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Research Article

Optimization through localized metric resolvability

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Abstract: The use of local metric resolvability can be realized in delivery services for optimal placement of existing and new resources like medical facilities, stores, and fire stations. The local metric basis produces codes for the facilities and regions to be served by these facilities in a network or a graph in such a way that the adjacent nodes get unique codes in terms of distances, so that each facility is used optimally. In this paper, the local metric dimension (LMD) has been computed for convex polytopes B_n, C_n, D_n, and Q_n. An algorithm to extend the number of resources in a distributed network and real-life applications of local metric resolvability have also been investigated.

Keywords: convex polytopes, metric dimension, local metric dimension.

AMS Subject classification: 05C12, 05C90

1. Introduction

Slater [31] and Melter et al. [15] introduced the concept of metric dimension (MD) with multi-purpose applications and uses in robotics [19], sensor networks [30], chemistry [13], optimization [28] and for detecting invaders in the networks [31]. There are several variants of MD that have been introduced since 1975, which comprise dominating sets [1, 10], resolving sets [14, 26], mixed metric dimension [18], strong resolving generators [28], k-metric generators [27, 33], and simultaneous metric dimension [22]. The MD selects a set of nodes called metric basis (MB) with minimum cardinality such that each node of a graph gets a unique code in terms of distances

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from the nodes in the basis, but many times, only adjacent nodes are required to be identified. The need for such a basis motivated Okamoto et al. [21] to introduce a local metric basis. The local metric basis (LMB) has been used to assign unique codes in delivery services to expedite the services [20]. If the cardinality of LMB of a distributed network like convex polytopes is fixed, then the company can plan network setup more effectively because the number of required facilities does not grow unpredictably with the extension of the network. This leads to better route planning, faster localization of delivery vehicles, and reduced operating costs. For a general graph, computing the local metric dimension LMD is NP-hard [9], since the number of possible vertex sets grows exponentially, which can lead to higher computational time for larger graphs. On the other side, it can be computed by exploiting graph-theoretic properties like symmetry of graphs along with combinatorial techniques and pattern recognition, which inspired us to compute LMD for some families of convex polytopes. The LMD for edge corona, corona, and rooted products of graphs has been studied in [24, 25, 32]. Recent studies have explored the multiset and LMD in various graph-theoretic representations of rings, including zero-divisor, compressed zero-divisor, and fuzzy zero-divisor graphs [2–5, 23], highlighting how graph invariants can be used to characterize algebraic structures and their resolvability properties. Further study on LMD can be found in [11, 12, 17, 20].
For a connected graph G, we denote the node set and the edge set by $V(G)$ and $E(G)$, respectively. If u and v are two nodes then shortest distance between these two nodes is denoted by $d(u,v)$ whereas $d(v,\Pi) = \min\{d(v,x) \mid x \in \Pi\}$ denotes the distance of node v from a set of nodes Π. Consider an ordered set of nodes, $\Pi = \{v_1, v_2, \ldots, v_n\}$ then the representation or code of the node v is given as a vector, $r(v \mid \Pi) = (d(v,v_1), d(v,v_2), \ldots, d(v,v_n))$. If we get distinct codes for each node, then the ordered set Π is called the resolving set. The smallest set Π is called MB, and the number of nodes in it represents the MD denoted by $dim(G)$. If we get distinct codes for every pair of adjacent nodes, then Π is called the local metric generator (LMG). The smallest LMG is called the LMB, and the number of nodes in it represents the LMD of G, denoted by $dim_l(G)$.
The following Proposition 1 gives results on the LMD of bipartite and complete graphs.

Proposition 1. *[21] Consider a connected graph G with n nodes, then*

(a) For $n \geq 2$, G is a bipartite graph if and only if $\dim_\ell(G) = 1$.

(b) K_n is the complete graph if and only if $\dim_\ell(G) = n - 1$.

In this article, the convex polytopes B_n, C_n, D_n, and Q_n are described and LMD is computed for these families in Section 2. An algorithm is presented in Section 3 to calculate the LMG, which would extend the number of resources in a locality. The real-life applications of LMB and LMG are also discussed in Section 3. The conclusion and open problems are given in Section 4.

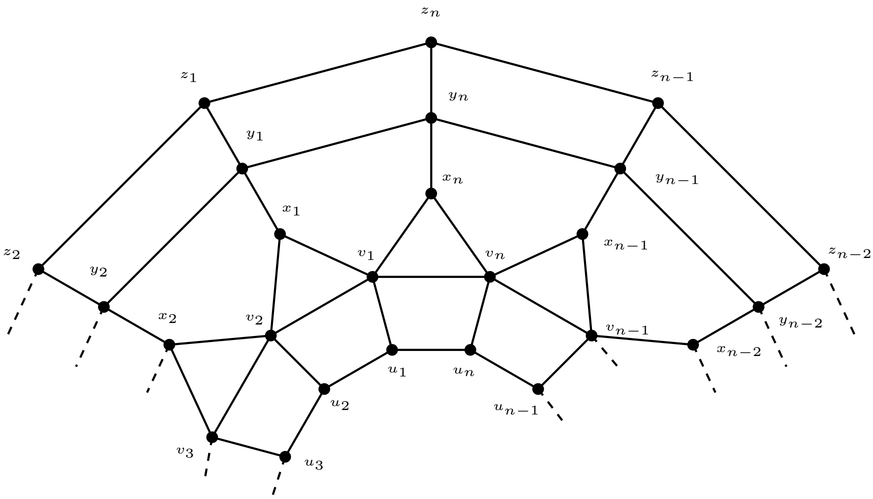

Figure 1. **Vertex–edge and combinatorial structure of B_n.**

2. Local metric dimension of convex polytopes

This section determines the exact LMD values of convex polytopes B_n, C_n, D_n, and Q_n. Some recent results regarding the partition resolvability of convex polytopes can be found in [6].

