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Research Article

Simple-intersection graphs of S-acts*

Xingliang Liang†, **Yujie Wang**‡, **Xiaojie Chen**§

Shaanxi University of Science and Technology, Xi'an, Shaanxi 710021, P.R. China
*lxl_119@126.com
†2391567128@qq.com
‡798863551@qq.com

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Abstract: The intersection graph of an algebraic structure plays a pivotal role in understanding and analyzing algebraic structures—such as groups, rings, modules, acts—by encoding substructural relationships into graph-theoretic frameworks. In this paper, we introduce a new intersection-graph type for an S-act A over a semigroup S, termed the *simple intersection graph* of A, denoted by $GS(A)$. We focus on the relationship between algebraic properties of A and graph-theoretic characteristics of $GS(A)$, including degree, cycles, cliques, connectivity, bipartiteness and dominaning sets. Specifically, we characterize S-acts A for which $GS(A)$ is complete, connected or complete bipartite, and determine key invariants such as degree, girth, diameter, clique number and domination number of $GS(A)$. Applications include solutions to coloring optimization problems and extensions to semigroup-based graphs $GS(S)$.

Keywords: S-act, simple-intersection graph, connectivity, clique, bipartiteness, domination number.

AMS Subject classification: 20M30, 05C15, 05C25, 05C40, 05C69, 16W22

1. Introduction

In graph theory, there is a well-known result that "every simple graph is an intersection graph" [5], and many such intersection graphs are not only interesting in their own right but also serve to provide insight into the structures from which they arise [3, 12, 16]. This leads to a natural and important study of the intersection graph to an algebraic structure. The interdisciplinary study allows us to obtain characterizations

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† *Corresponding Author*

and representations of special classes of algebraic structures in terms of graphs, and vice versa. A pioneering step in this direction was taken by Bosák [4] in 1964. He defined the intersection graph of semigroups. Inspired by his study, Csákány and Pollák [6] defined the intersection graph of a finite group. Then Zelinka [5] continued the work on the graph of finite abelian groups. Along this direction, the intersection graphs for rings, modules and acts are found in [2, 5, 15]. Recently, motivated by the coloring optimization problem, Ahmed and Moh'd [1, 13] developed a new type of intersection graph for rings and modules, known as *the simple-intersection graph* of rings and modules. This development leveraged the structures of simple rings and simple modules. In view of the close connection between acts and modules, this paper will naturally study the simple-intersection graph for acts by using the techniques similar to those in [1].

Let S be a semigroup and A an S-act. In this paper, we introduce and study the simple-intersection graph of A, denoted by $GS(A)$, as a subgraph of the intersection graph of A. Our purpose is to study the connection between the algebraic properties of A and the graph-theoretic properties of $GS(A)$. In Sec. 3, we define the simple-intersection graph $GS(A)$ of A as an undirected simple graph whose vertices are the subacts of A, with two distinct vertices being adjacent if and only if their intersection is a simple subact. First, we determine the degree of each vertex of $GS(A)$ for a semisimple S-act A. Then in Sec. 4, we show that the cycles with length at least 4 have only two special patterns and demonstrate that the girth of $GS(A)$ is either ∞, 3 or 4. Sec. 5 characterizes S-acts such that their simple-intersection graphs $GS(A)$ are connected and complete, and shows that the diameter of $GS(A)$ does not exceed 2. Furthermore, we investigate the cliques and dominating sets of $GS(A)$, and establish formulas computing the clique and domination numbers of $GS(A)$ in Sec. 6 and Sec. 7, respectively. By specializing the A to the semigroup S, we derive analogous results of the simple-intersection graph $GS(S)$ for S.

2. Preliminaries

In this section, we review the concepts and results related to S-acts and graphs that we need for this paper. For background material on S-acts and graphs, the reader can refer to [1, 3, 4, 7–11, 14] and the references contained therein.

Concerning *S*-acts: Let S be a semigroup. A nonempty set A is called a *left S-act* (denoted by ${}_SA$) if there exists a mapping $S \times A \to A$, $(s, a) \mapsto sa$ (a *left S-action*), such that $(ts)a = t(sa)$ for all $a \in A$ and $s, t \in S$. If S is a monoid with the identity 1, $1a = a$ for all $a \in A$. Right S-acts A_S are defined dually, and the categories **Act-S** and **S-Act** of right S-acts and left S-acts naturally are obtained, in which the morphisms are the obvious action-preserving mappings. We note that for a family $\{P_i \mid i \in I\}$ of left S-acts, the coproduct $\coprod_{i\in I} P_i$ in **S-Act** exists, being isomorphic to the disjoint union of the acts P_i, with a suitable action of S.

In this paper, we consider only left S-acts over a semigroup S. The sets of natural numbers, integers, and positive integers are denoted by $\mathbb{N}, \mathbb{Z}$, and $\mathbb{N}^*$, respectively.

A non-empty subset B of an S-act A is called a *subact* of A if $sb \in B$, for all $b \in B$ and $s \in S$. Every semigroup S can be considered as an S-act with the action given by its operation, and so subacts of S are exactly all its ideals. An S-act is said to be *simple* if it contains proper subacts (i.e., no subacts other than itself). Obviously, a monoid is simple if and only if it is a group. Also, an S-act is called *completely reducible* if it is a coproduct of simple subacts.

