

## On invariants for 1-factorizations of $K_{2n}$ : Description and computation

Saulo Matos<sup>1</sup>, Rafael A. Melo<sup>2,\*</sup>, Tiago Januario<sup>3</sup>, Sebastián Urrutia<sup>4</sup>,  
Dominique de Werra<sup>5</sup>

<sup>1</sup>Graduate Program in Computer Science, Federal University of Sergipe, São Cristóvão, Brazil  
[saulo.matos@ufba.br](mailto:saulo.matos@ufba.br)

<sup>2</sup>Institute of Computing, Universidade Federal da Bahia, Salvador, Brazil  
[\\*rafael.melo@ufba.br](mailto:*rafael.melo@ufba.br)

<sup>3</sup>Computer Science Department, Boston University, Boston, MA 02215, USA  
[januario@bu.edu](mailto:januario@bu.edu)

<sup>4</sup>Faculty of Logistics, Molde University College, Norway  
[sebastian.a.urrutia@himolde.no](mailto:sebastian.a.urrutia@himolde.no)

<sup>5</sup>Institut de Mathématiques, EPFL, Lausanne CH-1015, Switzerland  
[dominique.dewerra@epfl.ch](mailto:dominique.dewerra@epfl.ch)

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**Abstract:** A 1-factorization is a partition of the edge set of a graph into perfect matchings. The concept of 1-factorization is of great interest due to its applications in modeling sports tournaments. An invariant of a 1-factorization is a property that depends only on its structure such that isomorphic 1-factorizations are guaranteed to have equal invariant values. As such, non-isomorphic 1-factorizations may or may not have different invariant values. An invariant is complete when any two non-isomorphic 1-factorizations have distinct invariant values. We review seven invariants available in the literature to distinguish non-isomorphic 1-factorizations of  $K_{2n}$  (complete graphs with an even number of vertices). Additionally, considering that the invariants available in the literature are not complete, we propose two new ones, denoted lantern profiles and even-size bichromatic chains. We analyze the invariants concerning their sizes and calculation time complexity. Furthermore, we conduct computational experiments to assess their ability to distinguish non-isomorphic 1-factorizations. To accomplish that we use the sets of non-isomorphic 1-factorizations of  $K_{10}$  and  $K_{12}$ . We also consider the sets of non-isomorphic perfect 1-factorizations of  $K_{12}$ ,  $K_{14}$ , and  $K_{16}$ , as well as randomly generated 1-factorizations of  $K_{16}$  and  $K_{20}$ . Moreover, the experiments evaluate how the combination of invariants can increase the distinguishing ability.

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\* *Corresponding Author*

## 1. Introduction

Invariants are widely used when one wants to quickly verify whether two structures are non-isomorphic. In particular, invariants are needed to speed up classification algorithms. For instance, if the invariant values of two objects are different, then no further tests are needed to determine that they are structurally different.

A 1-factorization is a partition of the edge set of a graph into perfect matchings. The concept of 1-factorization is of great interest due to its applications in modeling sports tournaments. Several sports tournaments involving  $2n$  teams are organized as single round-robin tournaments in which teams face each other once. In such a tournament, there is a sequence of  $2n - 1$  rounds, with each team playing once in each round. In a basic sports scheduling problem, one has to determine in which round each pair of teams will face each other. It is natural to model such a problem as a complete graph  $K_{2n}$ , with each vertex representing a team and each edge representing the game between the teams corresponding to its endpoints. In this way, a schedule for such a tournament can be determined by computing a 1-factorization of the graph such that each of its perfect matchings represents a round of the schedule.

To tackle optimization problems in the context of sports scheduling, search procedures (such as local search and explicit or implicit enumeration) are often used to explore the distinct 1-factorizations of  $K_{2n}$ . These approaches may get stuck into a portion of the search space corresponding to 1-factorizations with the same structure. Thus, identifying whether two 1-factorizations are isomorphic becomes an important matter. This work reviews the invariants available in the literature to distinguish between non-isomorphic 1-factorizations of  $K_{2n}$ . Moreover, we propose two new invariants.

### 1.1. Basic definitions

Let  $G = (V, E)$  be a simple and undirected graph in which  $V$  is the set of vertices, and  $E \subset V \times V$  is the set of edges. A subgraph  $G' = (V', E')$  of  $G$  is a graph such that  $V' \subseteq V$  and  $E' \subseteq E$ . The degree of a vertex  $v \in V$  is given by the number of vertices adjacent to it. Two graphs  $G = (V_G, E_G)$  and  $H = (V_H, E_H)$  are *isomorphic* if there exists a bijective function  $\varphi$  from  $V_G$  to  $V_H$  such that  $xy \in E_G$  if and only if  $\varphi(x)\varphi(y) \in E_H$ . Informally, graphs  $G$  and  $H$  are isomorphic if it is possible to obtain  $H$  from  $G$  (and vice versa) just by renaming its vertices. A *chain* in a simple graph is a sequence of adjacent edges that links two vertices in the graph.

A mapping  $I : G \rightarrow \mathbb{R}^m$  that extracts properties from a graph and maps them to an  $m$ -dimensional vector is an *invariant* if it assigns equal values to isomorphic graphs. More specifically, given the graphs  $G_1$  and  $G_2$ , if  $G_1$  is isomorphic to  $G_2$ , then  $I(G_1) = I(G_2)$ . In case  $I(G_1) = I(G_2)$  if and only if  $G_1$  and  $G_2$  are isomorphic, then the invariant is said to be *complete*. In other words, if  $G_1$  and  $G_2$  are not isomorphic and  $I$  is a complete invariant, then  $I(G_1) \neq I(G_2)$ . Consider the two graphs illustrated in Figure 1. Both have the same number of vertices and edges. Thus, a simple invariant that considers only these properties cannot distinguish between them. However, when considering the sorted degree distributions at their vertices, we

can define a new invariant capable of distinguishing such graphs. Due to their potential low computational complexity, invariants may allow quick testing if two graphs are not isomorphic. It is much more efficient to test for isomorphism by checking the invariant values instead of checking for all possible isomorphism functions.

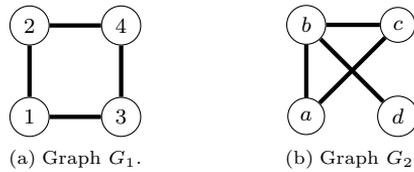


Figure 1. The graph  $G_1$  (on the left) and the graph  $G_2$  (on the right) have the same number of vertices and edges. Therefore, an invariant based on those values would not be able to distinguish them. By considering an invariant based on their sorted degree distributions, i.e.,  $I(G_1) = (2, 2, 2, 2)$  and  $I(G_2) = (1, 2, 2, 3)$ , we can assert that they are not isomorphic since their invariant values are different.

A *1-factorization* of  $G$  is a partition of  $E$  into perfect matchings. Each one of these matchings is called a *1-factor* of the 1-factorization. Figure 2 depicts a 1-factorization of  $K_4$ . Notice that not every graph has a 1-factorization. For instance, graphs with an odd number of vertices do not have perfect matchings and, thus, cannot have 1-factorizations. Complete graphs with an even number of vertices ( $K_{2n}$ ) do have 1-factorizations. The union of any two 1-factors of a given 1-factorization is a 2-regular graph consisting of a set of even-size cycles. A 1-factorization is said to be *perfect* if the union of any two of its 1-factors forms a Hamiltonian cycle.

A *proper edge coloring* of a graph is an assignment of colors to its edges such that adjacent edges do not have the same color. Notice that a 1-factorization provides a proper edge coloring of a graph by associating a color to each 1-factor. This is illustrated in the example of Figure 2.

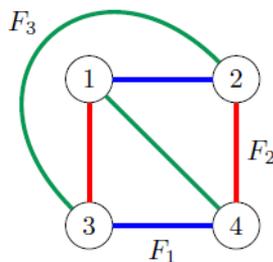


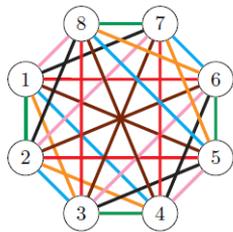
Figure 2. A 1-factorization  $\mathcal{F} = \{F_1, F_2, F_3\}$  of  $K_4$ , with  $F_1 = \{12, 34\}$ ,  $F_2 = \{13, 24\}$ , and  $F_3 = \{14, 23\}$ .

Two 1-factorizations  $\mathcal{F} = \{F_1, \dots, F_k\}$  and  $\mathcal{H} = \{H_1, \dots, H_k\}$  of  $G$  are called *isomorphic* if there exists a bijective function  $\varphi$  from the vertex set  $V$  of  $G$  onto itself

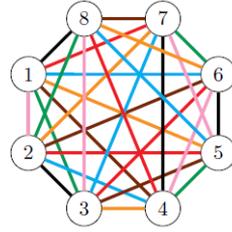
such that  $\{F_1^\varphi, \dots, F_k^\varphi\} = \{H_1, \dots, H_k\}$ , where  $F_i^\varphi$  is the set of all edges  $\varphi(x)\varphi(y)$  in which  $xy$  is an edge in  $F_i$ . We say that two isomorphic 1-factorizations belong to the same *isomorphism class* (or equivalence class). For instance, there are 6,240 distinct 1-factorizations of  $K_8$ , but they can be classified into only six isomorphism classes. Figure 3 illustrates two isomorphic 1-factorizations of  $K_8$ . Table 1 presents six non-isomorphic 1-factorizations of  $K_8$ , each representing its isomorphism class.