2.1. LMD of B_n

The convex polytope B_n is defined in [16] with the following node and edge sets.
$V(B_n) = \{u_i, v_i, x_i, y_i, z : 1 \leq i \leq n\}$ and
$E(B_n) = \{u_i u_{i+1}; v_i v_{i+1}; y_i y_{i+1}; z_i z_{i+1}; u_i v_i; v_i x_i; v_{i+1} x_i; x_i y_i; y_i z_i : 1 \leq i \leq n\}$, where $u_{n+1} = u_1, v_{n+1} = v_1, x_{n+1} = x_1, y_{n+1} = y_1$ and $z_{n+1} = z_1$. Figure 1 illustrates the graph of B_n. In the forthcoming result, we compute the LMD B_n.

Theorem 1. *Consider convex polytope B_n with $n \geq 6$, then* $\dim_\ell(B_n) = 2$.

Proof. We divide the proof into the following two cases.

Case 1. If n is odd, $n = 2\phi + 1$ and $\phi \geq 3$.

Consider the subset $\Pi = \{u_1, u_{\phi+1}\}$ of $V(B_n)$. The codes assigned to each node of B_n in relation to Π when n is odd are given below.

$$r(u_\epsilon \mid \Pi) = \begin{cases} (0,\phi) & \epsilon = 1, \\ (\epsilon-1, \phi-\epsilon+1) & 2 \le \epsilon \le \phi, \\ (\phi, 0) & \epsilon = \phi+1, \\ (n-\epsilon+1, \epsilon-\phi-1) & \phi+2 \le \epsilon \le n. \end{cases}$$

$$r(v_\epsilon \mid \Pi) = \begin{cases} (\epsilon, \phi-\epsilon+2) & 1 \le \epsilon \le \phi+1, \\ (n-\epsilon+2, \epsilon-\phi) & \phi+2 \le \epsilon \le n. \end{cases}$$

$$r(x_\epsilon \mid \Pi) = \begin{cases} (\epsilon+1, \phi-\epsilon+2) & 1 \le \epsilon \le \phi, \\ (n-\epsilon+2, \epsilon-\phi+1) & \phi+1 \le \epsilon \le n. \end{cases}$$

$$r(y_\epsilon \mid \Pi) = \begin{cases} (\epsilon+2, \phi-\epsilon+3) & 1 \le \epsilon \le \phi, \\ (n-\epsilon+3, \epsilon-\phi+2) & \phi+1 \le \epsilon \le n. \end{cases}$$

$$r(z_\epsilon \mid \Pi) = \begin{cases} (\epsilon+3, \phi-\epsilon+4) & 1 \le \epsilon \le \phi, \\ (n-\epsilon+4, \epsilon-\phi+3) & \phi+1 \le \epsilon \le n. \end{cases}$$

Case 2. If n is even, $n = 2\phi$ and $\phi \ge 3$.

Consider the subset $\Pi = \{u_1, x_\phi\}$ of $V(B_n)$. The codes assigned to each node of B_n in relation to Π when n is even are given below.

$$r(u_\epsilon \mid \Pi) = \begin{cases} (0,\phi+1) & \epsilon = 1, \\ (\epsilon-1, \phi-\epsilon+2) & 2 \le \epsilon \le \phi, \\ (n-\epsilon+1, \epsilon-\phi+1) & \phi+1 \le \epsilon \le n. \end{cases}$$

$$r(v_\epsilon \mid \Pi) = \begin{cases} (\epsilon, \phi-\epsilon+1) & 1 \le \epsilon \le \phi, \\ (n-\epsilon+2, \epsilon-\phi) & \phi+1 \le \epsilon \le n. \end{cases}$$

$$r(x_\epsilon \mid \Pi) = \begin{cases} (\epsilon+1, \phi-\epsilon+1) & 1 \le \epsilon \le \phi-1, \\ (\phi+1, 0) & \epsilon = \phi, \\ (n-\epsilon+2, \epsilon-\phi+1) & \phi+1 \le \epsilon \le n. \end{cases}$$

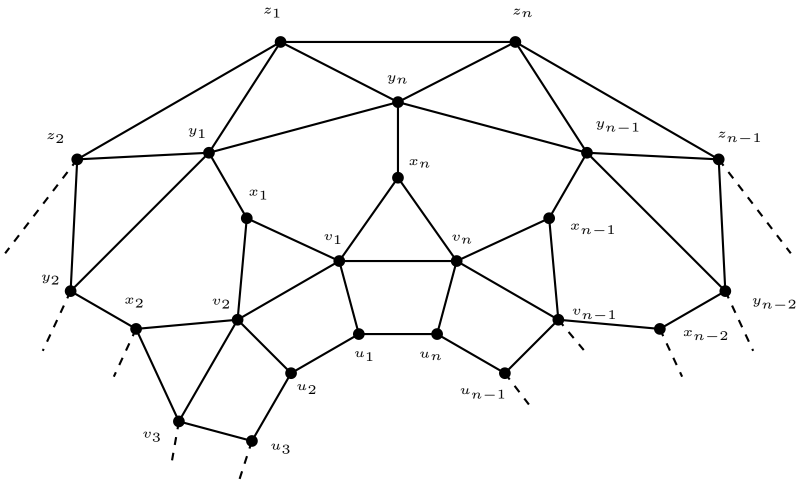

Figure 2. **Vertex–edge and combinatorial structure of C_n.**

$$r(y_\epsilon \mid \Pi) = \begin{cases} (\epsilon+2, \phi-\epsilon+1) & 1 \leq \epsilon \leq \phi, \\ (n-\epsilon+3, \epsilon-\phi+1) & \phi+1 \leq \epsilon \leq n. \end{cases}$$

$$r(z_\epsilon \mid \Pi) = \begin{cases} (\epsilon+3, \phi-\epsilon+2) & 1 \leq \epsilon \leq \phi, \\ (n-\epsilon+4, \epsilon-\phi+2) & \phi+1 \leq \epsilon \leq n. \end{cases}$$