An S-act A is *semisimple* if it is the coproduct of its simple subacts, each of which is called a *component* of A. We call each simple subact in this decomposition a *component* of A. The *socle* of A, denoted by $Soc(A)$, is defined to be the coproduct of all simple subacts of A. So, $A = Soc(A)$ iff A is a semisimple S-act. An S-act A is called *artinian* if every descending chain of subacts of A is eventually stationary. It is easy to see that A is artinian if and only if every non-empty set of subacts of A contains a minimal element.

Next, we turn to preliminaries from graph theory. By convention, all graphs are undirected and simple.

Concerning graphs: Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. By *order* of G, we mean the number of vertices of G denoted by $|G|$. If B and C are two adjacent vertices of G, then we write $B \leftrightarrow C$. The *degree* of a vertex v in G, denoted by $deg(v)$, is the number of edges incident with v. Let v be a vertex in G. The *neighborhood* $N(v)$ of v is the set of all vertices adjacent to v (i.e., linked to v by an edge). Let u and v be two distinct vertices of G. An *u,v-path* is a path with starting vertex u and ending vertex v, and by $d(u, v)$ we denote the least length of an u, v-path. In particular, if G has no such *u,v-path*, then $d(u, v) = \infty$. The *diameter* of G, i.e. the supremum of the set $\{d(u, v) : \ u, v \in V(G), \ u \neq v\}$, is denoted by $diam(G)$. The *girth* of G, denoted by $g(G)$, is the length of its shortest cycle. A graph with no-cycle has infinite girth.

By a *null graph*, we mean a graph with no edges. A graph G is *connected* if there is a path between every pair of its vertices. A *tree* is a connected graph which does not contain a cycle. A *star graph* is a tree consisting of one vertex adjacent to all the others. A *complete graph* is a graph in which every pair of distinct vertices is adjacent. We denote the complete graph with n distinct vertices by K_n, $n \in \mathbb{N}$. By a *clique* in G, we mean a complete subgraph of G and the number of vertices in a largest clique of G, is called the *clique number* of G and is denoted by $\omega(G)$. A graph G is called *bipartite* if its vertex set can be partitioned into two disjoint subsets U and W such that every edge has one end point in U and the other in W; such a partition (U, W) is called a *bipartition* of G, with U and W as its parts. A *complete bipartite graph* is a bipartite graph where each vertex in one part is adjacent to each vertex in the other part. A complete bipartite graph is denoted by $K_{m,n}$ or $K_{n,m}$, where m is the cardinality of one part and n is the cardinality of the other part.

3. Simple-intersection graphs of S-acts

In this section, we introduce simple-intersection graphs of S-acts over a semigroup S. We first examine the degree of each vertex in these graphs, focusing on semisimple S-acts.

Definition 1. The *simple-intersection graph* of an S-act A, denoted by $GS(A)$, is a simple graph whose vertices are the subacts of A, and two distinct vertices A_1 and A_2 are adjacent if and only if $A_1 \cap A_2$ is a simple subact.

Recall that *the intersection graph* of an S-act A, denoted by $G(A)$, is defined to be the undirected simple graph whose vertices are in a one-to-one correspondence with all nontrivial subacts of A, and two distinct vertices are adjacent if and only if the corresponding subacts of A have a non-empty intersection (see [15]). It is clear that $GS(A)$ must be a subgraph of $G(A)$. Now let's provide concrete examples of simple-intersection graphs for S-acts.

Example 1. Consider the cyclic semigroup S of order n with generating element $s \in S$, where $s^{n+1} = s^n$. Then all distinct ideals of S form the chain:

$$\langle s^n \rangle \subset \langle s^{n-1} \rangle \subset \cdots \subset \langle s^2 \rangle \subset \langle s \rangle,$$

where $\langle s^m \rangle = \{s^i |\ m \le i \le n\}$ for $m = 1, 2, \cdots, n$. So the simple-intersection graph $GS(S)$ is a star graph (see Figure 1).

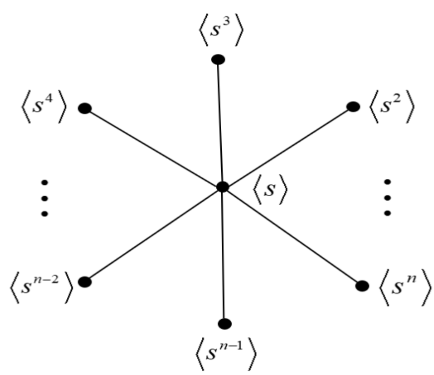

Figure 1. **A simple-intersection graph of a cyclic semigroup**

Example 2. Consider $\mathbb{Z}_p$ (where p is any prime number) as an S-act, where $S = \mathbb{Z}$ or $S = \mathbb{Z}_p$. In either case, $GS(\mathbb{Z}_p)$ is $\mathbb{Z}_p \leftrightarrow p\mathbb{Z}$. In fact, it is a complete graph (see Figure 2).

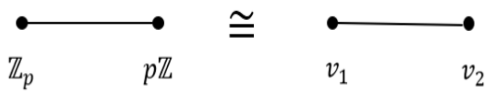

Figure 2. $GS(\mathbb{Z}_p) \cong K_2$

In the following, we give a condition on an S-act A under which the graph $GS(A)$ is null.