Table 1. Six non-isomorphic 1-factorizations of  $K_8$ . Each 8-digit block represents a 1-factor. For instance, 12345678 represents the 1-factor  $\{12, 34, 56, 78\}$ .

	$F_1$	$F_2$	$F_3$	$F_4$	$F_5$	$F_6$	$F_7$
$\mathcal{F}_0$	12345678	13245768	14235867	15263748	16253847	17283546	18273645
$\mathcal{F}_1$	12345678	13245768	14235867	15263748	16253847	17283645	18273546
$\mathcal{F}_2$	12345678	13245768	14235867	15273648	16283745	17253846	18263547
$\mathcal{F}_3$	12345678	13245768	14235867	15273846	16283745	17253648	18263547
$\mathcal{F}_4$	12345678	13245768	14253867	15273648	16283745	17234658	18263547
$\mathcal{F}_5$	12345678	13254768	14273658	15283746	16234857	17263845	18243567



(a) 1-factorization  $\mathcal{F}$  of  $K_8$ .



(b) 1-factorization  $\mathcal{H}$  of  $K_8$ .

Figure 3. Examples of isomorphic 1-factorizations of  $K_8$ . The seven 1-factors of each 1-factorization are characterized by different colors.  $\mathcal{H}$  (on the right) can be obtained from  $\mathcal{F}$  (on the left) with the function  $\varphi$  defined as follows:  $\varphi(1) = 4$ ,  $\varphi(2) = 5$ ,  $\varphi(3) = 3$ ,  $\varphi(4) = 1$ ,  $\varphi(5) = 2$ ,  $\varphi(6) = 8$ ,  $\varphi(7) = 7$ , and  $\varphi(8) = 6$ .

The concept of invariant can be extended to 1-factorizations. A mapping  $I_f : \mathcal{F} \rightarrow \mathbb{R}^m$  that extracts properties of a 1-factorization and maps them to an  $m$ -dimensional vector is a *1-factorization invariant* if it assigns equal values to isomorphic 1-factorizations. This means that, given the 1-factorizations  $\mathcal{F}_1$  and  $\mathcal{F}_2$ , if  $\mathcal{F}_1$  is isomorphic to  $\mathcal{F}_2$ , then  $I_f(\mathcal{F}_1) = I_f(\mathcal{F}_2)$ . In case  $I_f(\mathcal{F}_1) = I_f(\mathcal{F}_2)$  if and only if  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are isomorphic, then the invariant is said to be *complete*.

## 1.2. Contributions and organization

In this work, we concentrate on ways to distinguish between non-isomorphic 1-factorizations of  $K_{2n}$ . First, we review seven invariants in the literature. Second, we propose two new invariants. Finally, computational results are performed to evaluate the strength of the described invariants. The tests analyze their capacity to

distinguish non-isomorphic 1-factorizations. For that purpose, we consider the sets of 1-factorizations of  $K_{10}$  and  $K_{12}$ , as well as the sets of perfect 1-factorizations of  $K_{12}$ ,  $K_{14}$ , and  $K_{16}$ . Different combinations of the invariants are also analyzed to evaluate their complementarity. Finally, we consider a benchmark set composed of randomly generated 1-factorizations of  $K_{16}$  and  $K_{20}$ .

The remainder of this paper is organized as follows. Section 2 reviews the related literature. Section 3 details seven invariants for 1-factorizations from the literature. Section 4 proposes two new invariants: lantern profiles and even-size bichromatic chains. Section 5 summarizes the computational experiments. Section 6 discusses some concluding remarks.

## 2. Literature review

This section reviews some of the literature on graph and 1-factorization invariants and applications.

### 2.1. Graph isomorphism and graph invariants

Graph isomorphism as a computational problem first appeared in the chemistry literature of the 1950s as the problem of matching a molecular graph against a database of such graphs [13]. Whether graph isomorphism is solvable in polynomial time remains open. However, polynomial algorithms are known for testing the isomorphism of many classes of graphs. Additionally, it is claimed that the general graph isomorphism problem can be solved in quasipolynomial time [1, 2, 14]. Subgraph isomorphism, on the other hand, is known to be NP-complete [8]. Although an efficient algorithm for the graph isomorphism problem is not known, there are software available that can be executed in low computational times in practice for certain graphs. [18] lists various software for isomorphism testing.

There are several invariants for graphs [3]; some are trivial as the number of vertices and edges. Other examples are the maximum, minimum, and average degrees, connectivity, chromatic number, chromatic index, and the existence of a cycle. Although these invariants are not complete, we can construct invariants for 1-factorizations based on them that can be used to distinguish non-isomorphic 1-factorizations of  $K_{2n}$ .

### 2.2. 1-factorizations and 1-factorization isomorphism

Different works are concerned with the computation of the largest possible set of non-isomorphic 1-factorizations of complete graphs. It is known that this task has a high computational cost, since the number of non-isomorphic 1-factorizations of  $K_{2n}$  increases very fast with  $n$ . [6] state that the computation of all non-isomorphic 1-factorizations of  $K_{12}$  would require over 160 MIPS-years on a single computer.

Regarding the number of 1-factorizations, there is a unique 1-factorization of  $K_4$ , six of  $K_6$ , and 6,240 of  $K_8$  [28]. There are 1,255,566,720 1-factorizations

of  $K_{10}$  [9], 252, 282, 619, 805, 368, 320 of  $K_{12}$  [6], and a very large number of 98, 758, 655, 816, 833, 727, 741, 338, 583, 040 of  $K_{14}$  [17]. Nevertheless, the number of isomorphism classes of 1-factorizations for such complete graphs can be much smaller. There is a unique isomorphism class of 1-factorizations for  $K_4$  and  $K_6$ , six for  $K_8$  [4], 396 for  $K_{10}$  [10], 526, 915, 620 for  $K_{12}$  [6], and 1, 132, 835, 421, 602, 062, 347 for  $K_{14}$  [17].

Among the studies on the number of non-isomorphic perfect 1-factorizations, the ones conducted in [22], [5], [20], and [11] stand out for the graphs  $K_{12}$ ,  $K_{14}$ , and  $K_{16}$ . The reported results indicate that the numbers of isomorphism classes of perfect 1-factorizations are 5 for  $K_{12}$ , 23 for  $K_{14}$ , and 3155 for  $K_{16}$ . In addition, for  $4 \leq 2n \leq 10$  there is a unique perfect 1-factorization of  $K_{2n}$ . A survey on perfect 1-factorizations can be found in [23].

A survey [19] and an entire book [27] were devoted exclusively to 1-factorizations.

### 3. Existing invariants for 1-factorizations

In this section, we detail seven invariants for 1-factorizations described in the literature: cycle profiles [9], tricolor vectors [12], divisions [26, 27], trains [7], and the three invariants proposed in [11], denoted by trains-path, row-cycles, and row-cycles-per-row. To illustrate how each invariant works, we will use the six non-isomorphic 1-factorizations of  $K_8$  presented in Table 1.

#### 3.1. Cycle profiles

Given a 1-factorization  $\mathcal{F}$ , let  $c(v, F_i, F_j)$  be the size of the cycle containing the vertex  $v \in V$  in the subgraph formed by the two 1-factors  $F_i$  and  $F_j$ . Moreover, let  $c_k(v) = |\{(i, j) : c(v, F_i, F_j) = k, 1 \leq i < j \leq 2n - 1\}|$  be the number of  $k$ -cycles containing the vertex  $v \in V$  considering all possible pairs of distinct 1-factors. Given a 1-factorization  $\mathcal{F}$  and an integer  $k$ ,  $4 \leq k \leq 2n$ , the *cycle profiles* invariant [9] is defined as the sorted sequence  $(c_k(v_{\pi_k(1)}), c_k(v_{\pi_k(2)}), \dots, c_k(v_{\pi_k(2n)}))$ , where  $v_{\pi_k(\ell)}$  is the vertex participating in the  $\ell^{\text{th}}$  largest number of  $k$ -cycles. In this work, we consider the cycle profiles invariant, which we denote by  $I_f^{\text{cp}}(\mathcal{F})$ , to be the vector formed from the concatenation of the sequences for all the relevant values of  $k$ . Consequently, if two 1-factorizations have distinct sequences, they are not isomorphic. Notice that this invariant has size  $\Theta(n^2)$ . Furthermore, observe that cycle profiles does not distinguish between perfect 1-factorizations since for any vertex and pair of 1-factors,  $k = 2n$  is the only possible value.

Table 2 shows that cycle profiles is a complete invariant for the 1-factorizations of  $K_8$ , as the values for the six considered non-isomorphic 1-factorizations are pairwise different. Notice that for the graph  $K_8$ , the size of any cycle formed by two 1-factors will be either four or eight. Thus, the choice for the parameter  $k$  is restricted to  $k \in \{4, 8\}$ . Moreover, each vertex belongs to the same number of cycles of size  $k$  in every 1-factorization of  $K_8$  for both values of  $k$ .

Table 2. Cycle profiles invariant values for the six non-isomorphic 1-factorizations of  $K_8$ . The value of  $c_k(v)$  is the same for every  $v \in V(K_8)$ .