In both cases, the codes make it clear that Π is an LMG, therefore, $\dim_\ell(B_n) \leq 2$. Hence, from Proposition 1, we conclude that $\dim_\ell(B_n) = 2$ for $n \geq 6$ as B_n is not a bipartite graph. □

2.2. LMD of C_n

The convex polytope C_n is defined in [16] with the following node and edge sets.
$V(C_n) = \{u_i, v_i, x_i, y_i, z : 1 \leq i \leq n\}$ and
$E(C_n) = \{u_iu_{i+1}; v_iv_{i+1}; y_iy_{i+1}; z_iz_{i+1}; u_iv_i; v_ix_i; v_{i+1}x_i; x_iy_i; y_iz_i; y_iz_{i+1} : 1 \leq i \leq n\}$, where $u_{n+1} = u_1, v_{n+1} = v_1, x_{n+1} = x_1, y_{n+1} = y_1$ and $z_{n+1} = z_1$. Figure 2 illustrates the graph of C_n.
In the next results, we compute the LMD C_n.

Theorem 2. *Consider convex polytope C_n with $n \geq 6$, then $\dim_\ell(C_n) \geq 3$ when n is even.*

Proof. Suppose $\dim_\ell(C_n) = 2$ such that $\mid \Pi \mid = 2$ with $n = 2\phi$ then we have following possible cases.

Case 1. If we take both nodes from $u_i's$. We may take u_1 as the first node and u_t as the second node where $2 \leq t \leq n$.

Case 1.1. If $2 \leq t \leq \phi$ then $r(v_{t+1} \mid \Pi) = (t+1, 2) = r(x_t \mid \Pi)$ which leads to a contradictory state.

Case 1.2. If $t = \phi + 1$ then $r(y_\phi \mid \Pi) = (\phi + 2, 3) = r(y_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 1.3. If $\phi + \lambda \leq t \leq n$ and $2 \leq \lambda \leq \phi$ then $r(v_\lambda \mid \Pi) = (\lambda, \phi + 1) = r(x_{\lambda-1} \mid \Pi)$ which is again a contradictory fact.

Case 2. If we take both nodes from $v_i's$. We may take v_1 as the first node and v_t as the second node where $2 \leq t \leq n$.

Case 2.1. If $2 \leq t \leq \phi$ then $r(v_{t+1} \mid \Pi) = (t, 1) = r(x_t \mid \Pi)$ which leads to a contradictory state.

Case 2.2. If $t = \phi + 1$ then $r(y_\phi \mid \Pi) = (\phi + 1, 2) = r(y_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 2.3. If $\phi + \lambda \leq t \leq n$ and $2 \leq \lambda \leq \phi$ then $r(v_\lambda \mid \Pi) = (\lambda - 1, \phi) = r(x_{\lambda-1} \mid \Pi)$ which is again a contradictory fact.

Case 3. If we take both nodes from $x_i's$. We may take x_1 as the first node and x_t as the second node where $2 \leq t \leq n$.

Case 3.1. If $2 \leq t \leq \phi$ then $r(x_{t+1} \mid \Pi) = (t+1, 2) = r(y_{t+1} \mid \Pi)$ which leads to a contradictory state.

Case 3.2. If $\phi + 1 \leq t \leq n - 1$ then $r(x_{t+1} \mid \Pi) = (n - t + 1, 2) = r(y_{t+1} \mid \Pi)$ which leads to a contradictory state.

Case 3.3. If $t = n$ then $r(y_2 \mid \Pi) = (2, 3) = r(z_2 \mid \Pi)$ which is again a contradictory fact.

Case 4. If we take both nodes from $y_i's$. We may take y_1 as the first node and y_t as the second node where $2 \leq t \leq n$.

Case 4.1. If $2 \leq t \leq \phi$ then $r(y_{t+1} \mid \Pi) = (t, 1) = r(z_{t+1} \mid \Pi)$ which leads to a contradictory state.

Case 4.2. If $t = \phi + 1$ then $r(z_1 \mid \Pi) = (1, \phi) = r(z_2 \mid \Pi)$ which leads to a contradictory state.

Case 4.3. If $\phi + 2 \leq t \leq n$ then $r(v_2 \mid \Pi) = (2, n - t + 3) = r(x_2 \mid \Pi)$ which is again a contradictory fact.

Case 5. If we take both nodes from $z_i's$. We may take z_1 as the first node and z_t as the second node where $2 \leq t \leq n$.

Case 5.1. If $2 \leq t \leq \phi - 1$ then $r(y_{t+1} \mid \Pi) = (t+1, 2) = r(z_{t+2} \mid \Pi)$ which leads to a contradictory state.

Case 5.2. If $t = \phi$ then $r(y_\phi \mid \Pi) = (\phi, 1) = r(z_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 5.3. If $t = \phi + 1$ then $r(y_1 \mid \Pi) = (1, \phi) = r(y_n \mid \Pi)$ which leads to a contradictory state.

Case 5.4. If $\phi + 2 \leq t \leq n$ then $r(y_1 \mid \Pi) = (1, n - t + 2) = r(z_2 \mid \Pi)$ which is again a contradictory fact.

Case 6. If one of the nodes of Π is from $u_i^\prime s$ and the second one is from $v_i^\prime s$. We may take u_1 as the first node and v_t as the second node where $1 \leq t \leq n$.

Case 6.1. If $1 \leq t \leq \phi$ then $r(v_{t+1} \mid \Pi) = (t + 1, 1) = r(x_t \mid \Pi)$ which leads to a contradictory state.