Proposition 1. *The graph $GS(A)$ is a null graph if and only if A is a simple S-act or A has no simple subacts.*

Proof. **Necessity.** Suppose that A is not a simple S-act and it contains a simple subact B, but $GS(A)$ is a null graph. Then A and B are adjacent and hence $GS(A)$ is not null, which is the desired contradiction.
Sufficiency. This is obvious. □

The following example illustrates the above theorem.

Example 3. We note that $GS(\Theta)$ for a one-element S-act $\Theta = \{\theta\}$, where S is any semigroup, and $GS(\mathbb{N}^*)$ for the monoid $\mathbb{N}^*$, are null graphs.

Here we investigate the degree of the vertices of $GS(A)$ for a semisimple S-act A which will be used frequently later on.

Remark 1. It is easily seen that the graph $GS(A)$ has an edge $A \leftrightarrow B$ if and only if B is a simple subact of A. So, the subgraph of $GS(A)$ consisting of A and its simple subacts forms a star graph with A as the center. This implies that $deg(A)$ equals the number of simple subacts of A. Consequently, if A is semisimple with n components, then $deg(A) = n$. On the other hand, if B is a simple subact of A and C is a subact of A, then $B \leftrightarrow C$ if and only if $B \subset C$. Furthermore, every pair of distinct simple subacts is not adjacent in $GS(A)$, meaning that the subgraph of $GS(A)$ consisting of simple subacts is a null graph.

Theorem 1. *Let A be a semisimple S-act, and its completely reducible decomposition be $A = \coprod_{i=1}^{n} A_i$. Then*

(1) *$deg(A) = n$.*

(2) *$deg(A_i) = 2^{n-1} - 1$, for $i = 1, 2, \cdots, n$.*

(3) *$deg(B) = m \cdot 2^{n-m}$, where B is a subact of A with m components and $2 \leq m \leq n$.*

Proof. (1). See Remark 1.
(2). Since A is a semisimple S-act, every subact of A is a coproduct of components of A, and so A_i $(i = 1, 2, \cdots, n)$ is adjacent to every subact with at least 2 components containing it. Thus

$$deg(A_i) = |N(A_i)| = C(n-1, 1) + C(n-1, 2) + \cdots + C(n-1, n-1) = 2^{n-1} - 1.$$

Notice that for every $m = 1, 2, \cdots, n-1, C(n-1, m)$ represents the number of subacts with $m+1$ components, one of them is A_i and the rest are different from A_i.
(3). Let B be a subact of A with m components $(2 \leq m \leq n)$. Fix one component of B, and count the number of subacts C of A that are adjacent to B via this component. Since C contains this component and its other components are different from the m components of B, we obtain that, for each $t = 0, 1, \cdots, n-m$, there are $C(n-m, t)$ subacts C with $t+1$ components, one of them is the fixed component of B, while the remaining t components of C are not among the components of B. This implies that the number of the subacts C equals $C(n-m, 0) + C(n-m, 1) + \cdots + C(n-m, n-m) = 2^{n-m}$. Repeating the procedure for each component of J, we see that $deg(B) = m \cdot 2^{n-m}$. □

Remark 2. From the previous theorem it follows that the degree of any component of a semisimple S-act is odd, while the degree of any proper subact of a semisimple S-act with at least two components is even. Therefore, if A is a semisimple S-act with at least two components then $GS(A)$ is not a regular graph. If A is not semisimple, then $GS(A)$ may be regular (see Example 2).

Now we establish the relationship between the simple-intersection graph of semisimple S-acts and the Euler path. For this, we need the following lemma.

Lemma 1 ([1, Theorem 2.14]). *A graph has an Euler circuit if and only if every vertex has an even degree. A graph has an Euler path if and only if exactly two vertices have an odd degree (and all other vertices have even degrees).*

Theorem 2. *Let A be a semisimple S-act with n components. Then*

(1) *If $n \geq 3$, then $GS(A)$ has neither Euler circuits nor Euler paths.*

(2) *If $n = 2$, then $GS(A)$ has only one Euler path of length 2 which connects A and its two components.*

Proof. It is true by Remark 2 and Lemma 1. □

4. Cycles and girth of $GS(A)$

In this section, we investigate the cycles in the simple-intersection graphs $GS(A)$ of an S-act A, and show that the girth of $GS(A)$ is either ∞, 3 or 4.

Lemma 2. *The graph $GS(A)$ has a cycle of length 3 if and only if $GS(A)$ contains at least two adjacent non-simple subacts.*

Proof. **Necessity.** Suppose that $GS(A)$ contains a cycle of length 3-cycle (cycle of length 3), say $B \leftrightarrow C \leftrightarrow D \leftrightarrow B$. By Remark 1, at most one of the vertices is a simple subact. This means that at least two vertices in this cycle are adjacent non-simple subacts of A.

Sufficiency. Let B and C be two distinct non-simple subacts of A, where $B \cap C$ is simple. Then, we obtain the cycle $B \leftrightarrow C \leftrightarrow B \cap C \leftrightarrow B$. □

Corollary 1. *Let A be a semisimple S-act. If A has more than two components, then $g(GS(A)) = 3$. Otherwise, $g(GS(A)) = \infty$.*

Proof. We need to discuss the number of components of A. If A has at most two components, then $GS(A)$ is null or has without cycles, which implies that $g(GS(A)) = \infty$. If A has at least three components, namely B, C, and D, then $B \amalg C$ and $C \amalg D$ are distinct non-simple subacts whose intersection is the simple subact C. Applying Lemma 2, $GS(A)$ has a cycle of length 3, and so $g(GS(A)) = 3$. □

The following theorem states that there are only two kinds of graphs that contain circles according to their girth, and its proof provides a technique for constructing a cycle of minimum length.