	4-cycles: sorted $c_4(v)$ values	8-cycles: sorted $c_8(v)$ values
$I_f^{\text{CP}}(\mathcal{F}_0)$	21 21 21 21 21 21 21 21	0 0 0 0 0 0 0 0
$I_f^{\text{CP}}(\mathcal{F}_1)$	13 13 13 13 13 13 13 13	8 8 8 8 8 8 8 8
$I_f^{\text{CP}}(\mathcal{F}_2)$	7 7 7 7 7 7 7 7	14 14 14 14 14 14 14 14
$I_f^{\text{CP}}(\mathcal{F}_3)$	9 9 9 9 9 9 9 9	12 12 12 12 12 12 12 12
$I_f^{\text{CP}}(\mathcal{F}_4)$	3 3 3 3 3 3 3 3	18 18 18 18 18 18 18 18
$I_f^{\text{CP}}(\mathcal{F}_5)$	0 0 0 0 0 0 0 0	21 21 21 21 21 21 21 21

**Proposition 1.** *The cycle profiles invariant for a given 1-factorization can be computed in  $O(n^3)$ .*

*Proof.* Consider that each 1-factor is represented by adjacency lists. A traversal on the union of each pair of distinct 1-factors  $(F_i, F_j)$ , with  $i < j$ , can thus be performed in  $O(n)$  to obtain all the disjoint cycles. After obtaining each cycle, it can be traversed, and the value of  $c(v, F_i, F_j)$  can be set for every vertex  $v$  contained in it, which can be done in  $O(n)$  for all the cycles. Thus, we have  $O(n^2) \times O(n)$ , resulting in  $O(n^3)$ .  $\square$

### 3.2. Tricolor vectors

Given a 1-factorization  $\mathcal{F}$  and three distinct vertices, the edges between them belong to exactly three distinct 1-factors. Denote this property by  $\mathcal{T}(u, v, w) = \{F_i, F_j, F_k\}$  where  $u, v$ , and  $w$  are the vertices and  $F_i, F_j$ , and  $F_k$  are the corresponding 1-factors. Let  $N(F_i, F_j, F_k) = |\{(u, v, w) : 1 \leq u < v < w \leq 2n, \mathcal{T}(u, v, w) = \{F_i, F_j, F_k\}\}|$  be the number of unordered triples of distinct vertices  $\{u, v, w\}$  such that  $\mathcal{T}(u, v, w) = \{F_i, F_j, F_k\}$ . The *tricolor vectors* invariant [12] of a 1-factorization, which we denote by  $I_f^{\text{TV}}(\mathcal{F})$ , is defined as the vector corresponding to the sequence  $(\mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_{2n})$ , where  $\mathcal{T}_q$  is the number of triples  $\{F_i, F_j, F_k\}$  such that  $N(F_i, F_j, F_k) = q$ . The first element  $\mathcal{T}_0$  is called the *tricolor number* [27]. Notice that the maximum number of times a given set of three 1-factors may be the image of the function  $\mathcal{T}$  is limited by the number of vertices  $2n$ . To see this, observe that in the 3-regular graph formed by three specific 1-factors, a given vertex may be part of at most  $\binom{3}{2} = 3$  triangles. The result follows since each triangle is composed of three distinct vertices. Notice that this invariant has size  $\Theta(n)$ .

Table 3 presents each one of the tricolor vectors of the six non-isomorphic 1-factorizations of  $K_8$ . Observe that tricolor vectors is complete for the 1-factorizations of  $K_8$ .

**Proposition 2.** *The tricolor vectors invariant for a given 1-factorization can be computed in  $O(n^3)$ .*

*Proof.* Consider the 1-factorization represented as an adjacency matrix with each

**Table 3.** Tricolor vectors invariant for the six non-isomorphic 1-factorizations of  $K_8$ .

	$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$	$\mathcal{T}_7$	$\mathcal{T}_8$
$I_f^{tv}(\mathcal{F}_0)$	28	0	0	0	0	0	0	0	7
$I_f^{tv}(\mathcal{F}_1)$	24	0	0	0	8	0	0	0	3
$I_f^{tv}(\mathcal{F}_2)$	18	0	8	0	8	0	0	0	1
$I_f^{tv}(\mathcal{F}_3)$	22	0	0	0	12	0	0	0	1
$I_f^{tv}(\mathcal{F}_4)$	9	8	12	0	6	0	0	0	0
$I_f^{tv}(\mathcal{F}_5)$	0	14	21	0	0	0	0	0	0

element indicating the 1-factor in which each pair of vertices is adjacent. For each vertex  $u \in V$ , we consider all the pairs  $(v, w)$  of vertices in  $V \setminus u$ , with  $v \neq w$ . Let  $F_i$ ,  $F_j$ , and  $F_k$  be the 1-factors of the edges  $uv$ ,  $vw$ , and  $wu$ , respectively. We just have to account this triangle towards the value  $N(F_i, F_j, F_k)$ . This can be done in  $O(n^2)$  for each vertex, implying  $O(n^3)$  for all the vertices. Notice that the value of each  $N(F_i, F_j, F_k)$  must be divided by three since each triangle will be found three times, once from each one of its vertices. Given the values of  $N(F_i, F_j, F_k)$  for every triplet  $\{F_i, F_j, F_k\}$ , we now compute the invariant using counting in  $O(n^3)$ . In such a counting technique, the frequency of each value is recorded by incrementing the corresponding entry in an array of counters.  $\square$

This  $O(n^3)$  bound was mentioned in [11].

### 3.3. Divisions

Given a 1-factorization  $\mathcal{F}$ , a  $d$ -division is a set of  $d$  1-factors whose union is a disconnected graph. A  $d$ -division is considered *maximal* if it is not contained in a  $(d+1)$ -division, i.e., any 1-factor added to this  $d$ -division will make their union connected. As an example, for  $d=2$ , taking  $F_i$  and  $F_j$  as two 1-factors of a 1-factorization, their union will have at least one cycle. If the subgraph induced by the union of these two 1-factors has more than one cycle, we have a 2-division. Let the value  $\alpha_d(\mathcal{F})$  be the number of maximal  $d$ -divisions in the 1-factorization  $\mathcal{F}$ . The *divisions* invariant [19, 27], which we denote by  $I_f^d(\mathcal{F})$ , is given by the vector formed by the values  $\alpha_d$  for all the possible  $d$ . Notice that this invariant has size  $O(n)$ , since  $d \leq n-1$ . The so called binary 1-factorizations have at least one  $(n-1)$ -division [24]. Observe that the divisions value for perfect 1-factorizations is always zero as the union of any two 1-factors is a Hamiltonian cycle.

Table 4 shows the divisions invariant values for the six non-isomorphic 1-factorizations of  $K_8$ . The table also provides the maximal divisions corresponding to those values. Notice that the 1-factorization  $\mathcal{F}_2$  of  $K_8$  has a unique maximal 3-division,  $\{\{F_1, F_2, F_3\}\}$ , and four 2-divisions,  $\{\{F_3, F_4\}, \{F_3, F_6\}, \{F_4, F_6\}, \{F_5, F_7\}\}$ . Except for  $F_1 \cup F_2$ ,  $F_1 \cup F_3$ ,  $F_2 \cup F_3$ , and the four combinations that constitute a maximal 2-division, the remaining 14 pairs of 1-factors in  $\mathcal{F}_2$  of  $K_8$  each form a Hamiltonian cycle. Notice that divisions is a complete invariant for the 1-factorizations of  $K_8$ .

Table 4. Divisions invariant for the six non-isomorphic 1-factorizations of  $K_8$  (Adapted from [28]).

	$\alpha_3$	$\alpha_2$	3-division maximal	2-division maximal
$I_f^d(\mathcal{F}_0)$	7	0	$\{F_1, F_2, F_3\}, \{F_1, F_4, F_5\}, \{F_1, F_6, F_7\},$ $\{F_2, F_4, F_6\}, \{F_2, F_5, F_7\}, \{F_3, F_4, F_7\}, \{F_3, F_5, F_6\}$	
$I_f^d(\mathcal{F}_1)$	3	4	$\{F_1, F_2, F_3\}, \{F_1, F_4, F_5\}, \{F_1, F_6, F_7\}$	$\{F_2, F_4\}, \{F_2, F_5\}, \{F_3, F_4\},$ $\{F_3, F_5\}$
$I_f^d(\mathcal{F}_2)$	1	4	$\{F_1, F_2, F_3\}$	$\{F_3, F_4\}, \{F_3, F_6\}, \{F_4, F_6\},$ $\{F_5, F_7\}$
$I_f^d(\mathcal{F}_3)$	1	6	$\{F_1, F_2, F_3\}$	$\{F_4, F_5\}, \{F_4, F_6\}, \{F_4, F_7\},$ $\{F_5, F_6\}, \{F_5, F_7\}, \{F_6, F_7\}$
$I_f^d(\mathcal{F}_4)$	0	3		$\{F_1, F_2\}, \{F_3, F_6\}, \{F_5, F_7\}$
$I_f^d(\mathcal{F}_5)$	0	0		