Case 6.2. If $t = \phi + 1$ then $r(y_\phi \mid \Pi) = (\phi + 2, 2) = r(y_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 6.3. If $\phi+\lambda \leq t \leq n$ and $2 \leq \lambda \leq \phi$ then $r(v_\lambda \mid \Pi) = (\lambda, \phi) = r(x_{\lambda-1} \mid \Pi)$ which is again a contradictory fact.

Case 7. If one of the nodes of Π is from $u_i^\prime s$ and the second one is from $x_i^\prime s$. We may take u_1 as the first node and x_t as the second node where $1 \leq t \leq n$.

Case 7.1. If $1 \leq t \leq \phi - 1$ then $r(v_{t+2} \mid \Pi) = (t + 2, 2) = r(x_{t+1} \mid \Pi)$ which leads to a contradictory state.

Case 7.2. If $t = \phi$ then $r(z_1 \mid \Pi) = (4, \phi + 1) = r(z_n \mid \Pi)$ which leads to a contradictory state.

Case 7.3. If $t = \phi + 1$ then $r(z_1 \mid \Pi) = (4, \phi + 1) = r(z_2 \mid \Pi)$ which leads to a contradictory state.

Case 7.4. If $\phi + 2 \leq t \leq n$ then $r(y_2 \mid \Pi) = (4, n - t + 3) = r(z_2 \mid \Pi)$ which is again a contradictory fact.

Case 8. If one of the nodes of Π is from $u_i^\prime s$ and the second one is from $y_i^\prime s$. We may take u_1 as the first node and y_t as the second node where $1 \leq t \leq n$.

Case 8.1. If $1 \leq t \leq \phi - 1$ then $r(y_{t+1} \mid \Pi) = (t + 3, 1) = r(z_{t+1} \mid \Pi)$ which leads to a contradictory state.

Case 8.2. If $t = \phi$ then $r(z_1 \mid \Pi) = (4, \phi) = r(z_n \mid \Pi)$ which leads to a contradictory state.

Case 8.3. If $t = \phi + 1$ then $r(z_1 \mid \Pi) = (4, \phi) = r(z_2 \mid \Pi)$ which leads to a contradictory state.

Case 8.4. If $\phi + 2 \leq t \leq n$ then $r(y_2 \mid \Pi) = (4, n - t + 2) = r(z_2 \mid \Pi)$ which is again a contradictory fact.

Case 9. If one of the nodes of Π is from $u_i^\prime s$ and the second one is from $z_i^\prime s$. We may take u_1 as the first node and z_t as the second node where $1 \leq t \leq n$.

Case 9.1. If $t = 1$ then $r(y_\phi \mid \Pi) = (\phi + 2, \phi) = r(y_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 9.2. If $2 \leq t \leq \phi$ then $r(y_{t+\phi-1} \mid \Pi) = (n - t - 2, \phi) = r(z_{t+\phi} \mid \Pi)$ which leads to a contradictory state.

Case 9.3. If $t = \phi + 1$ then $r(y_\phi \mid \Pi) = (\phi + 2, 1) = r(y_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 9.4. If $\phi + 2 \leq t \leq n - 1$ then $r(y_t \mid \Pi) = (n - t + 3, 1) = r(z_{t+1} \mid \Pi)$ which leads to a contradictory state.

Case 9.5. If $t = n$ then $r(y_\phi \mid \Pi) = (\phi + 2, \phi) = r(z_\phi \mid \Pi)$ which is again a contradictory fact.

Case 10. If one of the nodes of Π is from $v_i's$ and the second one is from $x_i's$. We may take v_1 as the first node and x_t as the second node where $1 \leq t \leq n$.

Case 10.1. If $1 \leq t \leq \phi - 1$ then $r(v_{t+2} \mid \Pi) = (t + 1, 2) = r(x_{t+1} \mid \Pi)$ which leads to a contradictory state.

Case 10.2. If $t = \phi$ then $r(z_1 \mid \Pi) = (3, \phi + 1) = r(z_n \mid \Pi)$ which leads to a contradictory state.

Case 10.3. If $t = \phi + 1$ then $r(z_1 \mid \Pi) = (3, \phi + 1) = r(z_2 \mid \Pi)$ which leads to a contradictory state.

Case 10.4. If $\phi + 2 \leq t \leq n$ then $r(y_2 \mid \Pi) = (3, n - t + 3) = r(z_2 \mid \Pi)$ which is again a contradictory fact.

Case 11. If one of the nodes of Π is from $v_i's$ and the second one is from $y_i's$. We may take v_1 as the first node and y_t as the second node where $1 \leq t \leq n$.

Case 11.1. If $1 \leq t \leq \phi$ then $r(v_{\phi+1} \mid \Pi) = (\phi, n - t - 4) = r(x_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 11.2. If $t = \phi + 1$ then $r(z_1 \mid \Pi) = (3, \phi) = r(z_2 \mid \Pi)$ which leads to a contradictory state.

Case 11.3. If $\phi + 2 \leq t \leq n$ then $r(y_2 \mid \Pi) = (3, n - t + 2) = r(z_2 \mid \Pi)$ which is again a contradictory fact.

Case 12. If one of the nodes of Π is from $v_i's$ and the second one is from $z_i's$. We may take v_1 as the first node and z_t as the second node where $1 \leq t \leq n$.

Case 12.1. If $t = 1$ then $r(y_1 \mid \Pi) = (2, 1) = r(y_n \mid \Pi)$ which leads to a contradictory state.

Case 12.2. If $2 \leq t \leq \phi$ then $r(v_{\phi+1} \mid \Pi) = (\phi, n - t - 3) = r(x_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 12.3. If $\phi + 1 \leq t \leq n - 1$ then $r(v_{t+1} \mid \Pi) = (n - t, 3) = r(x_{t+1} \mid \Pi)$ which leads to a contradictory state.

