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Research Article

Majority sets in graphs

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The authors would like to dedicate this paper to Dr. Odile Favaron, in recognition of her outstanding career in graph theory, acknowledging that each of us have coauthored papers with Dr. Favaron and consider her such a good friend and coauthor.

Abstract: A set of vertices $S \subseteq V$ in a graph $G = (V, E)$ is called an internal majority set if for every vertex $v \in S$, a majority of the neighbors of v are in S, or equivalently, every vertex $v \in S$ has fewer neighbors in $V - S = \overline{S}$ than it has in S. A set S is called an external majority set if for every vertex $v \in \overline{S}$, a majority of the neighbors of v are in S, or equivalently, every vertex $v \in \overline{S}$ has more neighbors in S than it has in $\overline{S}$. A set of vertices $S \subseteq V$ in a graph $G = (V, E)$ is called a total majority set if for every vertex $v \in V$, a majority of the neighbors of v are in S, or equivalently, every vertex $v \in V$ has more neighbors in S than it has in $\overline{S}$. In this paper we show that majority sets in graphs are closely related to, but different than, a variety of sets that have been studied, such as offensive alliances, cost effective and very cost effective sets and unfriendly partitions in graphs. We also prove that the decision problems associated with external majority sets and total majority sets are NP-complete. Finally, we present a list of open problems related to majority sets in graphs.

Keywords: majority sets, independent sets, hereditary properties, dominating sets in graphs.

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1. Introduction

Let $G = (V, E)$ be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Two vertices v and w are *neighbors* in G if they are adjacent, that is, if $vw \in E$. The *open neighborhood* $N(v) = \{u \in V \mid uv \in E\}$ of a vertex $v \in V$ is the set of neighbors u of v, while the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v in G is $\deg(v) = |N(v)|$. For any $S \subseteq V$, the *open neighborhood* of S is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S = \cup_{v \in S} N[v]$. Moreover, throughout this paper we will use the following notation: $\overline{S} = V \setminus S = V - S$ denotes the vertices in V but not in S, called the *complement* of S in G, $G[S]$ is the *subgraph* of G induced by S, and $\delta(G) = \min\{\deg(v) \mid v \in V\}$ is the *minimum degree* of a vertex in V.

A set of vertices $S \subset V$ is called *independent* if no two vertices in S are adjacent, that is, for every $u, v \in S$, $u \notin N(v)$ and $v \notin N(u)$, or equivalently, $uv \notin E$. Let $i(G)$ and $\alpha(G)$, called the *independent domination number* and the *vertex independence number*, respectively, equal the minimum and maximum cardinalities among all maximal independent sets of G. A set of vertices $S \subset V$ is called a *dominating set* if $N[S] = V$, that is, for every vertex $v \in V$, either $v \in S$ or $v \in N(S)$. Let $\gamma(G)$ and $\Gamma(G)$, called the *domination number* and the *upper domination number*, respectively, equal the minimum and maximum cardinalities among all minimal dominating sets of G.

A property $\mathcal{P}$ of a set S of vertices is called *hereditary* (resp. *superhereditary*) if every subset $S' \subset S$ of S (resp. every superset S' of S, $S \subset S'$) also has property $\mathcal{P}$. For instance, the property of being an independent set is hereditary since every subset of an independent set is also independent, while the property of being a dominating set is superhereditary, since every superset of a dominating set is also a dominating set. Studies of hereditary and superhereditary properties related to dominating sets in graphs can be found in the book on domination in graphs by Haynes, Hedetniemi and Slater in 1998 [13], and in papers by Cockayne et al. in 1997 [6] and [7].

The following well known result was established in [7].

Proposition 1. *Let S be a set of vertices having some hereditary or superhereditary property $\mathcal{P}$.*

1. *If $\mathcal{P}$ is hereditary, then S is maximal with respect to $\mathcal{P}$ if and only if for every vertex $w \in \overline{S}$, $S \cup \{w\}$ does not have property $\mathcal{P}$.*
2. *If $\mathcal{P}$ is superhereditary, then S is minimal with respect to $\mathcal{P}$ if and only if for every vertex $v \in S$, $S - \{v\}$ does not have property $\mathcal{P}$.*

In the following we will focus on hereditary and superhereditary properties of sets of vertices, since, according to Proposition 1, for these types of sets it is easy to define the conditions under which a set is either maximal with respect to an hereditary property $\mathcal{P}$ or minimal with respect to a superhereditary property $\mathcal{P}$.

A *domination chain* expresses relationships that exist among independent sets, dominating sets, and irredundant sets in graphs. Irredundance is the concept that describes the minimality of a dominating set. If a dominating set S is *minimal*, then for every vertex $u \in S$, the set $S - \{u\}$ is no longer a dominating set. This means that vertex u dominates some vertex, which could be itself, that no other vertex in S dominates. Formally, a nonempty set S is *irredundant* if for every vertex $v \in S$ either v has no neighbor in S or there exists a vertex $w \in \overline{S}$ such that $N(w) \cap S = \{v\}$. The *irredundance numbers*, $ir(G)$ and $IR(G)$, equal the minimum and maximum cardinalities among all maximal irredundant sets of G. For a comprehensive treatment of irredundance in graphs the reader is referred to two 2021 chapters on this subject by Mynhardt and Roux [23] and Hedetniemi, McRae, and Mohan [21].
From the various definitions above, it can be easily seen that in every graph G, every maximal independent set is a minimal dominating set in G, and every minimal dominating set in G is a maximal irredundant set in G, leading to the domination chain:

Theorem 1 (The Domination Chain). *For every graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \alpha(G) \leq \Gamma(G) \leq IR(G).$$

Since its introduction by Cockayne, Hedetniemi, and Miller in 1978 [8], the Domination Chain of Theorem 1 has become one of the major focal points in the study of domination in graphs, having been studied in several hundred papers.