**Proposition 3.** *The divisions invariant for a given 1-factorization can be computed in  $O(n^{n+1})$ .*

*Proof.* Observe that, for a given value  $2 \leq d \leq n - 1$ , there are  $\binom{2n-1}{d} = O(n^d)$  combinations of  $d$  1-factors. For each one of these combinations, connectivity can be evaluated in  $O(nd)$ . Thus, all the calculations for a fixed  $d$  can be performed in  $O(n^d \times nd) = O(n^{d+1} \times d)$ . In the largest possible value,  $d = n - 1$ , this leads to  $O(n^n \times (n - 1)) = O(n^{n+1})$ . Given that this is the most dominant term in the summation for all the candidate values of  $d$ ,  $2 \leq d \leq n - 1$ , we have a total running time in  $O(n^{n+1})$ .  $\square$

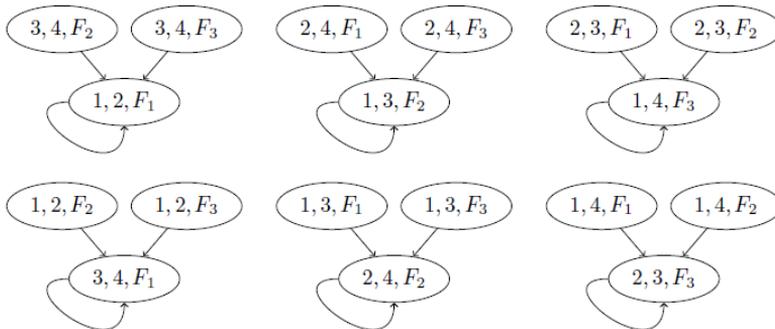
### 3.4. Trains

Given a 1-factorization  $\mathcal{F}$ , its associated *train graph*  $T(\mathcal{F})$  is a directed graph with  $n(2n - 1)^2$  vertices in which each vertex is a triple  $\{u, v, F\}$ , where  $\{u, v\}$  is an unordered pair of vertices and  $F$  is a 1-factor of  $\mathcal{F}$ . In a  $T(\mathcal{F})$ , exactly one arc leaves from each vertex, and the destination of each arc is determined as follows:

1. a loop at the vertex  $\{u, v, F_i\}$ , if  $uv \in F_i$ ;
2. an arc from the vertex  $\{u, v, F_i\}$  to  $\{w, z, F_j\}$ , if  $uw \in F_i, vz \in F_i$  and  $uv \in F_j$ .

Figure 4 illustrates the train graph associated with the 1-factorization of  $K_4$  depicted in Figure 2.

Two isomorphic 1-factorizations have isomorphic associated train graphs. However, whether verifying the isomorphism of train graphs can be solved in polynomial time is unknown. The simplified trains invariant [7] is based on the sequence of indegrees of the vertices in the train graph. Thus, for a given 1-factorization  $\mathcal{F}$ , let  $(t_0, t_1, \dots, t_{\ell_{max}})$  be a sequence, where  $t_\ell$  is equal to the number of vertices in  $T(\mathcal{F})$  that have  $\ell$  input arcs, with  $\ell_{max}$  defining the index of the last nonzero element. The trains invariant, which we denote by  $I_f^t(\mathcal{F})$ , is defined as the vector corresponding to the sequence  $(t_0, t_1, \dots, t_{\ell_{max}})$ . [7] showed that  $\ell_{max} \leq 2n - 1$  and, thus, the invariant has size  $\Theta(n)$ . The trains invariant associated with the 1-factorization of  $K_4$  depicted in Figure 2, whose train graph is provided in Figure 4, is given by  $(12, 0, 0, 6)$ . This



**Figure 4.** A train graph obtained from the 1-factorization of  $K_4$  in Figure 2.

means that twelve vertices have an indegree equal to zero, and six vertices have an indegree equal to three.

Table 5 shows the simplified trains invariant values for the six non-isomorphic 1-factorizations of  $K_8$ . It shows that, like the previous invariants, trains is complete for  $K_8$ .

**Table 5.** Trains invariant values for the six non-isomorphic 1-factorizations of  $K_8$ .

	$t_0$	$t_1$	$t_2$	$t_3$	$t_4$	$t_5$	$t_6$	$t_7$
$I_f^t(\mathcal{F}_0)$	168	0	0	0	0	0	0	28
$I_f^t(\mathcal{F}_1)$	144	0	16	8	8	16	0	4
$I_f^t(\mathcal{F}_2)$	112	16	36	24	4	4	0	0
$I_f^t(\mathcal{F}_3)$	108	48	0	12	28	0	0	0
$I_f^t(\mathcal{F}_4)$	72	64	48	12	0	0	0	0
$I_f^t(\mathcal{F}_5)$	42	112	42	0	0	0	0	0

**Proposition 4.** The simplified trains invariant for a given 1-factorization can be computed in  $O(n^3)$ .

*Proof.* Assume that each 1-factor is represented by an adjacency list. In addition, assume an adjacency matrix is available with each element indicating the 1-factor in which each pair of vertices is adjacent. For each combination of pair of vertices and 1-factor  $\{u, v, F\}$  forming a vertex of the trains graph, the outgoing arc must be computed. The destination of this arc can be computed in constant time by using the adjacency matrix to determine the 1-factor where the edge  $[u, v]$  is and the adjacency lists to consult the vertices that are adjacent to them in  $F$ . After computing the indegree of each vertex in  $O(n^3)$ , the invariant can be computed by using counting sort to determine the number of vertices with each possible indegree in  $O(n^3)$ .  $\square$

Such an  $O(n^3)$  bound was mentioned in [11].

### 3.5. Trains-path

For each vertex  $\{u, v, F_i\}$  of the train graph (see Section 3.4), define  $p(\{u, v, F_i\})$  as the length of the shortest directed path from  $\{u, v, F_i\}$  to any vertex  $\{w, z, F_j\}$  belonging to a directed cycle. Notice that if  $\{u, v, F_i\}$  itself is in a cycle then  $p(\{u, v, F_i\}) = 0$ . Given a 1-factorization  $\mathcal{F}$ , the *trains-path* invariant [11], which we denote by  $I_f^{\text{tp}}(\mathcal{F})$ , is defined as the vector  $(p_0, p_1, \dots, p_{\ell_{\max}})$ , where  $p_\ell$  is the number of vertices in  $T(\mathcal{F})$  that have  $p(\{u, v, F\}) = \ell$ . Notice that trains-path has size  $O(n^3)$ . For the train graph in Figure 4, the value of the invariant is  $(p_0, p_1) = (6, 12)$ . This is because the six vertices that are in a cycle (they have a loop) have  $p(\{u, v, F\}) = 0$ , while the other twelve vertices have  $p(\{u, v, F\}) = 1$ .

Table 6 presents the value of the trains-path invariant for each one of the six non-isomorphic 1-factorizations of  $K_8$ . It shows that trains-path is complete for  $K_8$ .

Table 6. The trains-path invariant values for the six non-isomorphic 1-factorizations of  $K_8$ .

	$p_0$	$p_1$	$p_2$	$p_3$	$p_4$	$p_5$
$I_f^{\text{tp}}(\mathcal{F}_0)$	28	168	0	0	0	0
$I_f^{\text{tp}}(\mathcal{F}_1)$	28	104	64	0	0	0
$I_f^{\text{tp}}(\mathcal{F}_2)$	28	56	80	32	0	0
$I_f^{\text{tp}}(\mathcal{F}_3)$	28	72	96	0	0	0
$I_f^{\text{tp}}(\mathcal{F}_4)$	28	24	48	48	24	24
$I_f^{\text{tp}}(\mathcal{F}_5)$	154	42	0	0	0	0

**Proposition 5.** *The trains-path invariant for a given 1-factorization can be computed in  $O(n^3)$ .*

*Proof.* Recall that the train graph has  $O(n^3)$  vertices and arcs. Besides, every vertex has an outdegree one. The value  $p(\{u, v, F\})$  for all the vertices of the train can be computed with a DFS traversal of the train graph as follows. Starting from an unvisited vertex, the search recursively visits its successor. Whenever a directed cycle is found, the search starts traversing the path backward, labeling the vertices in the cycle and setting their  $p(\{u, v, F\})$  values to 0. Besides, for the remaining vertices in the current DFS-tree (that is, in fact, a path) that are not in the cycle, the  $p(\{u, v, F\})$  values are determined by adding one unit to that of its DFS-tree descendant stopping in the original unvisited vertex. Then the search continues from another unvisited vertex if there is any. Whenever the traversal reaches a vertex that was already visited before corresponding to a previous explored DFS-tree (i.e., it is not related to a new found cycle), the search starts to traverse the path (current DFS-tree) backward, setting the value  $p(\{u, v, F\})$  as the value of its successor plus one. Recall that the construction of the train graph takes  $O(n^3)$ . Thus, as the DFS

traversal on the train graph can be performed in linear time in the size of the graph, that is  $O(n^3)$ , the whole procedure takes  $O(n^3)$ .  $\square$

We remark that Proposition 5 answers a question of the authors in [11], that claimed that trains-path could not be obviously computed in cubic time.

### 3.6. Row-cycles and row-cycles-per-row

A *Latin square* of order  $m$  is an  $m \times m$  array containing  $m$  different symbols such that each symbol occurs exactly once in each line and column of the array. In our notation, the lines (and columns) of a Latin square are indexed from 1 to  $m$ . A Latin square  $L$  is said to be *symmetric* if  $L(i, j) = L(j, i)$  for all  $1 \leq i, j \leq m$  and is said to be *unipotent* if  $L(i, i) = a$  for all  $1 \leq i \leq m$  and some value  $a$ . A symmetric and unipotent Latin square provides a natural way to describe a 1-factorization  $\mathcal{F}$ , and it is denoted by  $\mathcal{U}(\mathcal{F})$ . The element  $L(u, v)$  of a symmetric and unipotent Latin square determines the 1-factor containing the edge  $uv$ .