Case 12.4. If $t = n$ then $r(y_\phi \mid \Pi) = (\phi + 1, \phi) = r(z_\phi \mid \Pi)$ which is again a contradictory fact.

Case 13. If one of the nodes of Π is from $x_i's$ and the second one is from $y_i's$. We may take x_1 as the first node and y_t as the second node where $1 \leq t \leq n$.

Case 13.1. If $1 \leq t \leq \phi$ then $r(y_n \mid \Pi) = (2, t) = r(z_1 \mid \Pi)$ which leads to a contradictory state.

Case 13.2. If $t = \phi + 1$ then $r(z_1 \mid \Pi) = (2, \phi) = r(z_2 \mid \Pi)$ which leads to a contradictory state.

Case 13.3. If $\phi + 2 \leq t \leq n$ then $r(y_2 \mid \Pi) = (2, n - t + 2) = r(z_2 \mid \Pi)$ which is again a contradictory fact.

Case 14. If one of the nodes of Π is from $x_i's$ and the second one is from $z_i's$. We may take x_1 as the first node and z_t as the second node where $1 \leq t \leq n$.

Case 14.1. If $t = 1$ then $r(y_{\phi+1} \mid \Pi) = (\phi + 1, \phi) = r(z_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 14.2. If $2 \leq t \leq \phi + 1$ then $r(y_{\phi+1} \mid \Pi) = (\phi + 1, n - t - 4) = r(z_{\phi+2} \mid \Pi)$ which leads to a contradictory state.

Case 14.3. If $\phi + \lambda \leq t \leq n$ and $2 \leq \lambda \leq \phi$ then $r(y_{\phi+1} \mid \Pi) = (\phi + 1, \lambda - 1) = r(z_{\phi+1} \mid \Pi)$ which is again a contradictory fact.

Case 15. If one of the nodes of Π is from $y_i's$ and the second one is from $z_i's$. We may take y_1 as the first node and z_t as the second node where $1 \leq t \leq n$.

Case 15.1. If $t = 1$ then $r(y_{\phi+1} \mid \Pi) = (\phi, \phi) = r(z_{\phi+1} \mid \Pi)$ which leads to a contradictory state.

Case 15.2. If $2 \leq t \leq \phi + 1$ then $r(y_{\phi+1} \mid \Pi) = (\phi, n - t - 4) = r(z_{\phi+2} \mid \Pi)$ which leads to a contradictory state.

Case 15.3. If $\phi + \lambda \leq t \leq n$ and $2 \leq \lambda \leq \phi$ then $r(y_{\phi+1} \mid \Pi) = (\phi, \lambda - 1) = r(z_{\phi+1} \mid \Pi)$ which is again a contradictory fact.

Hence, from the above cases, we conclude that $\dim_\ell(C_n) \geq 3$ when n is even. □

Theorem 3. *Consider convex polytope C_n with $n \geq 6$,*

$$\dim_\ell(C_n) = \begin{cases} 2, & \text{if } n \text{ is odd}, \\ 3, & \text{if } n \text{ is even}. \end{cases}$$

Proof. We divide the proof into the following two cases.

Case 1. If n is odd, $n = 2\phi + 1$ and $\phi \geq 3$.

Consider the subset $\Pi = \{y_1, y_{\phi+1}\}$ of $V(C_n)$. The codes assigned to each node of C_n in relation to Π when n is odd are given below.

$$r(u_\epsilon \mid \Pi) = \begin{cases} (3, \phi - i + 4) & 1 \leq \epsilon \leq 2, \\ (\epsilon + 1, \phi - \epsilon + 4) & 3 \leq \epsilon \leq \phi + 1, \\ (n - \epsilon + 4, \epsilon - \phi + 1) & \phi + 2 \leq \epsilon \leq n. \end{cases}$$

$$r(v_\epsilon \mid \Pi) = \begin{cases} (2, \phi - i + 3) & 1 \leq \epsilon \leq 2, \\ (\epsilon, \phi - \epsilon + 3) & 3 \leq \epsilon \leq \phi + 1, \\ (n - \epsilon + 3, \epsilon - \phi) & \phi + 2 \leq \epsilon \leq n. \end{cases}$$

$$r(x_\epsilon \mid \Pi) = \begin{cases} (\epsilon, \phi - i + 2) & 1 \leq \epsilon \leq \phi + 1, \\ (n - \epsilon + 2, \epsilon - \phi) & \phi + 2 \leq \epsilon \leq n. \end{cases}$$

$$r(y_\epsilon \mid \Pi) = \begin{cases} (0, \phi) & \epsilon = 1, \\ (\epsilon - 1, \phi - \epsilon + 1) & 2 \leq \epsilon \leq \phi, \\ (\phi, 0) & \epsilon = \phi + 1, \\ (n - \epsilon + 1, \epsilon - \phi - 1) & \phi + 2 \leq \epsilon \leq n. \end{cases}$$

$$r(z_\epsilon \mid \Pi) = \begin{cases} (1, \phi - \epsilon + 2) & 1 \leq \epsilon \leq 2, \\ (\epsilon - 1, \phi - \epsilon + 2) & 3 \leq \epsilon \leq \phi + 1, \\ (n - \epsilon + 2, \epsilon - \phi - 1) & \phi + 2 \leq \epsilon \leq n. \end{cases}$$

In the above cases, the codes make it clear that Π is an LMG, therefore, $\dim_\ell(C_n) \leq 2$. Hence, from Proposition 1, we conclude that $\dim_\ell(C_n) = 2$ for $n \geq 6$ when n is odd as C_n is not a bipartite graph.

Case 2. If n is even with $n = 2\phi$ and $\phi \geq 3$.