Theorem 3. *If the graph $GS(A)$ contains cycles, then either $g(GS(A)) = 3$ or $g(GS(A))$ is even.*

Proof. Suppose that $GS(A)$ has the cycle of minimum length $n > 3$: $A_1 \leftrightarrow A_2 \leftrightarrow \cdots \leftrightarrow A_n \leftrightarrow A_1$. In view of Remark 1 and Lemma 2, if two vertices in the cycle are adjacent, then one of them is simple and the other is non-simple. That is to say, the vertices of the cycle alternating between simple and non-simple subacts. Without loss of generality, assume A_1 is simple. Then A_2 is not simple, A_3 is simple, and so on. Thus, for each $1 \leq k \leq n$, A_k is simple if k is odd, and A_k is non-simple if k is even. If n is odd, then A_n is simple, and so $A_{n+1} = A_1$ is non-simple, which contradicts the assumption that A_1 is simple. Therefore, n must be even. Since the length of the cycle equals n, this implies that $g(GS(A))$ is even. □

The next example demonstrates that the vertices of a cycle with length at least 4 in $GS(A)$ may all be non-simple subacts or may alternate between simple and non-simple subacts. However, from the proof of Theorem 3 we know that the cycle of minimum length must have alternating vertices between simple and non-simple subacts.

Example 4. Let $A = B \amalg C \amalg D \amalg E$ be a semisimple S-act with 4 components. Then,

$$C_1: \; B \leftrightarrow B \amalg C \leftrightarrow C \leftrightarrow B \amalg C \amalg D \leftrightarrow B$$

and

$$C_2: \; B \amalg E \leftrightarrow B \amalg C \leftrightarrow C \amalg D \leftrightarrow D \amalg E \leftrightarrow B \amalg E$$

are two different cycles of length 4. Observe that C_1 is constructed using the method in the proof Theorem 3; its vertices alternate between simple and non-simple subacts. But, C_2 was constructed in a different manner where all its vertices being non-simple subacts. Further, using Corollary 1, C_1 is not a cycle of minimum length. Also, by Theorem 3, C_2 cannot be of minimum length.

The following theorem is an extension of the previous work.

Theorem 4. *The girth of $GS(A)$ equals ∞, 3, or 4.*

Proof. First, an obvious fact is that, if A has no simple subacts or A is a simple S-act, then $g(GS(A)) = \infty$.
Now assume that A contains exactly one simple subact B. If B is a maximal subact, then $GS(A)$ can only be $A \leftrightarrow B$ and so $g(GS(A)) = \infty$. If B is not maximal, then it is contained in proper non-simple subacts of A. In this case, if B is contained in a unique proper non-simple subact C, then the only paths in $GS(A)$ are $A \leftrightarrow B$ and $C \leftrightarrow B$, and hence $g(GS(A)) = \infty$. Other than that, if B is contained in more than one proper non-simple subact, then either there exist two proper non-simple subacts that intersect at B this implies by Lemma 2 that $g(GS(A)) = 3$, or no two proper non-simple subacts intersect at B, which implies that $GS(A)$ is a star graph centered at B and hence $g(GS(A)) = \infty$.
Next, suppose that A has at least two different simple subacts. Let B and C be two different simple subacts. We will consider two cases. In the first case, if $A = B \amalg C$ then $g(GS(A)) = \infty$ by Corollary 1. In the second case, if $A \neq B \amalg C$, then we get the cycle $A \leftrightarrow B \leftrightarrow B \amalg C \leftrightarrow C \leftrightarrow A$ of length 4. But this cycle has the minimum length unless A possesses two different non-simple subacts whose intersection is a simple subact, which yields by Lemma 2 the existence of a cycle of length 3. The proof is complete. □

Example 5. Consider the nilpotent semigroup $S = \{0, s\}$, where 0 is a zero element and $s^2 = 0$. Then S has only one proper ideal $I = \{0\}$. Given the S-act $S^1 \amalg I^2$, the cycle $S^1 \amalg I^2 \leftrightarrow I^1 \leftrightarrow I^1 \amalg I^2 \leftrightarrow I^2 \leftrightarrow S^1 \amalg I^2$ is a shortest cycle in $GS(S^1 \amalg I^2)$. Thus $g(GS(S^1 \amalg I^2)) = 4$.

5. Paths, connectedness and completeness of $GS(A)$

This section is devoted to describing the characteristic paths of $GS(A)$, and characterizing S-acts such that their simple-intersection graphs $GS(A)$ are connected and complete. We started with connectivity.

Lemma 3. *The distance between two simple subacts is* 2.