2. Majority, balanced, and minority sets in graphs

The standard definition of a *majority* is: a larger number or part, or a number that is more than half of the whole number. Similarly, a *minority* is: a smaller number or part, or a number that is less than half of the whole number.
The use of the word majority has two basic meanings, one can be considered generic and the other academic usage. Numerically, the word indicates a statistical majority; however, it also applies to power differences among groups rather than the numbers of people in a population category. Some aspects of majorities and minorities can be found in many areas of human activity, and a non-exhaustive list has been given in [5].
Given a set $S \subseteq V$ of vertices in a graph $G = (V, E)$ we say that a vertex $v \in S$ is: (i) a *majority vertex* (with respect to S) if a majority of its neighbors are also in S, (ii) a *minority vertex* if a minority of its neighbors are in S, or (iii) a *balanced vertex*, if it has the same number of neighbors in S as it has in $\overline{S}$. Balanced vertices, of course, must have even degree. In a cycle C_n for $n \geq 4$ any two adjacent vertices form a balanced set. More examples can be found in [5].
In [5], seven types of sets were defined depending on which the three types of vertices (minority, balanced, or majority) they contain. These sets have been denoted by a

triple (x, y, z), where the value 1: for x denotes the presence of one or more minority vertices in a set; for y denotes the presence of one or more balanced vertices;, and for z denotes the presence of one or more majority vertices in a set, all with respect to a given set $S \subset V$. A value of zero for x, y, or z indicates the absence of a vertex of that type in a set S. Once again, we refer the reader to [5] for further details.
For the remainder of this section, we list a variety of sets related to majority and minority sets.
A vertex v in a set S is said to be *cost effective* if it is adjacent to at least as many vertices in $\overline{S}$ as it is in S, and is *very cost effective* if it is adjacent to more vertices in $\overline{S}$ than to vertices in S. A set S is a *(very) cost effective dominating set* if S is both a (very) cost effective set and a dominating set. Cost effective sets in graphs were introduced by Haynes et al. in 2012 [14] and have been studied in [4, 17, 18].
Cost effective domination is derived from the study of unfriendly partitions of graphs, as follows. An unfriendly partition of a graph is a partition of the vertices into two nonempty sets so that every vertex has at least many neighbors in the other class as in its own class. These types of partitions were defined and studied by Borodin and Koshtochka [3], Aharoni, Milner and Prikry [1], and Shelah and Milner [26], who called these unfriendly partitions. They observed the following.

Theorem 2. *Every finite connected graph G of order $n \geq 2$ has an unfriendly partition.*

Unfriendly partitions have shown up indirectly in several other lines of research. In [9, 10] the concept of α-domination in graphs is defined and studied. A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called an *α-dominating set* if for every vertex $v \in \overline{S}$, $|N(v) \cap S|/|N(v)| \geq \alpha$, where $0 \leq \alpha < 1$. In the case where $\alpha \geq 1/2$, every vertex in $\overline{S}$ meets the *unfriendly condition* in that it has at least as many neighbors in S as it has in $\overline{S}$. However, no unfriendly condition is imposed on the vertices in S.
Define $\gamma_{\frac{1}{2}}(G)$ and $\Gamma_{\frac{1}{2}}(G)$ to equal the minimum and maximum cardinalities of a minimal $\frac{1}{2}$-dominating set in G. Similarly, in [15, 16, 24] global offensive alliances in graphs are defined and studied. A set $S \subseteq V$ of vertices is called a *global offensive alliance* if for every vertex $v \in \overline{S}$, $|N(v) \cap S| \geq |N[v] \cap \overline{S}|$. As with α-domination, if S is a global offensive alliance, then every vertex $v \in \overline{S}$ satisfies the unfriendly condition, in that it has at least as many neighbors in S as it has in $\overline{S}$ if you count the vertex v as one of its own neighbors. But no unfriendly condition is imposed on the vertices in S.
A partition that is in some sense dual to an unfriendly partition is called a *satisfactory partition* in which the vertices of the graph are partitioned into two nonempty sets so that every vertex has at least many neighbors in its own set as in the other set. Satisfactory partitions have been studied in [11, 12, 25]. However, unlike unfriendly partitions, not every graph has a satisfactory partition. In fact, it is an NP-complete problem to decide if an arbitrary graph has a satisfactory partition [2].

3. Majority Sets

In this and the following sections we introduce the study of majority sets in graphs, as defined in Section 2. The study of minority sets is addressed in [5].

3.1. Internal, external, and total majority sets

Given a set of vertices $S \subset V$, we say that a vertex $v \in S$ is an *internal majority vertex* if $|N(v) \cap S| > |N(v) \cap \overline{S}|$, that is, vertex v has more neighbors in S than it has in $\overline{S}$. In the remainder of this paper, we will restrict attention to extending the definition of a majority vertex to sets S of vertices as follows.