Consider the subset $\Pi = \{u_1, u_2, u_{\phi+1}\}$ of $V(C_n)$. The codes assigned to each node of C_n in relation to Π when n is even are given below.

$$r(u_\epsilon \mid \Pi) = \begin{cases} (0, 1, \phi) & \epsilon = 1, \\ (1, 0, \phi - 1) & \epsilon = 2, \\ (\epsilon - 1, \epsilon - 2, \phi - \epsilon + 1) & 3 \leq \epsilon \leq \phi, \\ (\phi, \phi - 1, 0) & \epsilon = \phi + 1, \\ (n - \epsilon + 1, n - \epsilon + 2, \epsilon - \phi - 1) & \phi + 2 \leq \epsilon \leq n. \end{cases}$$

$$r(v_\epsilon \mid \Pi) = \begin{cases} (\epsilon, 3-\epsilon, \phi-\epsilon+2) & 1 \le \epsilon \le 2, \\ (\epsilon, \epsilon-1, \phi-\epsilon+2) & 3 \le \epsilon \le \phi+1, \\ (n-\epsilon+2, n-\epsilon+3, \epsilon-\phi) & \phi+2 \le \epsilon \le n. \end{cases}$$

$$r(x_\epsilon \mid \Pi) = \begin{cases} (\epsilon+1, 2, \phi-\epsilon+2) & 1 \le \epsilon \le 2, \\ (\epsilon+1, \epsilon, \phi-\epsilon+2) & 3 \le \epsilon \le \phi, \\ (n-\epsilon+2, \phi+1, \epsilon-\phi+1) & \phi+1 \le \epsilon \le n-1,\ \phi=3, \\ (n-\epsilon+2, \phi+1, \epsilon-\phi+1) & \phi+1 \le \epsilon \le \phi+3,\ \phi \ge 4, \\ (n-\epsilon+2, n-\epsilon+3, \epsilon-\phi+1) & \epsilon=n,\ \phi=3, \\ (n-\epsilon+2, n-\epsilon+3, \epsilon-\phi+1) & \phi+4 \le \epsilon \le n.\ \phi \ge 4. \end{cases}$$

$$r(y_\epsilon \mid \Pi) = \begin{cases} (\epsilon+2, 3, \phi-\epsilon+3) & 1 \le \epsilon \le 2, \\ (\epsilon+2, \epsilon+1, \phi-\epsilon+3) & 3 \le \epsilon \le \phi, \\ (n-\epsilon+3, \phi+2, \epsilon-\phi+2) & \phi+1 \le \epsilon \le \phi+2, \\ (n-\epsilon+3, n-\epsilon+4, \epsilon-\phi+2) & \phi+3 \le \epsilon \le n. \end{cases}$$

$$r(z_\epsilon \mid \Pi) = \begin{cases} (4, 4, \phi-\epsilon+4) & 1 \le \epsilon \le 2, \\ (\epsilon+2, \epsilon+1, \phi-\epsilon+4) & 3 \le \epsilon \le \phi, \\ (n-\epsilon+4, \epsilon+1, 4) & \phi+1 \le \epsilon \le \phi+2, \\ (n-\epsilon+4, n-\epsilon+5, \epsilon-\phi+2) & \phi+3 \le \epsilon \le n. \end{cases}$$

The codes in the above case clearly specify that Π is an LMG, therefore, $\dim_\ell(C_n) \le 3$. Hence, from Theorem 2, we conclude that $\dim_\ell(C_n) = 3$ for $n \ge 6$ when n is even. □

2.3. LMD of D_n

In [7], Bača et al. defined convex polytope D_n for $n \ge 4$ consisting of $2n$ 5– sided faces. Here $V(D_n) = \{u_i, v_i, x_i, y_i : 1 \le i \le n\}$ and $E(D_n) = \{u_iu_{i+1}; y_iy_{i+1}; u_iv_i; v_ix_i; v_{i+1}x_i; x_iy_i : 1 \le i \le n\}$. Traditionally, we take $u_{n+1} = u_1, v_{n+1} = v_1, x_{n+1} = x_1$ and $y_{n+1} = y_1$. The graph of the convex polytope D_n is shown in Figure 5.

In the next result, we compute the LMD D_n.

Theorem 4. *Consider convex polytope D_n with $n \ge 6$, then $\dim_\ell(D_n) = 2$.*

Proof. We divide the proof into the following two cases.

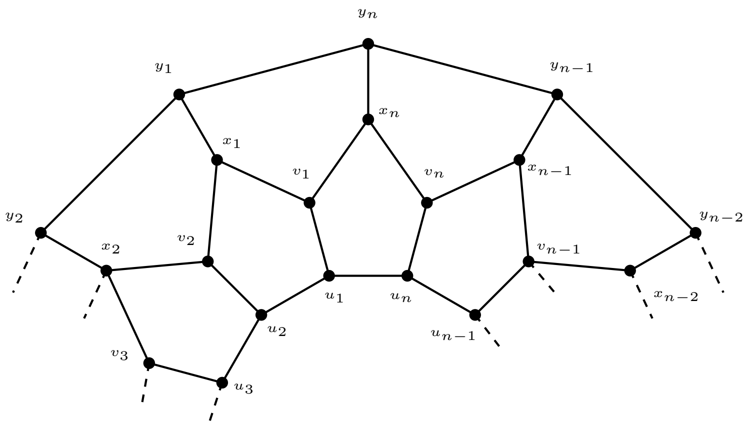

Figure 3. **Vertex–edge and combinatorial structure of D_n.**

Case 1. If n is odd, $n = 2\phi + 1$ and $\phi \geq 3$.

Consider the subset $\Pi = \{u_1, u_{\phi+1}\}$ of $V(D_n)$. The codes assigned to each node of D_n in relation to Π are the same as in the case of B_n due to symmetry in the first four cycles and choice of the subset Π in these convex polytopes.

