assigned at most one color. This allows to improve the lower bound for $k = 4$.

Lemma 6. *For any $\gamma_4(P_m \Box C_n)$-function we have $s_0 \ge n$, and $s_{m-1} \ge n$.*

Proof. Obviously there is no 4-dominating function f in which a vertex $v \in C^0$ is not assigned any color, i.e. $f(v) = 0$ because there must be 4 colors in the neighborhood of v, but there are only 3 neighbors, and any vertex can have at most one color. □

According to Lemma 6, we know that $s_0 \ge n$, and $s_{m-1} \ge n$, which gives rise to a proof of a better lower bound.

Proposition 2. *For $m \ge 2$ and $n \ge 3$, $\gamma_4(P_m \Box C_n) \ge (m+2)\lceil \frac{n}{2} \rceil$.*

Proof. As we know that all vertices in C^0 must be assigned a color, we consider C^1 and observe that at least one half of the vertices must be colored as each has only one neighbor in C^2, so it must have two colored neighbors in C^1. Hence $s_1 = \lceil \frac{n}{2} \rceil$. In C^2, $\lfloor \frac{n}{2} \rfloor$ vertices must be colored to fulfill the demans of vertices in C^1. If $\lfloor \frac{n}{2} \rfloor < \frac{n}{2}$, there are two uncolored adjacent vertices. One must be colored so $s_2 \geq \lceil \frac{n}{2} \rceil$. By induction, all $s_i \geq \lceil \frac{n}{2} \rceil$, and finally, $s_{m-1} \geq n$ by Lemma 6. □

4.4. Basic construction - upper bound for singleton rainbow domination

Recall from [4] that a 2RDF of *grid graphs* may be based on certain patterns of infinite grid. By grid graphs we mean the graphs that are constructed by taking a finite induced subgraph of the infinite grid, and then identify some pairs of the vertices on the boundary. Examples include, besides the cylindrical graphs that we study here also Cartesian products of cycles (that are examples of toroidal graphs) [4, 8, 9] and graph bundles of cycles over cycles [3]. The products of paths are obtained without any identifications needed.
In order to outline the two patterns, let us start with a function defined on $\mathbb{Z} \times \mathbb{Z}$,

$$f(i,j) = \begin{cases} 0, & i \not\equiv j \pmod 2 \\ 1 + (\frac{i+j}{2} \bmod 2), & i \equiv j \pmod 2 \text{ and } i,j \text{ even} \\ 3 + (\frac{i+j-2}{2} \bmod 2), & i \equiv j \pmod 2 \text{ and } i,j \text{ odd.} \end{cases} \tag{4.6}$$

Note that the case in the third line of definition (4.6), when both i and j are odd, can be defined also as $f(i,j) = 2 + f(i-1, j-1)$. Similarly, the values at even i, j satisfy $f(i,j) = f(i+1, j+1) - 2$.
Define the coordinate system, for example with

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & (0,0) & (0,1) & \dots \\ \dots & (1,0) & (1,1) & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{4.7}$$

to obtain

$$f = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & \dots \\ \dots & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & \dots \\ \dots & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & \dots \\ \dots & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & \dots \\ \dots & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & \dots \\ \dots & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & \dots \\ \dots & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & \dots \\ \dots & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{4.8}$$

It is straightforward to see the pattern can be seen as a tilling with quadrilaterals of size 4×4,

$$f = \left[\begin{array}{ccc|cccc|cccc|c}
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & \dots \\
\dots & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & \dots \\
\hline
\dots & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & \dots \\
\dots & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & \dots \\
\dots & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & \dots \\
\dots & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & \dots \\
\hline
\dots & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 1 & 0 & \dots \\
\dots & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}\right]. \tag{4.9}$$

illustrating the well-known fact $\gamma_{r4}(C_m \Box C_n) = \frac{mn}{4}$ when both m and n are multiples of 4 [8]. In fact, as the assignment (4.8) defines a singleton 4RDF, therefore we have $\tilde{\gamma}_{r4}(C_m \Box C_n) = \frac{mn}{4}$ when both m and n are multiples of 4.
Now consider application of the last pattern to domination of $P_m \Box C_n$. For example, a 4RDF of $P_{10} \Box C_4$ of weight 20 may be given by

$$\left[\begin{array}{cc|cccc|cccc|cc}
2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 \\
0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 \\
1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 \\
0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4
\end{array}\right] \longrightarrow \left[\begin{array}{c|cccc|cccc|c}
\rightarrow 2 & 1 & 0 & 2 & 0 & 1 & 0 & 2 & 0 & 1 \\
4 & 0 & 3 & 0 & 4 & 0 & 3 & 0 & 4 & 3 \leftarrow \\
\rightarrow 1 & 2 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 2 \\
3 & 0 & 4 & 0 & 3 & 0 & 4 & 0 & 3 & 4 \leftarrow
\end{array}\right]. \tag{4.10}$$

Another (smallest) example is a 4RDF for $P_2 \Box C_4$ of weight 8 :

$$\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 4 & 0 & 3 \\ 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} \rightarrow 2 & 1 \\ 4 & 3 \leftarrow \\ \rightarrow 1 & 2 \\ 3 & 4 \leftarrow \end{bmatrix}. \tag{4.11}$$

The observation can be generalized by the following Lemma.