Definition 1. A set of vertices $S \subseteq V$ in a graph $G = (V, E)$ is called an *internal majority set* if for every vertex $v \in S$, a majority of the neighbors of v are in S, or equivalently, every vertex $v \in S$ has more neighbors in S than it has in $\overline{S}$, or simply put, every vertex in S is a majority vertex with respect to S.

Notice that the property of a set S being an internal majority set is neither hereditary nor superhereditary. But since the entire set $S = V(G)$ is a trivial internal majority set of any isolate-free graph G, let $M(G)$ and $M^+(G)$ equal the minimum and maximum cardinalities of a minimal internal majority set in G.

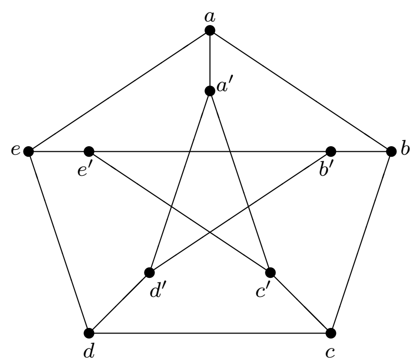

Figure 1. **The Petersen graph.**

As an illustrative example for these two parameters, consider the Petersen graph P as depicted in Figure 1. Observe that if S is a minimal internal majority set of P, then $G[S]$ is a subgraph of minimum degree at least two, since each vertex of S must have at least two neighbors in S. Moreover, since the smallest induced cycle in P has length 5, we deduce that $|S| \geq 5$. Now, since $\{a, b, c, d, e\}$ is a minimal internal majority set of P, we deduce that $M(P) \leq 5$, and thus $M(P) = 5$. On the other hand, one can see that $V(P) - \{a, b, e, a'\}$ is a minimal internal majority set of P, leading to $M^+(P) \geq 6$. Now, if there is a minimal internal majority set S of P with size at

least seven, then the vertices not in S induce a connected subgraph (else some vertex in S would have two neighbors in $\overline{S}$) and thus for some vertex $x \in S$, $S - \{x\}$ would contradict the minimality of S. Therefore $M^+(P) = 6$.
Now, similarly to internal majority sets, external majority sets can be defined by setting a similar condition that applies to each vertex $w \in \overline{S}$, giving rise, in turn, to *total majority* sets, where all vertices in V are involved.

Definition 2. A set of vertices $S \subseteq V$ in a graph $G = (V, E)$ is called an *external majority set* if for every vertex $v \in \overline{S}$, a majority of the neighbors of v are in S, or equivalently, every vertex $v \in \overline{S}$ has more neighbors in S than it has in $\overline{S}$. Let $\overline{M}(G)$ and $\overline{M}^+(G)$ equal the minimum and maximum cardinalities among all minimal external majority sets of G.

Definition 3. A set of vertices $S \subseteq V$ in a graph $G = (V, E)$ is called a *total majority set* if for every vertex $v \in V$, a majority of the neighbors of v are in S, or equivalently, every vertex $v \in V$ has more neighbors in S than it has in $\overline{S}$. Let $M_t(G)$ and $M_t^+(G)$ equal the minimum and maximum cardinalities among all minimal total majority sets of G.

Every isolate-free graph has internal, external and total majority sets. The entire vertex set of any isolate-free graph is both an internal and a total majority set. Moreover, the complement $\overline{S}$ of any maximal independent set S of an isolate-free graph, or equivalently any vertex cover set, is an external majority set. Therefore, $\overline{M}(G) \leq \beta(G)$ holds for any isolate-free graph G, where $\beta(G)$ stands for the vertex covering number of G.
Considering again the Peterson graph as depicted in Figure 1, the previous inequality yields $\overline{M}(P) \leq \beta(P) = 6$. However, this bound can be reduced to 4. Indeed, the set $\{a', e', b, d\}$ is a minimal external majority set of P. This bound seems to be the best possible since the complement $\overline{S}$ of any minimal external majority set S of P should induce an acyclic subgraph. Also, $\overline{M}^+(P) \geq 6$, since $\{b, c, e, a', d', e'\}$ is a is a minimal external majority set of P, and this bound might be the best possible. On the other hand, $V(P)$ minus any two adjacent vertices is the only minimal total majority set of P arising from the fact that any pair of non adjacent vertices of P have a common neighbor. This means that $M_t(P) = M_t^+(P) = 8$.
It is worth noting that the property of a set S being a total majority set or an external majority set is superhereditary, since moving a vertex $w \in \overline{S}$ from $\overline{S}$ to S creates even more neighbors in S. That is, (i) the vertices in $\overline{S} - \{w\}$ could have even more neighbors in $S \cup \{w\}$ than they do in S, (ii) S being a total majority set, vertex w has more neighbors in S than in $\overline{S}$ and will have the same number of neighbors in S when it is moved to S, and (iii) vertices $x \in \overline{S} - \{w\}$, having more neighbors in S than in $\overline{S}$ could have even more neighbors in $S \cup \{w\}$.
Accordingly, because of Proposition 1, an external majority set (or a total majority set) S is a minimal external majority set (or a total majority set) if and only if for every vertex $w \in S$, $S - \{w\}$ is no longer an external majority set (or a total majority set), that is, w or some neighbor $v \in \overline{S}$ of w can have at most as many neighbors in $S - \{w\}$ as in $\overline{S} \cup \{w\}$ (or some neighbor $v \in V$ of w can have at most as many

neighbors in $S - \{w\}$ as in $\overline{S} \cup \{w\}$, that is, v changes from a majority vertex with respect to S to a non-majority vertex with respect to $S - \{w\}$. All of this gives rise to the following concepts.