A *Latin rectangle* of  $L$  is a matrix in which each symbol occurs exactly once in each row and at most once in each column. A *Latin subrectangle* is a submatrix that is a Latin rectangle. If  $R$  is a  $2 \times l$ -Latin subrectangle of  $L$ , and  $R$  is minimal in that it does not contain any  $2 \times l'$ -Latin subrectangle for  $2 \leq l' < l$ , then, we say that  $R$  is a *row cycle* of length  $l$  in  $L$  [11, 29]. Table 7 shows two Latin subrectangles from  $\mathcal{U}(\mathcal{F}_2)$  of  $K_8$ . The cells that are double underlined form a  $2 \times 2$  Latin subrectangle, i.e., a row cycle of length two. On the other hand, the cells that are underlined form a  $2 \times 6$  Latin subrectangle, i.e., a row cycle of length six.

The last two invariants from [11] are based on the row cycles of a  $\mathcal{U}(\mathcal{F})$ , which were used to modify 1-factorizations in [16]. We will refer to these two invariants as row-cycles and row-cycles-per-row.

**Table 7.** Symmetric unipotent Latin square corresponding to 1-factorization  $\mathcal{F}_2$  of  $K_8$ .

·	<u>1</u>	2	<u>3</u>	4	5	6	7
1	·	3	2	6	7	4	5
2	<u>3</u>	·	<u>1</u>	7	4	5	6
<u>3</u>	<u>2</u>	<u>1</u>	·	<u>5</u>	<u>6</u>	<u>7</u>	4
4	6	7	5	·	1	2	3
5	7	4	6	1	·	3	2
6	4	5	7	2	3	·	1
<u>7</u>	<u>5</u>	<u>6</u>	<u>4</u>	<u>3</u>	<u>2</u>	<u>1</u>	·

#### 3.6.1. Row-cycles

Given a 1-factorization  $\mathcal{F}$ , for each pair of rows of  $\mathcal{U}(\mathcal{F})$ , determine the Latin subrectangles that can be formed. The *row-cycles* invariant, which we denote by  $I_r^f(\mathcal{F})$ , is defined as the vector formed by the number of row cycles of each size that can be found in  $\mathcal{U}(\mathcal{F})$ . Notice that row-cycles has size  $\Theta(n)$ .

In Table 7, the subrectangle in rows 1 and 3 (double underlined) forms a row-cycle of size two, and the subrectangle in lines 4 and 8 (underlined) forms a row-cycle of size six. Table 8 displays the value of the row-cycles invariant for each one of the six non-isomorphic 1-factorizations of  $K_8$ .

Table 8. The row-cycles invariant values for the six non-isomorphic 1-factorizations of  $K_8$ .

	size 2	size 3	size 4	size 6
$I_f^r(\mathcal{F}_0)$	84	0	0	0
$I_f^r(\mathcal{F}_1)$	52	0	16	0
$I_f^r(\mathcal{F}_2)$	28	0	16	8
$I_f^r(\mathcal{F}_3)$	36	0	0	16
$I_f^r(\mathcal{F}_4)$	12	8	12	12
$I_f^r(\mathcal{F}_5)$	0	14	0	21

**Proposition 6.** *The row-cycles invariant for a given 1-factorization can be computed in  $O(n^3)$ .*

*Proof.* We consider an auxiliary data structure that stores, for each row and symbol, the column in which that symbol is in the row. For each pair of rows  $(r_1, r_2)$ , the subrectangles formed from these rows can be obtained in linear time. Starting from the first unvisited column  $c$ , look for the symbol in  $s = \mathcal{U}(\mathcal{F})(r_1, c)$  and search in constant time, using the auxiliary data structure, the column having symbol  $s$  in row  $r_2$ . This becomes the new current column. Continue in this way until the symbol stored in the current column of the line  $r_1$  is the starting symbol  $s$ . At this point, a subrectangle is found, and it is accounted for based on its size. The procedure continues finding each subrectangle involving  $r_1$  and  $r_2$ .  $\square$

This  $O(n^3)$  bound was mentioned in [11].

### 3.6.2. Row-cycles-per-row

Given a 1-factorization  $\mathcal{F}$ , the *row-cycles-per-row* invariant, which we denote by  $I_f^{rr}(\mathcal{F})$ , is defined as the vector formed by the number of Latin subrectangles of each size that can be found in each line  $\mathcal{U}(\mathcal{F})$ . The values for each row should be sorted in lexicographical order. Row-cycles-per-row has size  $\Theta(n^2)$ .

In Table 7, the subrectangle in rows 1 and 3 (double underlined) forms a row-cycle of size two on rows 1 and 3, while the subrectangle in rows 4 and 8 (singly underlined) forms a row-cycle of size six on rows 4 and 8. Table 9 shows the value of the row-cycles-per-row invariant for each one of the six non-isomorphic 1-factorizations of  $K_8$ .

**Proposition 7.** *The row-cycles-per-row invariant for a given 1-factorization can be computed in  $O(n^3)$ .*

Table 9. The row-cycles-per-row invariant values for the six non-isomorphic 1-factorizations of  $K_8$ .

	row 1	row 2	row 3	row 4	row 5	row 6	row 7	row 8
$I_f^{rr}(\mathcal{F}_0)$	21000	21000	21000	21000	21000	21000	21000	21000
$I_f^{rr}(\mathcal{F}_1)$	13040	13040	13040	13040	13040	13040	13040	13040
$I_f^{rr}(\mathcal{F}_2)$	7042	7042	7042	7042	7042	7042	7042	7042
$I_f^{rr}(\mathcal{F}_3)$	9004	9004	9004	9004	9004	9004	9004	9004
$I_f^{rr}(\mathcal{F}_4)$	3233	3233	3233	3233	3233	3233	3233	3233
$I_f^{rr}(\mathcal{F}_5)$	0200	0206	0206	0206	0206	0206	0206	01406

*Proof.* This can be accomplished with a slight variation of the procedure described for row-cycles. The only difference is that we have to consider the size of the subrectangle obtained when accounting for the two rows under consideration.  $\square$

Such an  $O(n^3)$  bound was mentioned in [11].

## 4. New invariants for 1-factorizations

In this section, we propose two new invariants. In what follows, let  $G = (V, E)$  be a complete proper-edge-colored graph corresponding to a 1-factorization. Subsection 4.1 presents the lantern profiles invariant. Subsection 4.2 introduces the even-size bichromatic chains invariant. Subsection 4.3 summarizes the sizes and the computational complexities for the invariants described in our work.

### 4.1. Lantern profiles

Given a vertex  $v$  and  $W \subset N(v)$ , where  $N(v)$  is the set of vertices adjacent to  $v$ , define  $B(v, W) = \{vw \mid w \in W\}$ . Consider two vertices  $u$  and  $v$ , with  $u \neq v$ , and  $W \subset N(u) \cap N(v) \setminus \{u, v\}$ . Let  $C(E')$  be the set of colors occurring in  $E' \subseteq E$  in a proper edge coloring of  $K_{2n}$ . Recall that the connection between 1-factorizations and proper edge colorings was presented in Section 1.1. Consider the graph formed by  $B(u, W) \cup B(v, W)$  and assume a coloring  $C(E)$ . If  $C(B(u, W)) = C(B(v, W))$ , with  $W \neq \emptyset$  and inclusion-wise minimal for the equality to hold, the subgraph with vertices  $\{u, v\} \cup W$  and edges  $B(u, W) \cup B(v, W)$  is called a *colorful chordless lantern*  $L(u, v, W)$  [24]. Colorful chordless lanterns are illustrated in Figure 5.

Let the *degree of a lantern*  $L(u, v, W)$  be the degree of vertex  $u$  in the lantern. Note that for each pair of vertices  $(u, v)$ , the graph formed by  $B(u, V \setminus \{u, v\}) \cup B(v, V \setminus \{u, v\})$  is divided by the 1-factorization in a number of lanterns, each of them with some degree  $2 \leq l \leq 2n - 2$ . The sum of the degrees of such lanterns is  $2n - 2$ .

Let  $f(w, u, v)$ , with  $w \neq u$ ,  $w \neq v$ , and  $u \neq v$ , be the degree of the lantern  $L(u, v, W)$  with  $w \in W$ . Let  $f_k(w) = |\{(u, v) : f(w, u, v) = k, w \neq u, w \neq v, u \neq v\}|$  be the total number of lanterns of degree  $k$  containing the vertex  $w$  considering any pair of distinct vertices  $u$  and  $v$ . Given a 1-factorization  $\mathcal{F}$ , the *lantern profiles invariant*, which we denote by  $I_f^{lp}(\mathcal{F})$ , is defined as the vector corresponding to the



$c_j = c$ , a complete lantern  $L(u, v, W)$  was obtained, and its degree  $k$  is accounted for in  $f_k(w)$  for every  $w \in W$ . The procedure thus resumes by starting from an edge with a previously unselected color. Thus, the complete procedure can be implemented to run in  $O(n^3)$ .  $\square$

Notice that lanterns are alternative ways to see row-cycles. Thus, the lantern profiles invariant is strongly related to the row-cycles and row-cycles-per-row invariant. In fact, the lantern profiles invariant could also be seen as a row-cycles-per-symbol invariant.