Lemma 7. *For $m \geq 2$ and $n \equiv 0 \pmod 4$, it holds*

$$\tilde{\gamma}_{r4}(P_m \Box C_n)) \leq \frac{(m+2)n}{2} \tag{4.12}$$

Proof. Start with the infinite pattern and consider a $(m+2) \times n$ square. Merge the first two and the last two columns to obtain a 4RDF. Straightforward details are left to the reader. □

The basic construction can be used to obtain upper bounds for all n.

Lemma 8. *Let $m \geq 2$ and $n \geq 4$. Then we have*

$$\tilde{\gamma}_{r4}(P_m \Box C_n)) \leq \begin{cases} \frac{(m+2)n}{2}, & n \equiv 0 \pmod 4 \\ \frac{(m+2)(n+1)}{2}, & n \equiv 1,3 \pmod 4 \\ \frac{(m+2)(n+2)}{2}, & n \equiv 2 \pmod 4 \end{cases} \tag{4.13}$$

Proof. We will construct 4RD functions, starting from the subgrid of size $(m+2) \times n'$, where $n' = 4\left\lceil \frac{n}{4} \right\rceil$. Wlog, the coordinates are $-1 \leq i \leq m$ and $0 \leq j \leq n'-1$.

Case 1. $n \equiv 0 \bmod 4$. In this case, $n' = n$. Recall Lemma 7, and the result follows.

Case 2. $n \equiv 1 \bmod 4$. In this case, $n' = n+3$. Merge the rows $n-1$ and $n+2$, or, more formally, define $\tilde{f}(i, n-1) = f(i, n-1) + f(i, n+2)$ for $0 \leq i \leq m-1$. Then merge the first two and last two columns, i.e. define $\tilde{f}(0, j) = f(0, j) + f(-1, j)$ for columns 0 and -1; and $\tilde{f}(m-1, j) = f(m, j) + f(m-1, j)$ to merge columns m and $m-1$. Otherwise, set $\tilde{f}(i, j) = f(i, j)$, for $1 \leq i \leq m-2$ and $0 \leq j \leq n-1$. Observe that the $\tilde{f}$ defined on the square $0 \leq i \leq m-1$ and $0 \leq j \leq n-1$ gives rise to a 4RDF of $P_m \Box C_n$.

Case 3. $n \equiv 2 \bmod 4$. Now $n' = n+2$. In this case, we merge two pairs of rows, for example first two rows (0 and 1) and last two rows (n and $n+1 = n'-1$). In the other rows, merge the two pairs of columns as before. The 4RDF of $P_m \Box C_n$ is in this case obtained from the assignment on $0 \leq i \leq m-1$ and $1 \leq j \leq n$.

Case 4. $n \equiv 3 \bmod 4$. In this case, $n' = n+1$. Similarly to the case $n \equiv 1 \bmod 4$, merge rows n and $n-1$. □

Note that the construction in the proof of Lemma 8 works for $n \geq 4$ because otherwise, when merging the rows, we may assign more than one color to a vertex. If insteaad of singleton 4RDF we aim to construct a 4RDF, sets of colors may be assigned to vertices. So the construction in this case may be used for $n = 3$ as well.

Corollary 3. *Let $m \geq 2$ and $n \geq 3$. Then we have*

$$\gamma_{r4}(P_m \Box C_n)) \leq \begin{cases} \frac{(m+2)n}{2}, & n \equiv 0 \pmod 4 \\ \frac{(m+2)(n+1)}{2}, & n \equiv 1,3 \pmod 4 \\ \frac{(m+2)(n+2)}{2}, & n \equiv 2 \pmod 4 \end{cases} \tag{4.14}$$

4.5. Basic construction - upper bound for weak 4-domination

Recall the pattern (4.9) defining a 4RDF, but note that now we only count the number of colors in the neighborhood. So, ignore the numbers, just count the number of nonzero assignments.

$$f = \left[\begin{array}{cccccccccccc}
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \dots \\
\dots & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & \dots \\
\hline
\dots & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \dots \\
\dots & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & \dots \\
\dots & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \dots \\
\dots & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & \dots \\
\hline
\dots & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \dots \\
\dots & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}\right]. \tag{4.15}$$

Clearly, we have the following lemma.