Definition 4. A subset $S \subset V$ is an external majority-critical set if for every vertex $v \in S$, either (i) v has at most as many neighbors in S as in $\overline{S}$, or (ii) v has a neighbor $w \in \overline{S}$ such that w is a majority vertex with respect to S but has at most as many neighbors in $S - \{v\}$ as in $\overline{S} \cup \{v\}$. Let $M_{cr}(G)$ and $M^+_{cr}(G)$ equal the minimum and maximum cardinalities among all maximal external majority-critical sets of G.

Definition 5. A subset $S \subset V$ is a total majority-critical set if every vertex $v \in S$ has at least one neighbor w, either in S or in $\overline{S}$, such that w is a majority vertex with respect to S but is a non-majority vertex with respect to $S - \{v\}$. Let $M_{tcr}(G)$ and $M^+_{tcr}(G)$ equal the minimum and maximum cardinalities among all maximal total majority-critical sets of G.

3.2. First consequences

Proposition 2. *Every minimal external majority set S is a maximal majority-critical set.*

Proof. It follows from the definition that every minimal external majority set S is a majority-critical set. All that remains is to show that S is a maximal majority-critical set. For this purpose, suppose that a minimal external majority set S is not a maximal majority-critical set. Thus, there is a nonempty subset $A \subseteq \overline{S}$ such that $S \cup A$ is a majority-critical set of G. Let v be a vertex in A and note that v has more neighbors in S (and thus in $S \cup A$) than it has in $\overline{S}$, since S is a minimal external majority set in G. It follows from the definition of external majority-critical sets that v has a neighbor $w \in \overline{S} - A$ with more neighbors in $S \cup A$ than it has in $\overline{S} - A$, but is a non-majority vertex with respect to $S \cup A - \{v\}$. This means that w is a vertex of $\overline{S}$ with at most as many neighbors in S as in $\overline{S}$, which contradicts the fact that S is an external majority set in G, and this completes the proof. □

The following inequality chain is derived from Proposition 2.

Corollary 1. *For any graph G, $M_{cr}(G) \leq \overline{M}(G) \leq \overline{M}^+(G) \leq M^+_{cr}(G)$.*

Proposition 3. *Every minimal total majority set S is a maximal total majority-critical set.*

Proof. It follows from the definition that every minimal total majority set S is a majority-critical set. All that remains is to show that S is a maximal total majority-critical set. To do this, suppose that a minimal total majority set S is not a maximal total majority-critical set. Thus, there is a nonempty subset $A \subseteq \overline{S}$ such that $S \cup A$ is a total majority-critical set of G. Let v be a vertex in A. By the definition of total majority-critical sets, vertex v has a neighbor w, either in $S \cup A$ or in $\overline{S \cup A}$, such that w is a majority vertex with respect to $S \cup A$ but is non-majority vertex with respect to $(S \cup A) - \{v\}$. Clearly, such a vertex w cannot belong to either S or A, since S is a minimal total majority set in G. Hence $w \in \overline{S} - A$. But then the fact that w is a non-majority vertex with respect to $S \cup A - \{v\}$, contradicts the fact that S is a minimal total majority set, where $w \in \overline{S}$ has to be a majority vertex with respect to S. □

According to Proposition 3, the following inequality chain is obtained.

Corollary 2. *For any graph G, $M_{tcr}(G) \leq M_t(G) \leq M_t^+(G) \leq M_{tcr}^+(G)$.*

We close this subsection by giving the exact value of $M(G)$ and $M^+(G)$ when G is either a path P_n or a cycle C_n.

Proposition 4. *For any $n \geq 2$, $M(P_n) = M^+(P_n) = n$.*

Proof. From the definition, if S is a (minimal) internal majority set of P_n, then each vertex of S must have all its neighbors in S, and thus $M(P_n) = M^+(P_n) = n$. □

By a similar proof, we obtain the following.

Proposition 5. *For any $n \geq 3$, $M(C_n) = M^+(C_n) = n$.*

4. Bounds and relationships with some graph parameters

In this section we will assume that no graphs have isolated vertices, since, by definition, isolated vertices cannot be majority vertices with respect to any set S. Thus, for all graphs G, we will assume that $\delta(G) \geq 1$.