Case 2. If n is even, $n = 2\phi$ and $\phi \geq 3$. For $\phi = 3$, $\Pi = \{v_1, v_2\}$ and for $\phi = 4$, $\Pi = \{v_1, x_4\}$, it can be verified easily that Π in each case is an LMG.

For $\phi \geq 5$, Consider the subset $\Pi = \{v_1, v_\phi\}$ of $V(D_n)$. The codes assigned to each node of D_n in relation to Π when n is even are given below.

$$r(u_\epsilon \mid \Pi) = \begin{cases} (\epsilon, \phi - \epsilon + 1) & 1 \leq \epsilon \leq \phi, \\ (n - \epsilon + 2, \epsilon - \phi + 1) & \phi + 1 \leq \epsilon \leq n. \end{cases}$$

$$r(v_\epsilon \mid \Pi) = \begin{cases} (0, \phi + 1) & \epsilon = 1, \\ (2, \phi) & \epsilon = 2, \\ (\epsilon + 1, \phi - \epsilon + 2) & 3 \leq \epsilon \leq \phi - 2, \\ (\phi, 2) & \epsilon = \phi - 1, \\ (\phi + 1, 0) & \epsilon = \phi, \\ (\phi + 2, 2) & \epsilon = \phi + 1, \\ (n - \epsilon + 3, \epsilon - \phi + 2) & \phi + 2 \leq \epsilon \leq n - 1, \\ (2, \phi + 2) & \epsilon = n. \end{cases}$$

$$r(x_\epsilon \mid \Pi) = \begin{cases} (1, \phi+1) & \epsilon = 1, \\ (3, \phi) & \epsilon = 2, \\ (\epsilon+2, \phi-\epsilon+2) & 3 \leq \epsilon \leq \phi-3, \phi \geq 6, \\ (\phi, 3) & \epsilon = \phi-2, \\ (\epsilon+2, \phi-\epsilon+2) & \phi-1 \leq \epsilon \leq \phi, \\ (\phi+2, 3) & \epsilon = \phi+1, \\ (n-\epsilon+3, \epsilon-\phi+3) & \phi+2 \leq \epsilon \leq n-2, \\ (3, \phi+2) & \epsilon = n-1, \\ (1, \phi+2) & \epsilon = n. \end{cases}$$

$$r(y_\epsilon \mid \Pi) = \begin{cases} (\epsilon+2, \phi-\epsilon+1) & 1 \leq \epsilon \leq \phi, \\ (n-\epsilon+3, \epsilon-\phi+1) & \phi+1 \leq \epsilon \leq n. \end{cases}$$

$$r(z_\epsilon \mid \Pi) = \begin{cases} (\epsilon+1, \phi-\epsilon+1) & 1 \leq \epsilon \leq \phi-1, \\ (\phi+1, \epsilon-\phi+2) & \phi \leq \epsilon \leq \phi+1, \\ (n-\epsilon+2, \epsilon-\phi+2) & \phi+2 \leq \epsilon \leq n-2, \\ (n-\epsilon+2, \phi+1) & n-1 \leq \epsilon \leq n. \end{cases}$$

In the above cases, the codes make it clear that Π is an LMG, therefore, $\dim_\ell(D_n) \leq 2$. Hence, from Proposition 1, we conclude that $\dim_\ell(D_n) = 2$ for $n \geq 6$ as D_n is not a bipartite graph. □

2.4. LMD of Q_n

In [8], Bača et al. defined convex polytope Q_n for $n \geq 6$ consisting of 3 - sided, 4 - sided, 5 - sided and n - sided faces respectively. Here $V(Q_n) = \{u_i, v_i, x_i, y_i : 1 \leq i \leq n\}$ and $E(Q_n) = \{u_iu_{i+1}; v_iv_{i+1}; y_iy_{i+1}; u_iv_i; v_ix_i; v_{i+1}x_i; x_iy_i : 1 \leq i \leq n\}$. Traditionally, we take $u_{n+1} = u_1, v_{n+1} = v_1, x_{n+1} = x_1$ and $y_{n+1} = y_1$. The graph of the convex polytope Q_n is shown in Figure 4.
In the next result, we compute the LMD Q_n.

Theorem 5. *Consider convex polytope Q_n with $n \geq 6$, then* $\dim_\ell(Q_n) = 2$.

Proof. We divide the proof into the following two cases.

Case 1. If n is odd, $n = 2\phi + 1$ and $\phi \geq 3$.

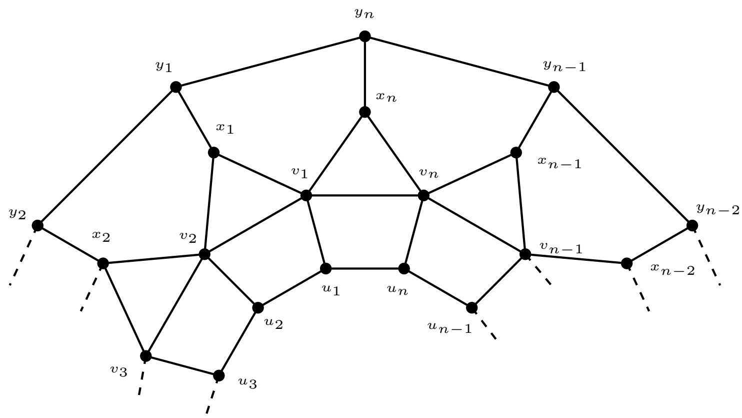

Figure 4. **Vertex–edge and combinatorial structure of Q_n.**

Consider the subset $\Pi = \{u_1, u_{\phi+1}\}$ of $V(Q_n)$. The codes assigned to each node of Q_n in relation to Π are the same as in the case of B_n due to symmetry in the first four cycles and choice of the subset Π in these convex polytopes.