Proof. Suppose that B and C are two simple subacts of A. Then, a shortest path connecting them is $B \leftrightarrow A \leftrightarrow C$, and so $d(B, C) = 2$. $\square$

Theorem 5. *The graph $GS(A)$ is connected if and only if A is an Artinian S-act.*

Proof. **Necessity.** Assume that $GS(A)$ is connected and A is not simple. Let B be a non-empty subact of A. Then there exists a path from A to B in $GS(A)$. If the length of this path equals 1, then $B \leftrightarrow A$, it follows that B is a simple subact of A. If the length of the path is at least 2, then there exists a subact $C \neq A$ such that $B \leftrightarrow C$. Consequently, $B \cap C$ is a simple subact contained within B. By [9, Proposition 2.1], this implies that A is an Artinian S-act.

Sufficiency. Suppose that A is an Artinian S-act. Let B and C be two non-empty subacts of A. Then, there exist simple subacts B_1 and C_1 such that $B_1 \subseteq B$ and $C_1 \subseteq C$. Therefore, we get the path $B \leftrightarrow B_1 \leftrightarrow B_1 \amalg C_1 \leftrightarrow C_1 \leftrightarrow C$ in $GS(A)$, and hence $GS(A)$ is connected. $\square$

Corollary 2. *If A is an Artinian S-act, and B, C are non-empty subacts of A, then* $1 \leq d(B, C) \leq 4$.

Proof. Suppose that A is an Artinian S-act, and B, C are non-empty subacts of A. According to Theorem 5, $d(B, C) \geq 1$. We have the following three cases to consider.

Case 1. Both B and C are simple. By Lemma 3, we have $d(B, C) = 2$.

Case 2. Only one of B and C is simple, say B. If $B \subset C$, then $B \leftrightarrow C$ and so $d(B, C) = 1$. Otherwise, $B \not\subset C$, then $B \cap C = \emptyset$. Let $\emptyset \neq K \subset C$ be a simple subact. Then, $B \leftrightarrow B \amalg K \leftrightarrow C$ is the shortest path between B and C in $GS(A)$. Thus, $d(B, C) = 2$.

Case 3. Neither B nor C is simple. If $B \cap C$ is simple, then $B \leftrightarrow C$ and so $d(B, C) = 1$. If $B \cap C$ is not simple, then it contains a simple subact K. It follows that $B \leftrightarrow K \leftrightarrow C$ is the shortest path between B and C and hence $d(B, C) = 2$. If $B \cap C = \emptyset$, then $B \leftrightarrow M \leftrightarrow M \amalg N \leftrightarrow N \leftrightarrow C$ is the shortest path between B and C, where M and N are simple subacts of A contained in B and C, respectively. Thus $d(B, C) = 4$ do the job. $\square$

Recall that a *path component* of a graph is a maximal path-connected subgraph (that is, it is a connected subgraph that is not a proper subgraph of any other connected subgraph). A graph may possesss more than one path component. A graph is connected if and only if it has a unique path component. Furthermore, the diameter of a graph equals to the supremum of the diameter of its path components.

Theorem 6. *All vertices in $GS(A)$ with positive degree belong to a path component containing vertex A. The rest of components are isolated vertices.*

Proof. Let B be a non-empty subact of A, with $deg(B) > 0$ in $GS(A)$. If B is simple, then there is a path $B \leftrightarrow A$ of length 1. If B is not simple, then there exists a non-empty subact C of A such that $B \leftrightarrow C$ since $deg(B) > 0$. In this case, we classify C into simple and non-simple cases, where we get a path $B \leftrightarrow C \leftrightarrow A$ when C is simple and a path $B \leftrightarrow B \cap C \leftrightarrow A$ when C is not simple. □

Definition 2. The component containing A (mentioned in Theorem 6) is called the *A-component* of $GS(A)$.

According to Theorem 6, we can, in many situations, identify $GS(A)$ with its A-component, since we can carry the discussion of a path disconnected $GS(A)$ to its A-component.

Corollary 3. *For an S-act A, we have $0 \leq diam(GS(A)) \leq 4$. Moreover, $diam(GS(A)) = 1$ if and only if the A-component of $GS(A)$ is a complete graph consisting of at least two vertices.*

Proof. It readily follows from Corollary 2 and Theorem 6. □

Remark 3. According to Theorem 6, if A is a semisimple but not simple S-act, then $GS(A)$ is connected since $GS(A)$ has no isolated vertex. However, $GS(A)$ is not complete. For instance, if $A = S \coprod S$ and S is a group, then $GS(A)$ is not complete.

Theorem 7. *The graph $GS(A)$ is complete if and only if A has a unique non-empty proper subact. In this case, $GS(A)$ is K_2.*

Proof. **Necessity.** For contradiction, suppose B and C are two distinct non-empty proper subacts of A. If both of them are simple, then B is not adjacent to C. If either B or C is not simple, say B, then A is not adjacent to B. In either case, we get a contradiction to the completeness of $GS(A)$.
Sufficiency. Assume A has a unique non-empty proper subact B. Then $V(GS(A)) = \{A, B\}$. This means that B is simple and thus $B \leftrightarrow A$. Hence $GS(A)$ is complete and $GS(A)$ is K_2. □

We have seen so far that if $GS(A)$ is not a null graph, then $1 \leq diam(GS(A)) \leq 4$. Next, we will further refine the $diam(GS(A))$ in the following theorem.

Theorem 8. *For the graph $GS(A)$, we have $diam(GS(A)) \leq 2$.*

Proof. Without loss of generality, let $GS(A)$ be not a null graph. Suppose that B and C are two non-empty subacts of A, and they are not isolated vertexs in $GS(A)$. If B is adjacent to C, then $d(B, C) = 1$. Otherwise, we will consider three cases.