Proposition 6. *Let G be a connected graph of order $n \geq 2$ with no vertex of even degree. Then*

$$\overline{M}(G) \leq \frac{n}{2} \leq \overline{M}^+(G).$$

Proof. Let $\pi = \{B(V), R(V)\}$ be an unfriendly partition G. Since G has no vertex of even degree, every vertex of $B(V)$ has fewer neighbors in $B(V)$ than it has in $R(V)$, and likewise every vertex of $R(V)$ has fewer neighbors in $R(V)$ than it has in $B(V)$. Therefore each part of π is both an external majority set in G.
The next step is to show that each of $B(V)$ and $R(V)$ is a minimal external majority set. Assume, without loss of generality, that $R(V)$ is not a minimal external majority set. Then for some nonempty subset $A \subset R(V)$, $R(V)-A$ is an external majority set of

G, that is for every vertex w in $B(V) \cup A$, a majority of the neighbors of w are in $R(V) - A$. But since $\{B(V), R(V)\}$ is an unfriendly partition, if $w \in A$, then w has more neighbors in $B(V)$ than it has in $R(V)$, leading to a contradiction. Therefore, each part of π is minimal with respect to the property of being an external majority set, and thus $\overline{M}(G) \leq \min\{|B(V)|, |R(V)|\} \leq \frac{n}{2}$ and $\overline{M}^+(G) \geq \max\{|B(V)|, |R(V)|\} \geq \frac{n}{2}$. □

The bound of Proposition 6 is sharp. Indeed, for a cycle C_4, we have $\overline{M}^+(G) = \overline{M}(G) = 2$. Moreover, the bound of Proposition 6 can be strict as can be seen for stars $K_{1,n-1}$ of even order $n \geq 4$, where $\overline{M}^+(G) = n - 1$ and $\overline{M}(G) = 1$.

Proposition 7. *If G is a connected bipartite graph of order $n \geq 2$, then*

$$\overline{M}(G) \leq \frac{n}{2} \leq \overline{M}^+(G).$$

Proof. Let X and Y be the partite sets of G. Since X and Y are independent sets, and each vertex has all its neighbors in the partite set that does not contain it, it can be seen that each of X and Y is a minimal external majority set in G, leading to the desired result. □

The next result provides a lower bound on $M(G)$ in terms of the minimum degree of a graph G.

Proposition 8. *For every graph G of minimum degree $\delta(G)$, $M(G) \geq \left\lceil \frac{\delta(G)+1}{2} \right\rceil + 1$. If further G is triangle-free, then $M(G) \geq \delta(G) + 1$.*

Proof. Let S be a minimum internal majority set of G. Since every vertex $v \in S$ has at least $\left\lceil \frac{\deg(v)+1}{2} \right\rceil$ neighbors in S, it follows that $|S| \geq \left\lceil \frac{\deg(v)+1}{2} \right\rceil + 1$, leading to $M(G) \geq \left\lceil \frac{\delta(G)+1}{2} \right\rceil + 1$. Assume now that G is triangle-free. If there is a vertex v in S with no neighbors in $\overline{S}$, then clearly $|S| \geq \deg(v) + 1 \geq \delta(G) + 1$. Otherwise, since G is triangle-free, no two adjacent vertices x and y in S can have common neighbors, and thus S contains at least $\left\lceil \frac{\deg(x)+1}{2} \right\rceil + \left\lceil \frac{\deg(y)+1}{2} \right\rceil$ vertices leading as before to the desired result. □

Corollary 3. *For every isolate-free graph G of order n, $M(G) \geq 2$, with equality if and only if $G = P_2$.*

Proof. The lower bound follows from Proposition 8. Moreover, if $M(G) = 2$ and S is an $M(G)$-set, then $\overline{S} = \emptyset$, and since G is isolate-free, we deduce that G is a path on two vertices. The converse is obvious. □

In the aim to characterize the isolate-free graphs G such that $M(G) = 3$, let $\mathcal{H} = \{P_3, C_3, K_{1,3}, K_4, K_4 - e, Cor(C_3), G_1, G_2, G_3\}$ be the family of graphs G, where G_1 and G_2 are obtained from $Cor(C_3)$ by deleting one or two leaves, respectively, while G_3 is obtained from $K_4 - e$ by adding a new vertex attached to a vertex of degree two of $K_4 - e$.

Proposition 9. *Let G be an isolate-free graph of order $n \geq 3$. Then $M(G) \geq 3$, with equality if and only if $G \in \mathcal{H}$.*

Proof. The lower bound follows from Proposition 8 and Corollary 3. Now, assume that $M(G) = 3$, and let S be an $M(G)$-set. Then $|\overline{S}| = n-3$. Moreover, the vertices of S induce a connected subgraph, and thus $G[S]$ is either a path P_3 or a cycle C_3. As a consequence, $|\overline{S}| \leq 3$ since every vertex in S can have at most one neighbor in $\overline{S}$. Now, if $n = 3$, then $G \in \{P_3, C_3\}$. Hence we assume that $n \geq 4$. If $G[S] = P_3$, then only the center vertex of $G[S]$ has a neighbor in $\overline{S}$ leading to $G = K_{1,3}$. In the next, we can assume that $G[S] = C_3$. Now according to whether or not vertices in S have common neighbors in $\overline{S}$, one can see that G is one of the graphs $K_4, K_4-e, Cor(C_3), G_1, G_2, G_3$. Conversely, by Corollary 3, $M(G) \geq 3$, and the equality holds since for each graph $G \in \mathcal{H}$, an internal majority set of size 3 can be easily constructed. □

It is easy to observe that every external majority set of an isolate-free graph G is a $\frac{1}{2}$-dominating set, leading to $\gamma_{\frac{1}{2}}(G) \leq \overline{M}(G)$. The inequality can be strict for instance when $G = K_5$, where $\gamma_{\frac{1}{2}}(K_5) = 2 < \overline{M}(K_5) = 3$. However, for a large family of graphs, we'll see that not only are these two parameters equal, but so are all the upper parameters $\Gamma_{\frac{1}{2}}(G)$ and $\overline{M}^+(G)$.