Lemma 9. *For $m, n \geq 2$ it holds*

$$\gamma_{w4}(P_m \square C_n)) \leq \begin{cases} \frac{(m+1)n}{2}, & n \equiv 0 \pmod 4 \\ \frac{(m+1)(n+1)}{2} + 1, & n \equiv 1 \pmod 4 \\ \frac{(m+1)n}{2} + 1, & n \equiv 2 \pmod 4 \\ \frac{(m+1)(n+1)}{2}, & n \equiv 3 \pmod 4). \end{cases} \tag{4.16}$$

Proof. Obviously, for n even, the pattern defines a (weak) 4-dominating assignment on the columns $i = 1, 2, \dots, m-2$. Consider the column 0. Replace every second $\star$ with two colors. Observe that this 4-dominates all vertices in C^0. Similarly for C^{m-1} and if $n \equiv 0 \pmod 4$, and the result in this case is proven.
Now let $n \equiv 2 \pmod 4$. In this case, there is an odd number of $\star$'s in columns 0 and $m-1$ so we need to replace $2\frac{n}{4} = \frac{n}{2} + 1$ of them. Hence we need $m\frac{n}{2} + \frac{n}{2} + 1 = (m+1)\frac{n}{2} + 1$, as claimed.
If n is odd, then we can start with an assignment for $n' = n + 1$ and merge two rows. The result follows. □

Now consider an application of the last pattern to domination of $P_m \square C_n$. For example, an assignment based on pattern (4.15), altered as suggested in the proof of Lemma 9, for $P_{10} \square C_4$ of weight 20 is given by

$$\left[\begin{array}{cc|cccc|cccc|cc}
\star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & 0 & \star \\
0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & \star & 0 \\
\star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & 0 & \star \\
0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & \star & 0
\end{array}\right] \longrightarrow \left[\begin{array}{r|cccc|cccc|l}
0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star \\
x\star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 \\
0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star y \\
\star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0
\end{array}\right]. \tag{4.17}$$

Recall that, as the colors are now irrelevant, the vertex assigned $x\star$ satisfies the missig demand of two neighbors, similarly the vertex assigned $\star y$.

As another example, an assignment for $P_{10}\Box C_5$ is given by

$$\left[\begin{array}{r|cccc|cccc|l} \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\ 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star \\ x\star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 \\ 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star y \\ \star & 0 & \star & 0 & \star & 0 & \star & 0 & \star & 0 \end{array}\right]. \tag{4.18}$$

5. Conclusions

We conclude with some open questions.

- We have found exact values for a subfamiliy for three invariants including 4-domination, weak 4-domination and the singleton 4-rainbow domination number. It is natural to ask for exact values in the missing cases. We beileve that, at least for small enough m, the algebraic approach [17] (see also [11, 18]) may be the tool of choice.
- We have lower and upper bounds for the 4-rainbow domination number. A significant gap is left between the lower and the upper bounds reported here. It seems that closing this gap may be the main challenge left after this study. However note that the lower bound provided by Lemma 8 is tight, as for $m = 3$ we know that $\gamma_{r4}(P_3\Box C_n)) = 2m = (m+1)\frac{n}{2}$ [10].
- As already mentioned, Lemma 1 can be naturally applied to products of cycles and to graph bundles (twisted products) of cycles over cycles. It seems that finding exact formulae at least in some restricted families are doable.
- Finally, after considering variations of 2-domination in [23] and variations of 4-domination studied here, it is natural to ask whether similar results hold for the case when $k = 3$. (Note that when $k > 4$, 4-domination is trivial because the cylindrical graphs are 4-regular.)

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References

[1] B. Brešar, *Rainbow domination in graphs*, Topics in Domination in Graphs (Cham) (T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, eds.), Springer International Publishing, 2020, pp. 411–443.

[2] B. Brešar, M.A. Henning, and D.F. Rall, *Rainbow domination in graphs*, Taiwanese J. Math. **12** (2008), no. 1, 213–225.
https://doi.org/10.11650/twjm/1500602498.

[3] S. Brezovnik, D. Rupnik Poklukar, and J. Žerovnik, *The 2-rainbow domination number of cartesian bundles over cycles*, CEJOR Cent. Eur. J. Oper. Res. **33** (2025), no. 3, 641–659.
https://doi.org/10.1007/s10100-024-00949-6.