Case 1. Both B and C are simple. Lemma 3 implies $d(B, C) = 2$.

Case 2. Only one of B and C is simple, say B. Then $B \cap C = \emptyset$; Since C is not isolated, there exists a non-empty subact D of A such that $C \leftrightarrow D$ (so $C \cap D$ is simple). In this case, we can only get $B \cap (C \cap D) = \emptyset$ (because if not, the simplicity of B and $C \cap D$ implies $B = C \cap D$, which yields that B is adjacent to C, a contradiction). Now we have a path $B \leftrightarrow B \coprod (C \cap D) \leftrightarrow C$ which is the shortest path of length 2 connecting B and C in $GS(A)$.

Case 3. Neither B nor C is simple. Then there exist two non-empty subacts D and E of A such that $B \leftrightarrow D$ and $E \leftrightarrow C$. This means that $B \cap D$ and $C \cap E$ are simple subacts. If at least one of $B \cap (C \cap E)$ or $(B \cap D) \cap C$ is not the empty set, we might as well say that $B \cap C \cap E \neq \emptyset$, then from the simplicity of $C \cap E$ it follows that $B \cap (C \cap E) = C \cap E$, so we get $C \cap E \subseteq B$. This implies that there exists a path $B \leftrightarrow (C \cap E) \leftrightarrow C$ of length 2 connecting B and C. If $B \cap (C \cap E) = \emptyset$ and $(B \cap D) \cap C = \emptyset$, here we take $F = (B \cap D) \amalg (C \cap E)$, then $B \cap F = B \cap D$ and $C \cap F = C \cap E$ which are simple subacts. Hence we have the path $B \leftrightarrow F \leftrightarrow C$ of length 2 connecting B and C. In conclusion, we obtain $diam(GS(A)) \leq 2$. □

The following corollary is straightforward result from the proof of Theorem 8.

Corollary 4. *If $GS(A)$ has no isolated vertices, then $GS(A)$ is a connected and $diam(GS(A)) \leq 2$.*

6. The "Bipartite" property of $GS(A)$

The "bipartite" problem is deeply connected to graph coloring. Now we study the conditions under which $GS(A)$ is bipartite. Since the null graph is obviously bipartite, we will only consider the non-null graph $GS(A)$.

Proposition 2. *If $3 < g(GS(A)) < \infty$, then $GS(A)$ is bipartite.*

Proof. Suppose that $3 < g(GS(A)) < \infty$. Then from the proof of Theorem 3, we conclude that no simple subacts are adjacent to each other, and also no non-simple subacts are adjacent to each other. Thus, the graph $GS(A)$ is bipartite with the partitions U consisting of all simple subacts of A, and W consisting of all non-simple subacts of A. $\square$

We note that in Example 5 the graph $GS(S^1 \amalg I^2)$ is bipartite by Proposition 2
For the following discussion, let $GS(A,B)$ denote the subgraph of $GS(A)$ with vertices are $B \cup N(B)$ (where $N(B)$ is the neighborhood of B)and edges incident to these vertices, where B is a subact of an S-act A.

Theorem 9. *$GS(A)$ is bipartite if and only if for every simple subact B of A, the subgraph $GS(A,B)$ is a star-graph with center B.*

Proof. **Necessity.** Assume $GS(A)$ is a bipartite graph, with bipartition (U,W). Let B be a simple subact of A, and suppose $B \in U$. Then no vertex in $N(B)$ is in U (as bipartite graph graphs have no adjacent vertices in the same partition), so all vertices in $N(B) \in W$. It follows at once that all vertices in $N(B)$ are pairwise non-adjacent. Thus $GS(A,B)$ is a star graph with center B.
Sufficiency. Assume that $GS(A,B)$ is a star graph with center B, for every simple subact $B \subset A$. Let U be the set of all simple subacts of A, and W be the set of the remaining (non-simple) subacts of A. From Remark 1 it follows that the vertices in U are not adjacent to each other. On the other hand, let $C, D \in W$. Assume, for contrary, that $C \leftrightarrow D$. Then, $C \cap D$ is a simple subact, and so $C, D \in V(GS(A, C \cap D))$ which contradicts the fact that $GS(A, C \cap D)$ is a star graph with center $C \cap D$. Thus, we conclude that any two vertices in W are not adjacent. Consequently, $GS(A)$ is bipartite with the bipartition (U,W). $\square$

In general, a bipartite graph may have more than one bipartition. However, the existence of the (U,W)-bipartition of $GS(A)$, where U is the set of all simple subacts of A and W is the set of all non-simple subacts of A, is necessary and sufffcient for $GS(A)$ to be bipartite. We would prove this in the next corollary, but before we do that it is worth reminding the reader that if a subact does not include any simple subact, then it is an isolated vertex, which can be freely added to any part of a bipartition of $GS(A)$. Therefore, when considering by partitions, what matters are the simple subacts and the non-simple subacts that contain some simple subact.