Proposition 10. *Let G be an isolate-free graph G with no vertices of even degree. Then $\gamma_{\frac{1}{2}}(G) = \overline{M}(G)$ and $\Gamma_{\frac{1}{2}}(G) = \overline{M}^+(G)$.*

Proof. It suffices to see that a subset of vertices S is a minimal external majority set of G if and only if it is a minimal $\frac{1}{2}$-dominating set. Indeed, if S is an external majority set (resp. $\frac{1}{2}$-dominating set), then since the degree of every vertex is odd, every vertex in $\overline{S}$ has more neighbors in S than it has in $\overline{S}$, and thus S is a $\frac{1}{2}$-dominating set (resp. an external majority set). Now, if S is not a minimal $\frac{1}{2}$-dominating set (resp. a minimal external majority set), then there is a vertex $y \in S$ such that $S - \{y\}$ is a $\frac{1}{2}$-dominating set of G (resp. external majority set) and thus an external majority set (resp. $\frac{1}{2}$-dominating set), a contradiction. This completes the proof. □

5. Complexity results

Consider the following two decision problems associated with external majority sets and total majority sets.

EXTERNAL MAJORITY SET (EMS)
Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.
Question: Does G have an external majority set of cardinality at most k?

TOTAL MAJORITY SET (TMS)
Instance: Graph $G = (V, E)$, positive integer $k \leq |V|$.
Question: Does G have a total majority set of cardinality at most k?

We show that these problems are NP-complete by reducing the special case of Exact Cover by 3-sets (X3C), to which we refer as X3C3, to each of EMS and TMS. Note that the NP-completeness of X3C3 was proven in 2008 by Hickey et al. [22].

X3C3
Instance: A set of elements X with $|X| = 3q$, and a collection C of $3q$ 3-element subsets of X, such that each element occurs in exactly 3 members of C.
Question: Does C contain an exact cover for X, i.e. does there exist a subcollection $C' \subset C$ such that every element of X occurs in exactly one member of C'?

Theorem 3. *EMS is NP-complete for bipartite graphs.*

Proof. EMS belongs to $\mathcal{NP}$, since we can check in polynomial time that a set S of cardinality at most k is an external majority set. Next we show how to transform any instance of X3C3 into an instance G of EMS so that one of the instances has a solution if and only if the other instance has a solution. Let $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{C_1, C_2, ..., C_{3q}\}$ be an arbitrary instance of X3C3.
For each $x_i \in X$, we build a "gadget" graph of order 5 whose vertices are labeled as depicted in Figure 2, and let $W = \{w_3^1, w_3^2, ..., w_3^{3q}\}$. For each $C_j \in C$, we build a gadget graph of order 9 whose vertices are labeled as depicted in Figure 2, and let $Y = \{c_1, c_2, ..., c_{3q}\}$. Now to obtain a graph G, we add edges $c_j w_3^i$ if $x_i \in C_j$. Clearly, G is a bipartite graph. Set $k = 19q$.

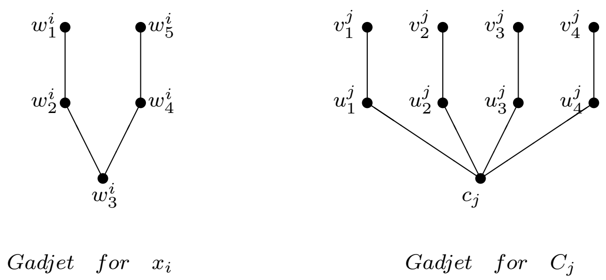

Figure 2. **Gadjets for an element x_i and a triplet C_j**

Suppose that the instance X, C of X3C3 has a solution C'. We construct an external majority set S of G of size k as follows. Put in S all support vertices of G, as well as every vertex c_j such that $C_j \in C'$. Observe that since C' exists, $|C'| = q$ and thus $|S| = 12q + 6q + |C'| = 19q = k$. The existence of C' also implies that the number of c_j's that belong to S is q, having disjoint neighborhoods in W. Accordingly, one can check that c_j not in S has 4 neighbors in S and 3 neighbors outside S, and every w_3^i has 3 neighbors in S and 2 outside S. Therefore, S is an external majority set of G of size k.

Conversely, suppose that G has an external majority set of size at most k. Among all such sets, let S be one that contains the most support vertices of G. The following two facts are noted.

(i) Each support or its leaf belongs to S, and certainly S does not contain both (else, it can be reduced by keeping the support vertex and removing the leaf).

(ii) If $u_r^j \notin S$ and $v_r^j \in S$ for some $r \in \{1, \dots, 3q\}$, then replacing v_r^j by u_r^j in S produces another set that is an external majority set of the same cardinality but contains more support vertices than S, contradicting our choice of S.