#### 4.2. Even-size bichromatic chains

Let  $\mathcal{F}$  be a 1-factorization, and  $u$  and  $v$  be a pair of distinct vertices. Denote by  $q(u, v)$  the number of even-size bichromatic chains connecting  $u$  and  $v$ , where an even-size bichromatic chain is one composed of an even number of edges such that the edge colors alternate between two values. Figure 6 illustrates four even-size bichromatic chains connecting the vertices  $u$  and  $v$ , implying that they contribute four units to the value  $q(u, v)$ .

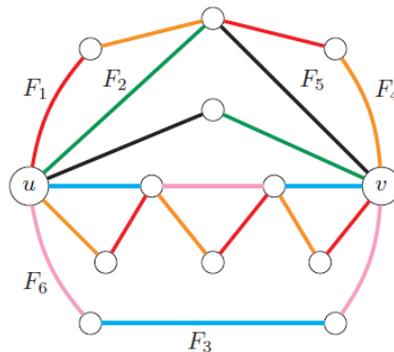


Figure 6. Vertices  $u$  and  $v$  are linked by even-size bichromatic chains corresponding to the pairs of 1-factors  $(F_1, F_4)$  and  $(F_2, F_5)$ , and possibly others.

Let  $q_k = |\{\{u, v\} : q(u, v) = k, u, v \in V\}|$  for  $2n - 2 \leq k \leq 2\binom{2n-2}{2}$ . The lower bound for  $k$  comes from the fact that any of the other  $2n - 2$  vertices  $w \in V \setminus \{u, v\}$  form the even-size bichromatic chain  $(u, w, v)$ . The upper bound is defined as twice the number of possible pairs of colors, not taking into account the color of the edge  $[u, v]$ . The *even-size bichromatic chains* invariant, which we denote by  $I_f^{ec}(\mathcal{F})$ , is defined as the vector formed by the sequence of  $q_k$  for the possible values of  $k$ . Notice that this invariant has size  $\Theta(n^2)$ .

Table 11 shows the value of the even-size bichromatic chains invariant for each one of the six non-isomorphic 1-factorizations of  $K_8$ . It can be noticed that the invariant is complete for the 1-factorizations of  $K_8$ .

Table 11. Even-size bichromatic chains invariant values for the six non-isomorphic 1-factorizations of  $K_8$ .

	6	10	12	14	16	18	22
$I_f^{ec}(\mathcal{F}_0)$	28	0	0	0	0	0	0
$I_f^{ec}(\mathcal{F}_1)$	8	16	0	0	0	0	4
$I_f^{ec}(\mathcal{F}_2)$	0	8	8	4	0	4	4
$I_f^{ec}(\mathcal{F}_3)$	0	0	16	12	0	0	0
$I_f^{ec}(\mathcal{F}_4)$	0	0	4	0	12	12	0
$I_f^{ec}(\mathcal{F}_5)$	0	0	0	0	0	28	0

**Proposition 9.** *The even-size bichromatic chains invariant for a given 1-factorization can be computed in  $O(n^4)$ .*

*Proof.* For each pair of colors, consider the subgraph induced by the edges of those colors. Such a subgraph decomposes into disjoint even-length cycles. For every vertex  $v$ , we traverse the cycle containing  $v$ . For each vertex  $u$  located at an even distance from  $v$  along this cycle, we incorporate the corresponding chain into the computation of  $q(u, v)$ . This procedure can be carried out in  $O(n^2)$  for each pair of colors, resulting in a total complexity of  $O(n^4)$ . The values of  $q_k$  are then obtained by iterating over all pairs  $\{u, v\}$  in  $O(n^2)$  time and applying a counting technique, where the number of occurrences of each value is recorded by incrementing the corresponding entry in an array of counters. Consequently, the overall algorithm runs in  $O(n^4)$ .  $\square$

### 4.3. Summary of the sizes, calculation times, and classification of the considered invariants

Table 12 summarizes the invariants detailed in our work. It provides their sizes, complexities for calculation, and classifications (i.e., what they are based on).

Table 12. Invariant sizes, running times for their calculation, and classifications.

Invariant	Size	Calculation time	Classification (based on)
<i>cycle profiles</i>	$\Theta(n^2)$	$O(n^3)$	union of two 1-factors
<i>tricolor vectors</i>	$\Theta(n)$	$O(n^3)$	union of three 1-factors
<i>divisions</i>	$O(n)$	$O(n^{n+1})$	union of $d$ 1-factors
<i>trains</i>	$\Theta(n)$	$O(n^3)$	trains graph
<i>trains-path</i>	$O(n^3)$	$O(n^3)$	trains graph
<i>row-cycles</i>	$\Theta(n)$	$O(n^3)$	row-cycles / lanterns
<i>row-cycles-per-row</i>	$\Theta(n^2)$	$O(n^3)$	row-cycles / lanterns
<i>lantern profiles</i>	$\Theta(n^2)$	$O(n^3)$	row-cycles / lanterns
<i>even-size bichromatic chains</i>	$\Theta(n^2)$	$O(n^4)$	union of two 1-factors

## 5. Experimental results

In this section, we summarize the results of the experiments carried out to evaluate the strength of the invariants on different sets of non-isomorphic 1-factorizations. We define the strength of an invariant as the number of isomorphism classes identified, which is given by the amount of distinct invariant values for the non-isomorphic 1-factorizations in the set. The benchmark set consists of:

- (a) the set of 396 non-isomorphic 1-factorizations of  $K_{10}$ ;
- (b) the sets of five non-isomorphic perfect 1-factorizations of  $K_{12}$ , 23 of  $K_{14}$ , and 3,155 of  $K_{16}$ ;
- (c) the set of 526,915,620 non-isomorphic 1-factorizations of  $K_{12}$ ;
- (d) sets of 25,000,000 randomly generated 1-factorizations of  $K_{16}$  and  $K_{20}$ .

All the experiments were performed on a machine running under Ubuntu 22.04.1 LTS with an Intel Core i5-9300H 2.40 GHz processor and 8 GB of RAM. The codes were written in C++ and compiled with g++ version 11.3.0, using the options ‘-O3’ and ‘-std=c++20’.

Each one of the following subsections considers one of the enumerated benchmark sets. More specifically, Subsection 5.1 presents the results of the invariants for the non-isomorphic 1-factorizations of  $K_{10}$ . Subsection 5.2 shows the results for the non-isomorphic perfect 1-factorizations of  $K_{12}$ ,  $K_{14}$ , and  $K_{16}$ . Subsection 5.3 displays the results for the non-isomorphic 1-factorizations of  $K_{12}$  and also evaluates the combination of different invariants in an attempt to improve the distinguishing ability. Subsection 5.5 presents the results for some randomly generated 1-factorizations of  $K_{16}$  and  $K_{20}$ .

### 5.1. Non-isomorphic 1-factorizations of $K_{10}$

Table 13 shows the strength of the different invariants to distinguish between the 396 non-isomorphic 1-factorizations of  $K_{10}$ . The first column presents the invariant, and the second column the amount (absolute value and percentage) of isomorphism classes distinguished.

The table shows that trains-path and even-size bichromatic chains are complete for the non-isomorphic 1-factorizations of  $K_{10}$ . In addition, row-cycles-per-row and lantern profiles can distinguish all but a single pair of non-isomorphic 1-factorizations. Moreover, divisions has very poor performance for this benchmark set, as it can only distinguish 46 isomorphism classes (11.6%). Given the low performance of divisions for  $K_{10}$ , it will not be considered in the remaining tests.

### 5.2. Non-isomorphic perfect 1-factorizations of $K_{12}$ , $K_{14}$ , and $K_{16}$

Table 14 shows the strength of the different invariants to distinguish between the five non-isomorphic perfect 1-factorizations of  $K_{12}$ , 23 non-isomorphic perfect 1-

Table 13. Strength of the invariants to distinguish non-isomorphic 1-factorizations of  $K_{10}$ .

Invariant	Observable isomorphism classes
<i>cycle profiles</i>	346 (87.4%)
<i>tricolor vectors</i>	323 (81.6%)
<i>divisions</i>	46 (11.6%)
<i>trains</i>	394 (99.5%)
<i>trains-path</i>	<b>396</b> (100.0%)
<i>row-cycles</i>	374 (94.4%)
<i>row-cycles-per-row</i>	395 (99.7%)
<i>lantern profiles</i>	395 (99.7%)
<i>even-size bichromatic chains</i>	<b>396</b> (100.0%)

factorizations of  $K_{14}$ , and 3,155 non-isomorphic perfect 1-factorizations of  $K_{16}$ . The table does not present values for cycle profiles as it cannot distinguish between perfect 1-factorizations (see Section 3.1).

Table 14. Strength of the invariants to distinguish non-isomorphic perfect 1-factorizations of  $K_{12}$ ,  $K_{14}$ , and  $K_{16}$ .