[4] ———, *The 2-rainbow domination number of cartesian product of cycles*, Ars Math. Contemp. **25** (2025), no. 3, #P3.04.
https://doi.org/10.26493/1855-3974.3168.74d.

[5] J.J. Carreño, J.A. Martínez, and M.L. Puertas, *A general lower bound for the domination number of cylindrical graphs*, Bull. Malays. Math. Sci. Soc. **43** (2020), no. 2, 1671–1684.
https://doi.org/10.1007/s40840-019-00765-1.

[6] R. Erveš and J. Žerovnik, *On* 3*-rainbow domination number of generalized petersen graphs* $P(6k, k)$, Symmetry **13** (2021), no. 10, 1860.
https://doi.org/10.3390/sym13101860.

[7] J.F. Fink and M.S. Jacobson, *n-domination in graphs*, Graph Theory with Applications to Algorithms and Computer Science (USA), John Wiley & Sons, Inc., 1985, pp. 283–300.
https://api.semanticscholar.org/CorpusID:118533627.

[8] H. Gao, K. Li, and Y. Yang, *The k-rainbow domination number of* $C_n \square C_m$, Mathematics **7** (2019), no. 12, 1153.
https://doi.org/10.3390/math7121153.

[9] H. Gao, C. Xi, and Y. Yang, *The 3-rainbow domination number of the Cartesian product of cycles*, Mathematics **8** (2020), no. 1, 65.
https://doi.org/10.3390/math8010065.

[10] H. Gao, Y. Zhang, Y. Wang, Y. Guo, X. Liu, R. Liu, C. Xi, and Y. Yang, *Rainbow domination in Cartesian product of paths and cycles*, Internat. J. Found. Comput. Sci. **35** (2024), no. 8, 907–928.
https://doi.org/10.1142/S0129054123500272.

[11] E.M. Garzón, J.A. Martínez, J.J. Moreno, and M.L. Puertas, *Hpc acceleration of large (min,+) matrix products to compute domination-type parameters in graphs*, J. Supercomput. **78** (2022), no. 16, 17826–17843.
https://doi.org/10.1007/s11227-022-04574-5.

[12] ———, *On the 2-domination number of cylinders with small cycles*, Fund. Inform. **185** (2022), no. 2, 185–199.
https://doi.org/10.3233/FI-222107.

[13] D.R. Guichard, *A new lower bound for the domination number of complete cylindrical grid graphs*, J. Comb. Math. Comb. Comput. **120** (2024), 411–416. https://doi.org/10.61091/jcmcc120-38.

[14] R.H. Hammack, W. Imrich, and S. Klavžar, *Handbook of Product Graphs*, vol. 2, CRC press Boca Raton, 2011.

[15] T.W. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs*, CRC press, 2013.

[16] T.W. Haynes, S.T. Hedetniemi, and M.A. Henning, *Topics in Domination in Graphs*, vol. 64, Springer, 2020.

[17] S. Klavžar and J. Žerovnik, *Algebraic approach to fasciagraphs and rotagraphs*, Discrete Appl. Math. **68** (1996), no. 1-2, 93–100. https://doi.org/10.1016/0166-218X(95)00058-Y.

[18] J.A. Martínez, A.B. Castaño-Fernández, and M.L. Puertas, *The 2-domination number of cylindrical graphs*, Comput. Appl. Math. **41** (2022), no. 8, 424. https://doi.org/10.1007/s40314-022-02137-1.

[19] M. Nandi, S. Parui, and A. Adhikari, *The domination numbers of cylindrical grid graphs*, Appl. Math. Comput. **217** (2011), no. 10, 4879–4889. https://doi.org/10.1016/j.amc.2010.11.019.

[20] T. Pisanski, J. Shawe-Taylor, and J. Vrabec, *Edge-colorability of graph bundles*, J. Comb. Theory, B **35** (1983), no. 1, 12–19. https://doi.org/10.1016/0095-8956(83)90076-X.

[21] E. Sampathkumar and L.P. Latha, *Strong weak domination and domination balance in a graph*, Discrete Math. **161** (1996), no. 1-3, 235–242. https://doi.org/10.1016/0012-365X(95)00231-K.

[22] Z. Shao, Z. Li, R. Erveš, and J. Žerovnik, *The 2-rainbow domination numbers of $C_4\Box C_n$ and $C_8\Box C_n$*, Nat. Acad. Sci. Lett. **42** (2019), no. 5, 411–418. https://doi.org/10.1007/s40009-018-0779-y.

[23] J. Žerovnik, *On 2-domination and 2-rainbow domination of cylindrical graphs*, Comput. Appl. Math. **44** (2025), no. 5, 239. https://doi.org/10.1007/s40314-025-03201-2.