From items (i) and (ii) we deduce that S contains all support vertices and no leaves of G. Note that at this point any c_j already has the majority of its neighbors in S and any w_3^i not in S has only 2 neighbors in S and thus needs to have one of its neighbors in Y to be in S. Therefore, let $A = S \cap Y$ and $B = S \cap W$, and let $|A| = a$ and $b = |B|$. Since $|S| \leq k = 19q$, it follows that $a + b \leq q$. Also, since every $W - B$ must have a neighbor in A and every vertex of A has three neighbors in W, we must have $3a \geq 3q - b$. Together, these two inequalities imply that $a = q$ and $b = 0$. Consequently, one can easily show that X3C3 has a solution $C' = \{C_j : c_j \in S\}$. □

Theorem 4. *TMS is NP-complete.*

Proof. TMS is a member of $\mathcal{NP}$, since we can check in polynomial time that a set S has cardinality at most k and is a total majority set of a graph. We show how to transform any instance of X3C3 into an instance G of TMS so that one of the problems has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, ..., x_{3q}\}$ and $C = \{C_1, C_2, \dots, C_{3q}\}$ be an arbitrary instance of X3C3.

First, let H' be the connected graph of order 5 as illustrated in Figure 3. For each $x_i \in X$, we build a gadget graph T_i of order 11 obtained from two disjoint copies of H' as given in Figure 3 with a distinguished vertex labeled w_i. Let $W = \{w_1, w_2, ..., w_{3q}\}$. For each $C_j \in C$, we built a gadget graph H_j of order 9 as given in Figure 3 with a distinguished vertex labeled c_j, and let $Y = \{c_1, c_2, ..., c_{3q}\}$. Now to obtain a graph G, we add edges $c_j w_i$ if $x_i \in C_j$. Set $k = 31q$.

Suppose that the instance X, C of X3C3 has a solution C'. We construct a total majority set S of G of size k as follows. Put in S all support vertices of G, and for each copy of H' in G, put into S the common neighbor of the two support vertices of H'. We also put in S every vertex c_j such that $C_j \in C'$. Clearly, since C' exists, $|C'| = q$ and thus $|S| = 12q + 18q + |C'| = 31q = k$. The existence of C' also implies that the q vertices of $Y \cap S$ have disjoint neighborhoods in W, where every w_i has exactly three neighbors in S and two neighbors outside S. Moreover, we can verify

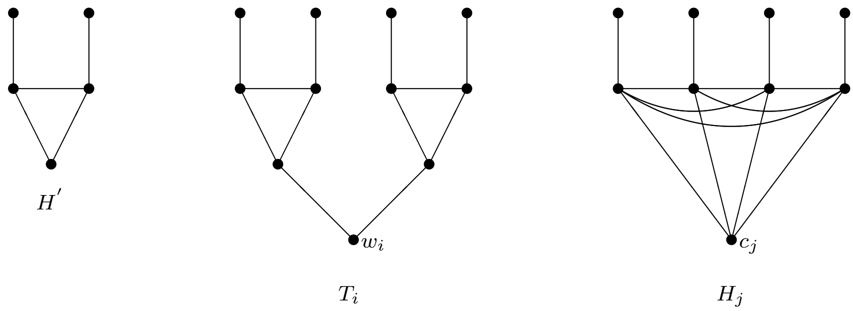

Figure 3. **The graphs H', T_i and H_j**

that any other vertex of the graph G, either in S or not, has more neighbors in S than it has outside S, and therefore, S is a total majority set of G of size k.
Conversely, suppose that G has a total majority set of size at most k. Among all such sets, let S be one that contains as few leaves as possible. Clearly, S contains all support vertices of G. If a leaf of some H_j is in S, then deleting it from S we still have a total majority set of G but with one less leaf, contradicting our choice of S. On the other hand, if a leaf of some copy of H' is in S, then we replace it with the vertex common to both support vertices of H'. Hence no leaf of H' is in S, and therefore S must contain this common neighbor of the support vertices of H'. Note that so far, S already contains $30q$ vertices. Now, if $w_i \in S$ for some i, then S must contain at least one c_j neighbor of w_i so that w_i can have the majority of its neighbors in S. But then deleting w_i from S we still have a total majority set of G. Therefore, we can assume that no w_i belongs to S. But in this case we need at least q vertices of Y to be in S so that each w_i can have more neighbors in S. Since only q other vertices can be included in S, we deduce that $|Y \cap S| = q$, and thus each w_i has exactly one neighbor in $Y \cap S$. Consequently, $C' = \{C_j : c_j \in S\}$ is a solution for X3C3. □

6. Internal majority partitions

Trivially, the entire set $S = V(G)$ of an isolate-free graph G is an internal majority set. Thus, it is natural to consider the problem of partitioning the vertices of a graph $\pi = \{V_1, V_2, \ldots, V_k\}$ such that every set V_i, $1 \le i \le k$, is an internal majority set, called an *internal majority partition*. Therefore, it makes sense to define the *internal majority partition number* $\psi_M(G)$ to equal the maximum order of an internal majority partition of a graph G. The notation $\psi_M(G)$ has been chosen by analogy to the *quorum coloring number* $\psi_q(G)$ of a graph G defined as the maximum order of a partition of $V(G)$ into defensive alliances. Note that Haynes and Lachniet in [19] were the first to introduce the problem of partitioning the vertex set V into defensive alliances, but the use of quorum coloring as well as the notation of $\psi_q(G)$ have been recommended and well justified by Hedetniemi et al. in [20].

Given a nontrivial connected graph G of order n, let $\pi = \{V_1, V_2, \ldots, V_{\psi_M(G)}\}$ be a maximum partition of $V(G)$ into internal majority sets. The following two facts should be noted.