Invariant	$K_{12}$	$K_{14}$	$K_{16}$
<i>tricolor vectors</i>	<b>5</b>	<b>23</b>	2,320
<i>trains</i>	<b>5</b>	<b>23</b>	3,104
<i>trains-path</i>	<b>5</b>	<b>23</b>	<b>3,155</b>
<i>row-cycles</i>	4	22	<b>3,155</b>
<i>row-cycles-per-row</i>	4	22	<b>3,155</b>
<i>lantern profiles</i>	4	22	<b>3,155</b>
<i>even size bichromatic chains</i>	4	<b>23</b>	<b>3,155</b>

The table shows that trains-path is complete for the non-isomorphic perfect 1-factorizations of  $K_{12}$ ,  $K_{14}$ , and  $K_{16}$ . Tricolor vectors and trains are complete for the non-isomorphic perfect 1-factorizations of  $K_{12}$  and  $K_{14}$ . Even size bichromatic chains is complete for the non-isomorphic perfect 1-factorizations of  $K_{14}$  and  $K_{16}$ . Row-cycles, row-cycles-per-row, and lantern profiles are only complete for the non-isomorphic perfect 1-factorizations of  $K_{16}$ .

### 5.3. Non-isomorphic 1-factorizations of $K_{12}$

This subsection reports the results for the non-isomorphic 1-factorizations of  $K_{12}$ . Initially, we consider a preliminary subset of 5,000,000 1-factorizations on which we test all the invariants. We constructed this subset by sequentially selecting the first 5,000,000 1-factorizations from the complete set of non-isomorphic 1-factorizations of  $K_{12}$  provided by Petteri Kaski [6]. Based on the results of this first experiment, we select the strongest invariants to be part of the experiments carried out in the second part of this section.

### 5.3.1. Preliminary subset

Table 15 shows the strength of the different invariants to distinguish between 5,000,000 non-isomorphic 1-factorizations of  $K_{12}$ . This is a proper subset of the set with 526,915,620 non-isomorphic 1-factorizations of  $K_{12}$ . The first column presents the invariant, and the second column the amount (absolute value and percentage) of isomorphism classes distinguished.

Table 15. Strength of the invariants to distinguish 5,000,000 non-isomorphic 1-factorizations of  $K_{12}$ .

Invariant	Observable isomorphism classes
<i>cycle profiles</i>	2,984,500 (59.690%)
<i>tricolor vectors</i>	283,044 (05.661%)
<i>trains</i>	1,698,355 (33.967%)
<i>trains-path</i>	4,999,812 (99.996%)
<i>row-cycles</i>	3,371,571 (67.431%)
<i>row-cycles-per-row</i>	4,999,564 (99.991%)
<i>lantern profiles</i>	4,999,624 (99.992%)
<i>even-size bichromatic chains</i>	4,999,375 (99.987%)

The table shows that trains-path, row-cycles-per-row, lantern profiles, and even-size bichromatic chains are much stronger than the others. All of them distinguish more than 99.9% of the isomorphism classes, while the others distinguish less than 68% of them. For that reason, in the following experiments, we only show the results for these four strongest invariants and their combinations.

### 5.3.2. The complete set of non-isomorphic 1-factorizations of $K_{12}$

Table 16 shows the strength of the selected invariants to distinguish between the 526,915,620 non-isomorphic 1-factorizations of  $K_{12}$ .

Table 16. Strength of the invariants to distinguish non-isomorphic 1-factorizations of  $K_{12}$ .

Invariant	Observable isomorphism classes
<i>trains-path</i>	<b>526,914,386 (99.9997%)</b>
<i>row-cycles-per-row</i>	526,912,461 (99.9994%)
<i>lantern profiles</i>	526,913,049 (99.9995%)
<i>even-size bichromatic chains</i>	512,617,397 (97.2864%)

The table shows that trains-path is slightly stronger than the other invariants. Among the invariants with guaranteed quadratic size, lantern profiles was the strongest, followed by row-cycles-per-row. Notice that, considering trains-path, there are still 1,234 unidentified isomorphism classes. Appendix A shows a pair of non-isomorphic 1-factorizations that was not distinguished by any of the considered invariants. The appendix also presents the corresponding invariant values.

### 5.3.3. How can the combination of invariants improve the distinguishing strength?

We now analyze the strength of different invariant combinations. We consider all the possible combinations of the four selected invariants.

Table 17 shows the average strength of the different combinations of the invariants to distinguish between the 526,915,620 non-isomorphic 1-factorizations of  $K_{12}$ .

Table 17. Strength of the invariant combinations to distinguish non-isomorphic 1-factorizations of  $K_{12}$ .

Invariant combination	Observable isomorphism classes
trains path $\oplus$ row-cycles-per-row	526,915,584 (99.9999931%)
trains path $\oplus$ lantern profiles	526,915,594 (99.9999950%)
trains path $\oplus$ even-size bichromatic chains	526,915,608 (99.9999977%)
row-cycles-per-row $\oplus$ lantern profiles	526,913,057 (99.9995135%)
row-cycles-per-row $\oplus$ even-size bichromatic chains	526,915,587 (99.9999937%)
lantern profiles $\oplus$ even-size bichromatic chains	526,915,584 (99.9999931%)
trains path $\oplus$ row-cycles-per-row $\oplus$ lantern profiles	526,915,594 (99.9999950%)
trains path $\oplus$ row-cycles-per-row $\oplus$ even-size bichromatic chains	<b>526,915,614 (99.9999988%)</b>
trains path $\oplus$ lantern profiles $\oplus$ even-size bichromatic chains	<b>526,915,614 (99.9999988%)</b>
row-cycles per row $\oplus$ lantern profiles $\oplus$ even-size bichromatic chains	526,915,587 (99.9999937%)
trains path $\oplus$ row-cycles-per-row $\oplus$ lantern profiles $\oplus$ even-size bichromatic chains	<b>526,915,614 (99.9999988%)</b>

The table shows that the strongest combination of two invariants was obtained by combining trains-path with even-size bichromatic chains. This shows that although the latter was shown to be the weakest of the selected invariants individually, when combined with a strong complementary invariant such as trains-path, it is able to obtain promising results. Notice that the combination of row-cycles-per-row with lantern profiles is much weaker than the combination of any other two invariants. In fact, it is slightly stronger than these invariants used individually. This shows that contrary to what happens with trains-path and even-size bichromatic chains, the invariants are strongly related, and their distinguishing capacities seem very similar. Notice that, with the strongest combination of two invariants, only 12 isomorphism classes remain unidentified. The two best combinations of three invariants are also the ones including trains-path and even-size bichromatic chains. These combinations have the same strength. With these two combinations, a very small number of six isomorphism classes remain unidentified. Notably, the combination of the four invariants does not improve upon the best combinations of three invariants. Appendix B displays the six undistinguished pairs of non-isomorphic 1-factorizations of  $K_{12}$ .

### 5.4. A small note on the six undistinguished pairs of non-isomorphic 1-factorizations

We performed an additional test with the six pairs of non-isomorphic 1-factorizations that remained undistinguished with the best invariant combinations. More specifically, we verified the values provided by the weaker invariants that were ruled out from the experiments in the previous subsections (cycle profiles, tricolor vectors, trains, and row cycles). The results showed that tricolor vectors could distinguish two of such pairs, while trains could distinguish one (the same as one of those distinguished by

tricolor vectors). This implies that, combined with the best combination from the previous subsection, tricolor vectors could distinguish all the isomorphism classes but four.

### 5.5. Randomly generated 1-factorizations of $K_{16}$ and $K_{20}$

In this experiment, we used a randomized variant of Vizing's algorithm [15, 21, 25] to randomly generate sets of 25,000,000 1-factorizations of  $K_{16}$  and  $K_{20}$ . The results in Table 18 show that the four invariants selected are able to distinguish between all the 25,000,000 1-factorizations for both sets.

Table 18. Strength of the invariants to distinguish 25,000,000 randomly generated 1-factorizations of  $K_{16}$  and  $K_{20}$ .

Invariant	$K_{16}$	$K_{20}$
<i>trains-path</i>	25,000,000	25,000,000
<i>row-cycles-per-row</i>	25,000,000	25,000,000
<i>lantern profiles</i>	25,000,000	25,000,000
<i>even size bichromatic chains</i>	25,000,000	25,000,000

## 6. Concluding remarks

We analyzed invariants for 1-factorizations of  $K_{2n}$ . We described seven of the main invariants available in the literature (cycle profiles, tricolor vectors, divisions, trains, trains-path, row-cycles, and row-cycles-per-row). Furthermore, we proposed two new invariants, denoted lantern profiles and even-size bichromatic chains. For all the nine invariants presented, we analyzed their size and computational complexity.

Furthermore, we performed experiments to evaluate the strength of the invariants. We used a benchmark set composed of the non-isomorphic 1-factorizations of  $K_{10}$  and  $K_{12}$ , as well as the non-isomorphic perfect 1-factorizations of  $K_{12}$ ,  $K_{14}$ , and  $K_{16}$ . The results show that trains-path and even-size bichromatic chains are complete for the non-isomorphic 1-factorizations of  $K_{10}$ . Besides, only trains-path is complete for all the three sets of non-isomorphic perfect 1-factorizations considered. Moreover, four of the invariants (trains-path, row-cycles-per-row, lantern profiles, and even-size bichromatic chains) have shown to be much stronger than the others when considering the non-isomorphic 1-factorizations of  $K_{12}$ . Last but not least, the strengths of the combinations of invariants were also evaluated, showing a complementarity in their distinguishing abilities.