Case 2. If n is even, $n = 2\phi$ and $\phi \geq 3$.

Consider the subset $\Pi = \{u_1, x_\phi\}$ of $V(Q_n)$. The codes assigned to each node of Q_n in relation to Π are the same as in the case of B_n due to symmetry in the first four cycles and choice of the subset Π in these convex polytopes.

In the above cases, the codes make it clear that Π is an LMG, therefore, $\dim_\ell(Q_n) \leq 2$. Hence, from Proposition 1, we conclude that $\dim_\ell(Q_n) = 2$ for $n \geq 6$ as Q_n is not a bipartite graph. □

3. Applications

In this section, we discuss the use of LMB and LMG in the delivery system. We also propose an algorithm to extend an LMB to get an LMG of desired size. Furthermore, since practical delivery systems often require cost-efficient connectivity, our approach aligns with optimization principles studied in network design. In particular, [29] introduced methods for creating network-state homomorphisms through optimization, which inspire strategies to minimize resource usage while maintaining structural constraints.

3.1. Application of LMB in optimizing a delivery system

For optimal placement of resources like hospitals, departmental stores, pharmacies, and fire stations etc., it is desirable to place a minimum number of resources at certain

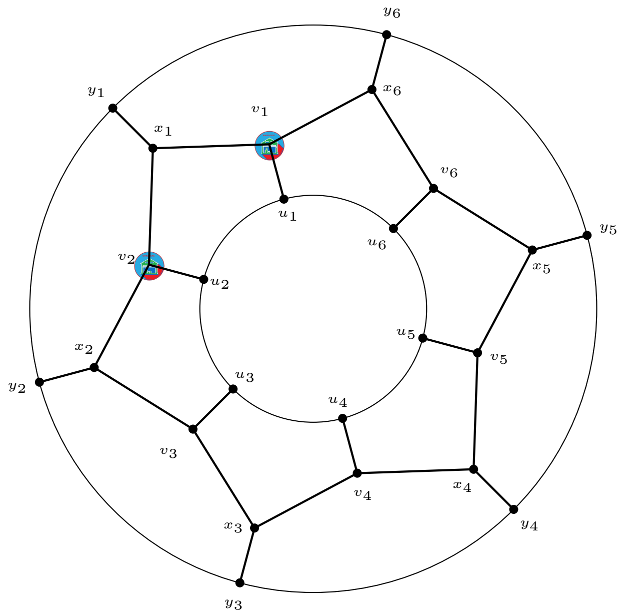

Figure 5. **Location of warehouses in Convex Polytope D_6.**

places so that each region has easy access to these resources. This problem can be seen in graph theory as choosing the minimum number of nodes MB from the graph such that each node is uniquely identified in terms of distances from the nodes in the MB. Here, the nodes are resources and regions with edges giving average distances among the nodes. The LMB ensures that adjacent nodes have unique identification codes, which can be used to select the closest resources and alternate choices of resources. Use of LMB in delivery service can be seen in [20].

Example 1. For further explanation, consider a distributed network in the form of a convex polytope D_6 in Figure 5. The LMB gives two locations at v_1 and v_2 for the warehouses such that adjacent nodes get distinct codes. The codes $u_5(3,4)$ and $u_6(2,3)$ indicate that u_5 is at a distance of 3 units from the warehouse v_1 and u_6 is 2 units from at v_1. The first coordinate of the codes gives the distance from v_1, whereas the second coordinate gives the distance from v_2. The number of warehouses can be increased using LMG.

3.2. Application of LMG in Extending a Delivery System

The number of nodes in an LMG is always greater than or equal to the number of nodes in an LMB. Suppose we want to extend the number of nodes in a graph to obtain an LMG when some of the nodes are already known. For this, we can use

Algorithm 1. If the nodes $v_1, v_2, \ldots, v_k$ are in the LMB, then Algorithm 1 selects m additional nodes from the remaining nodes so that we obtain a valid LMG.

Algorithm 1 Algorithm for generating an LMG from LMB

```
Require: Adjacency matrix A = [a_ij], LMB nodes {v_1, ..., v_k}, number of additional nodes m
Ensure: Local Metric Generator Π
  Compute the distance matrix D = [d_ij], where d_ij is the distance between nodes v_i and v_j
  Generate all subsets Π_j of size k + m that include the LMB nodes.
  for each subset Π_j do
      Compute r(v | Π_j) for all pairs of adjacent nodes
      if all adjacent nodes have distinct codes then
          return Π_j as an LMG
      end if
  end for
  return No LMG found with the given m additional nodes
```

The use of Algorithm 1 allows one to extend an existing LMB by adding m nodes to obtain an LMG while ensuring that adjacent nodes in the graph have distinct codes.

Example 2. In the previous Example 1 of convex polytope D_6, the two locations v_1 and v_2 are the LMB. If we want to add another warehouse, then by following the Algorithm 1, we will create all subsets of size three that would contain v_1 and v_2. The Algorithm 1 will check all of these sets whether they form LMG or not. Here, the Algorithm 1, after examining all the sets, shows that another warehouse can be constructed at any of the nodes except at $u_i's$.

4. Conclusion

We deduce from this paper that the LMD of the convex polytopes B_n, C_n, D_n and Q_n is fixed and unaffected by the number of nodes in these graphs. Applications for LMB are possible in the delivery services and in identifying the location for different facilities in an area. The LMG can be used to extend the existing setup without disturbing it with the help of the algorithm proposed. The algorithm proposed is helpful in computing the upper bounds for LMB.

Problem 1. Compute the LMD of generalized Petersen graphs.

Problem 2. Compute the LMD of mesh-related networks.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Use of AI tools declaration: The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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