Corollary 5. *The graph $GS(A)$ is bipartite if and only if the (U,W)-bipartition of $GS(A)$ exists, where U is the set of all simple subacts of A, and W is the set of all non-simple subacts of A.*

Proof. **Necessity.** Assume that $GS(A)$ is bipartite. Let (X,Y) be a bipartition of $GS(A)$. Since $GS(A)$ is not null, A contains at least one proper subact. Without loss of generality, assume $A \in Y$. By Remark 1, A is adjacent to every simple subact. This implies that all simple subacts must belong to X. Consequently, all non-simple subacts that contain simple subacts must belong to Y. The isolated vertices are distributed randomly between X and Y. By moving all isolated vertices from X to

Y we obtain a new bipartition which is actually the (U, W)-bipartition, where U is X after deleting the isolated vertices, and W is Y plus all the isolated vertices.
Sufficiency. This is trivial. □

Recall that a subact B of an S-act A is *essential* (or *large*) in A [17] if every subact of A has a non-empty intersection with B. Here we explore the relationship between the algebraic properties of an S-act A and the graphical properties of the bipartite graph $GS(A)$.

Theorem 10. *Let $GS(A)$ be bipartite. Then the following statements are equivalent:*

(1) *$GS(A)$ is complete bipartite.*

(2) *The intersection of all non-simple subacts is $Soc(A)$.*

(3) *$Soc(A)$ is an essential subact.*

(4) *A is Artinian and $diam(GS(A)) \leq 2$.*

(5) *$GS(A)$ is connected and $diam(GS(A)) \leq 2$.*

Proof. The proof is straightforward when A contains only one simple subact. Therefore, we assume that $GS(A)$ contains more than one simple subact. By Corollary 5, there exists a (U, W)-bipartition consisting of the set U of simple subacts and the set W consisting of non-simple subacts.
(1) ⇒ (2). Suppose that $GS(A)$ is complete bipartite. Then every non-simple subact is adjacent to every simple subact. Hence every non-simple subact contains the socle of A. Thus, $Soc(A)$ is contained in the intersection of all non-simple subacts. Now, $Soc(A)$ is not simple and hence it includes the intersection of all non-simple subacts. Therefore, we conclude that $Soc(A)$ is equal to the intersection of all non-simple subacts.
(2) ⇒ (3). This is trivial.
(3) ⇒ (4). Assume that $Soc(A)$ is an essential subact of A. Every non-empty family of subacts of A has a minimal element, which implies A is Artinian. We only need to prove that $diam(GS_S(A)) \leq 2$. Applying Lemma 3, the distance between any two simple subacts is 2. Let $B \in U$ and $C \in W$. If $B \subset C$, then $B \leftrightarrow C$, and so $d(B, C) = 1$. Now suppose $B \not\subset C$. Since $Soc(A)$ is an essential subact, $Soc(A) \cap C \neq \emptyset$, this implies that C contains a simple subact $C_1 \neq B$. Therefore, we have a path $B \leftrightarrow B \amalg C_1 \leftrightarrow C$ of length 2 between B and C. Next, assume $B, C \in W$. Obviously, $B \cap C$ is not simple. Again since $Soc(A)$ is an essential subact, the subact $B \cap C$ contains a simple subact C_1. Now, the path $B \leftrightarrow C_1 \leftrightarrow C$ between B and C has a length of 2.
(4) ⇒ (5). Follows from Theorem 5.
(5) ⇒ (1). Assume that $GS(A)$ is connected and $diam(GS(A)) \leq 2$. Let $B \in U$ and $C \in W$. Then there exists a shortest path connecting B and C. If the length of this path is 2, then this contradicts the assumption that $GS(A)$ is bipartite. Therefore, the length of this shortest path is 1, which means that B is adjacent to C, as required. □

As an application of Theorem 10, if $GS(A)$ is complete bipartite, then $GS(A) = K_{m,n} = K_{n,m}$, where $m = |U|$ and $n = |W|$.

7. Cliques of $GS(A)$

In this section, we examine the types of cliques and the clique number of $GS(A)$. First we introduce

Definition 3. Let B be a simple subact of A. We say that two *non-empty subacts* C *and* D *are adjacent through* B if $C \cap D = B$.

Using Remark 1 one obtains the following

Proposition 3. *Each clique in* $GS(A)$ *contains at most one simple subact.*

From the above proposition we learn that there are two types of cliques in $GS(A)$. The first is a clique that does not contain any simple subact, and the other is a clique that contains exactly one simple subact. The following example illustrates both types.

Example 6. Let $A = B \amalg C \amalg D$ be a semisimple S-act. The subgraph with edges $B \amalg C \leftrightarrow C \amalg D \leftrightarrow B \amalg D \leftrightarrow B \amalg C$ is a clique with no simple subact vertices. However, the subgraph $B \amalg C \leftrightarrow B \leftrightarrow B \amalg D \leftrightarrow B \amalg C$ is a clique with one simple subact as stated in Proposition 3 (see Figure 3).

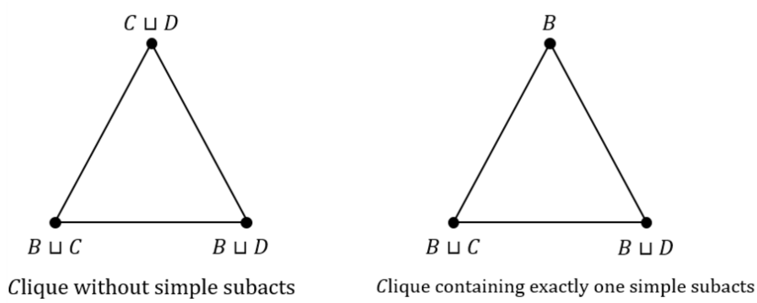

Figure 3. **Two typical types of cliques in** $GS(A)$

Next, we aim to study each type of clique in order to determine the clique number of $GS(A)$. The following theorem asserts that a clique containing a unique simple subact B is a subgraph of $GS(A, B)$ whose vertices are adjacent through B.