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## Appendix A Example of an undistinguished pair of non-isomorphic 1-factorizations and their corresponding invariant values

**Table 19.** An undistinguished pair of non-isomorphic 1-factorizations  $K_{12}$ .

$\mathcal{F}$	$F_1$	$F_2$	$F_3$	$F_4$	$F_5$	$F_6$
$\mathcal{F}_{122}$	123456789ABC	132457689BAC	142358679CAB	1526394A7B8C	16253B4C798A	192A35467C8B
$\mathcal{F}_{123}$	123456789ABC	132457689BAC	142358679CAB	1526394A7B8C	16253B4C798A	192A35467C8B

$\mathcal{F}$	$F_7$	$F_8$	$F_9$	$F_{10}$	$F_{11}$
$\mathcal{F}_{122}$	1B2C36457A89	17283A495C6B	18273C4B596A	1C2B37485A69	1A2938475B6C
$\mathcal{F}_{123}$	1C2B36457A89	17283A495B6C	18273C4B596A	1B2C37485A69	1A2938475C6B

**Table 20.** Cycle profiles invariant values for the two non-isomorphic 1-factorizations of  $K_{12}$  shown in Table 19. The value of  $c_k(v)$  is the same for every  $v \in V(K_{12})$ .

	4-cycles: $c_4(v)$ values	6-cycles: $c_6(v)$ values	8-cycles: $c_8(v)$ values	12-cycles: $c_{12}(v)$ values
$I_f^{cp}(\mathcal{F})$	19 19 $\dots$ 19	19 18 18 $\dots$ 18	18 4 4 $\dots$ 4	4 14 14 $\dots$ 14

**Table 21.** Trains invariant values for the two non-isomorphic 1-factorizations of  $K_{12}$  shown in Table 19.

	$t_0$	$t_1$	$t_2$	$t_3$	$t_4$	$t_5$	$t_6$	$t_7$	$t_8$	$t_9$	$t_{10}$	$t_{11}$
$I_f^t(\mathcal{F})$	340	208	112	28	16	16	0	0	0	0	0	6

**Table 22.** Tricolor vectors invariant for the two non-isomorphic 1-factorizations of  $K_{12}$  shown in Table 19.

	$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$	$\mathcal{T}_4$	$\mathcal{T}_5$	$\mathcal{T}_6$	$\mathcal{T}_7$	$\mathcal{T}_8$	$\mathcal{T}_9$	$\mathcal{T}_{10}$	$\mathcal{T}_{11}$	$\mathcal{T}_{12}$
$I_f^{tv}(\mathcal{F})$	94	0	36	0	34	0	0	0	0	0	0	0	1

**Table 23.** Even-size bichromatic chains invariant values for the two non-isomorphic 1-factorizations of  $K_{12}$  shown in Table 19.

	18	20	22	24	26	30	46
$I_f^{ec}(\mathcal{F})$	8	12	16	4	16	4	6

**Table 24.** Trains-path invariant values for the two non-isomorphic 1-factorizations of  $K_{12}$  shown in Table 19.

	$p_0$	$p_1$	$p_2$	$p_3$	$p_4$	$p_5$
$I_f^{tp}(\mathcal{F})$	186	292	88	80	48	32

Table 25. Lantern profiles invariant values for the two non-isomorphic 1-factorizations of  $K_{12}$  shown in Table 19.

	Lanterns with degree 2						Lanterns with degree 3						Lanterns with degree 4						Lanterns with degree 5																												
$I_{\mathcal{F}}^{\text{P}}(\mathcal{F})$	19	19	19	19	19	19	19	19	19	19	19	19	4	4	4	4	4	4	4	4	4	4	4	5	5	5	7	7	7	7	8	8	8	8	2	2	2	2	3	3	3	3	5	5	5	5	
	Lanterns with degree 6						Lanterns with degree 7						Lanterns with degree 8						Lanterns with degree 10																												
$I_{\mathcal{F}}^{\text{P}}(\mathcal{F})$	4	4	4	4	6	6	6	6	6	8	8	8	0	0	0	0	0	0	0	0	0	0	0	12	12	12	12	18	18	18	18	18	18	18	18	0	0	0	0	0	0	0	0	0	0	0	0

Table 26. Row-cycles-per-row invariant values for the two non-isomorphic 1-factorizations of  $K_{12}$  shown in Table 19.

	row 1			row 2			row 3			row 4			row 5			row 6																																
$I_{\mathcal{F}}^{\text{P}}(\mathcal{F})$	17	2	1	2	0	0	3	0	17	2	1	2	0	0	3	0	17	2	1	2	0	0	3	0	17	4	1	2	0	0	3	0	17	4	1	2	0	0	3	0								
	row 7			row 8			row 9			row 10			row 11			row 12																																
$I_{\mathcal{F}}^{\text{P}}(\mathcal{F})$	17	4	5	2	1	0	3	0	17	4	5	2	1	0	3	0	19	6	6	2	1	0	5	0	19	6	6	2	1	0	5	0	19	6	6	6	2	0	7	0	19	6	6	6	2	0	7	0

## Appendix B The six undistinguished pairs of non-isomorphic 1-factorizations

Table 27. Undistinguished tuples of non-isomorphic 1-factorizations  $K_{12}$ .

$\mathcal{F}$	$F_1$	$F_2$	$F_3$	$F_4$	$F_5$	$F_6$
$\mathcal{F}_{122}$	123456789ABC	132457689BAC	142358679CAB	1526394A7B8C	16253B4C798A	192A35467C8B
$\mathcal{F}_{123}$	123456789ABC	132457689BAC	142358679CAB	1526394A7B8C	16253B4C798A	192A35467C8B
$\mathcal{F}_{126}$	123456789ABC	132457689BAC	142358679CAB	1526394A7B8C	16253C4B798A	192A35467C8B
$\mathcal{F}_{128}$	123456789ABC	132457689BAC	142358679CAB	1526394A7C8B	16253B4C798A	192A35467B8C
$\mathcal{F}_{129,032}$	123456789ABC	132457689BAC	1423596A7B8C	152637489CAB	16253C4B798A	192A35467C8B
$\mathcal{F}_{129,034}$	123456789ABC	132457689BAC	1423596A7C8B	152637489CAB	16253B4C798A	192A35467B8C
$\mathcal{F}_{129,112}$	123456789ABC	132457689BAC	1423596A7B8C	152637489CAB	16253B4C798A	192A35467C8B
$\mathcal{F}_{129,115}$	123456789ABC	132457689BAC	1423596A7C8B	152637489CAB	16253C4B798A	192A35467B8C
$\mathcal{F}_{129,215}$	123456789ABC	132457689BAC	1423596A7B8C	152637489CAB	16253B4C798A	192A35467C8B
$\mathcal{F}_{129,216}$	123456789ABC	132457689BAC	1423596A7B8C	152637489CAB	16253C4B798A	192A35467C8B
$\mathcal{F}_{197,453,536}$	123456789ABC	132457689BAC	142358679CAB	1529364C7B8A	19253C467A8B	162B354A798C
$\mathcal{F}_{197,453,557}$	123456789ABC	132457689BAC	142358679CAB	1529364C7B8A	19253C467A8B	162B354A798C

$\mathcal{F}$	$F_7$	$F_8$	$F_9$	$F_{10}$	$F_{11}$
$\mathcal{F}_{122}$	1B2C36457A89	17283A495C6B	18273C4B596A	1C2B37485A69	1A2938475B6C
$\mathcal{F}_{123}$	1C2B36457A89	17283A495B6C	18273C4B596A	1B2C37485A69	1A2938475C6B
$\mathcal{F}_{126}$	1C2B36457A89	17283A495B6C	18273B4C596A	1B2C37485A69	1A2938475C6B
$\mathcal{F}_{128}$	1B2C36457A89	17283A495C6B	18273C4B596A	1C2B37485A69	1A2938475B6C
$\mathcal{F}_{129,032}$	1C2B36457A89	1728394A5B6C	18273B4C5A69	1A2938475C6B	1B2C3A495867
$\mathcal{F}_{129,034}$	1C2B36457A89	1728394A5B6C	18273C4B5A69	1A2938475C6B	1B2C3A495867
$\mathcal{F}_{129,112}$	1C2B36457A89	17283A495B6C	18273C4B5A69	1A2938475C6B	1B2C394A5867
$\mathcal{F}_{129,115}$	1B2C36457A89	17283A495C6B	18273B4C5A69	1A2938475B6C	1C2B394A5867
$\mathcal{F}_{129,215}$	1C2B36457A89	17283C4B5A69	18273A495C6B	1A2938475B6C	1B2C394A5867
$\mathcal{F}_{129,216}$	1C2B36457A89	17283B4C5A69	18273A495B6C	1A2938475C6B	1B2C394A5867
$\mathcal{F}_{197,453,536}$	1B263A457C89	172C38495A6B	1C2739485B6A	182A374B5C69	1A283B47596C
$\mathcal{F}_{197,453,557}$	1B263A457C89	172A384B596C	1A273B485C69	182C37495B6A	1C2839475A6B

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Out of the six pairs shown in Table 27,  $\{129, 112; 129, 115\}$  could be distinguished with tricolor vectors and  $\{197, 453, 536; 197, 453, 557\}$  with both tricolor vectors and trains.