- $\psi_M(G) \geq 1$, and the equality holds for every nontrivial complete graph K_n, as well as stars $K_{1,n}$.
- The partition π is also a quorum coloring of G, and so $\psi_q(G) \geq \psi_M(G) \geq 1$. Moreover, it is clear that if $\psi_q(G) = 1$, then $\psi_M(G) = 1$, but the converse is not true in general, as it can be seen by any complete graph of odd order.

It is of interest, therefore, to characterize the family of connected graphs G such that $\psi_M(G) = 1$. In this regard, we provide the following class of trees T for which $\psi_M(T) = 1$.
Given a nontrivial tree T, let $V_{\geq 3}(T)$ denote the set of all vertices of T of degree three or more.

Proposition 11. *If T is a nontrivial tree, then $\psi_M(T) = 1$ if and only if $V_{\geq 3}(T)$ is either empty or independent.*

Proof. Assume that $\psi_M(T) = 1$. Suppose that $V_{\geq 3}(T)$ is nonempty and let x and y be two adjacent vertices in $V_{\geq 3}(T)$. Let T_x and T_y be the subtrees of T resulting from the deletion of the edge xy, where $x \in V(T_x)$ and $y \in V(T_y)$. Note that $V(T_x) \cup V(T_y) = V(T)$ and that each of T_x and T_y has order at least three. Also, it is quite obvious that each of $V(T_x)$ and $V(T_y)$ is an internal majority set in T, leading to $\psi_M(T) \geq 2$, a contradiction. Hence $V_{\geq 3}(T)$ is independent.
Conversely, let T be a nontrivial tree such that $V_{\geq 3}(T)$ is either empty or independent. If $V_{\geq 3}(T) = \emptyset$, then T is a path P_n and clearly $\psi_M(P_n) = 1$. Hence we can assume that $V_{\geq 3}(T)$ is an independent set. Let $\psi_M(T) = k$ and $\pi = \{V_1, V_2, \ldots, V_k\}$ be a maximum partition of $V(T)$ into internal majority sets. If $k = 1$, then we are done. Hence we assume that $k \geq 2$. Since each V_i is nonempty, let u be a vertex in V_i having an external neighbor $v \in V_j$ for some $j \neq i$. From the definition of internal majority sets, u must have at least two neighbors in V_i, and thus u has degree at least three. Therefore $u \in V_{\geq 3}(T)$, and likewise $v \in V_{\geq 3}(T)$. But then $V_{\geq 3}(T)$ is not independent, a contradiction. This completes the proof. □

Our next result is an upper bound on the internal majority partition number of any connected graph G of order $n \geq 3$ in terms of n. Moreover, for the purpose of characterizing the extremal graphs attaining this upper bound, let $\mathcal{F}$ be the family of connected graphs G of order n that are obtained from the disjoint unions of r paths P_3 and s cycles C_3 with $r + s = n/3$, by adding edges between only vertices of degree two so that the added edges are pairwise independent and the resulting graph is connected.

Proposition 12. *For every connected graph G of order $n \geq 3$, $\psi_M(G) \leq \frac{n}{3}$, with equality if and only if $G \in \mathcal{F}$.*

Proof. Let $\psi_M(G) = k$, and $\pi = \{V_1, V_2, \ldots, V_k\}$ be a maximum partition of $V(G)$ into internal majority sets. Since G is connected, no V_i contains a single vertex. Since also $n \geq 3$, no V_i can contain exactly two vertices. Therefore, each V_i has order at least three, implying that

$$3k \leq |V_1| + |V_2| + ... + |V_k| = n.$$

Assume now that $\psi_M(G) = \frac{n}{3}$. Then clearly $|V_i| = 3$ for each $i \in \{1, ..., \frac{n}{3}\}$, where the subgraph induced by each V_i is either a path P_3 or a cycle C_3. Moreover, each vertex of degree two in $G[V_i]$ can have at most one external neighbor. Thus, edges connecting vertices belonging to different classes are pairwise independent and since G is connected we deduce that $G \in \mathcal{F}$.
Conversely, let $G \in \mathcal{F}$. Hence it is obtained from the disjoint unions of r paths P_3 and s cycles C_3 with $r + s = n/3$, by adding edges between only vertices of degree two so that the added edges are pairwise independent and the resulting graph is connected. It is easy to see that the vertices of each path P_3 or a cycle C_3 form an internal majority set in G, and thus $\psi_M(G) \geq r + s = n/3$. The equality follows from the upper bound already shown. □

Since every graph G in $\mathcal{F}$ has maximum degree three, the following corollary is derived.

Corollary 4. *If G is a connected graph of order n with maximum degree at least four or n is not a multiple of 3, then $\psi_M(G) < \frac{n}{3}$.*

7. Open Problems

1. What can you say about the value of any of $M, M^+, \overline{M}, \overline{M}^+, M_t, M_t^+$ for grid graphs of the form $P_m \square P_n$?

2. Characterize the graphs G such that every minimal external majority set is minimum, that is, $\overline{M}(G) = \overline{M}^+(G)$.

3. Study the complexity issues of decision problems related to the remaining majority parameters.

4. Let λ represent any of $M, M^+, \overline{M}, \overline{M}^+, M_t, M_t^+$. Can you design an algorithm for computing the value of $\lambda(T)$ for any tree T?

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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