Theorem 11. *Let Γ be a clique in $GS(A)$. Then Γ contains exactly one simple subact B if and only if $\Gamma = B \cup S$, where $S \subseteq N(B)$ and any two vertices in S are adjacent through B.*

Proof. Let C and $D \in V(\Gamma)$. Since Γ is a clique, $C \leftrightarrow D$, so $C \cap D$ is simple. By assumption Γ has one exactly one simple subact B, thus $C \cap D = B$. Since $B \leftrightarrow C$ and $B \leftrightarrow D$, we have $C, D \in N(B) \cup \{B\}$ as required. The converse is trivial. □

Corollary 6. *Let Γ be a clique that contains a unique simple subact B. Then $|V(\Gamma)| > 2$ if and only if $A \notin V(\Gamma)$, and $V(\Gamma)$ contains at least two proper non-simple subacts that are adjacent via B.*

Proof. Suppose that $|V(\Gamma)| \geq 3$. Since B is the only simple subact in Γ, and Theorem 11 implies that the remaining vertices are non-simple subacts that are adjacent to each other via B. By Remark 1, none of these non-simple subacts equals A. The converse is obvious. □

Definition 4. Let B be a simple subact of A. A maximal clique induced by B is a maximal clique (not contained in any larger clique) in $GS(A)$ that contains B.

Let B be a simple subact of A. Then the clique $B \leftrightarrow A$ is always a maximal clique induced by B, which we call the *trivial maximal clique induced* by B. It is not difficult to see from Corollary 6 that if $|N(B)| = 1$, then the trivial maximal clique induced by B is the only maximal clique induced by B. However, if $|N(B)| > 1$, then there are another maximal clique induced by B consisting, beside B, of all proper non-simple subacts in $N(B)$ that are adjacent to each other through B. We denote by $\Gamma(B)$ this non-trivial maximal clique induced by B. Notice that $|V(\Gamma(B))| > 2$.

Example 7. Consider $GS(\mathbb{Z}_2)$ where $S = \mathbb{Z}$ or $S = \mathbb{Z}_2$. In both case, the trivial maximal clique $2\mathbb{Z} \leftrightarrow \mathbb{Z}_2$ is the only maximal clique induced by the subact $2\mathbb{Z}$.

Example 8. In Example 6, $\Gamma(B)$ is $B \leftrightarrow B \amalg C \leftrightarrow B \amalg D \leftrightarrow B$. Therefore $|V(\Gamma(B))| = 3$.

In the subsequent work, we will investigate the second type of cliques which does not contain simple subacts as vertices. For this we need to introduce the following

Definition 5. Let X be a non-empty set of simple subacts of A. By a *clique in $GS(A)$ induced by X* we mean a clique such that any two of its vertices are adjacent through a member of X, and for each member of X there exist two vertices of this clique that are adjacent through it.

The following example shows that, the set of all cliques induced by a nonempty set X of simple subacts may be empty, singleton, or contain more than one clique.

Remark 4. In Example 6, the subgraph $\Gamma : B \amalg C \leftrightarrow B \amalg D \leftrightarrow C \amalg D \leftrightarrow B \amalg C$ is the unique clique induced by $X = \{B, C, D\}$. Meanwhile, the subgraphs $\Gamma_1 : B \leftrightarrow A$, $\Gamma_2 : B \amalg C \leftrightarrow B$, and $\Gamma_3 : B \leftrightarrow B \amalg C \leftrightarrow B \amalg D \leftrightarrow B$ are some cliques induced by $X = \{B\}$. However, there is no clique in $GS(A)$ induced by $X = \{B, C\}$.

Next, we demonstrate that if a nonempty set X of simple subacts induces cliques in $GS(A)$, then there exists a maximal clique (denoted by $\Gamma(X)$) induced by X, that is a clique induced by X which is not a subgraph of any other clique induced by X.

Theorem 12. *Let X be a non-empty set of simple subacts which induces cliques in $GS(A)$. There exists a maximal clique induced by X.*

Proof. Let Σ be the set of all cliques in $GS(A)$ induced by X. By assumption, $\Sigma \neq \emptyset$. Let $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$ be a chain in Σ. The union $\bigcup_{n=1}^{\infty} \Gamma_n$ of these subgraphs is in Σ and is an upper bound for the chain. Therefore, Σ has a maximal element by Zorn's Lemma. □

If $X = \{B\}$, where B is a simple subact of A, then $\Gamma(X)$ is either the trivial maximal clique $A \leftrightarrow B$ or $\Gamma(X) = \Gamma(B)$. In general, a maximal clique of $GS(A)$ induced by X is not necessarily unique, this point can be illustrated by Example 6. In this example, there is a unique maximal clique $\Gamma : B \amalg C \leftrightarrow B \amalg D \leftrightarrow C \amalg D \leftrightarrow B \amalg C$ induced by $X = \{B, C, D\}$. However, the subgraphs $\Gamma_1 : B \leftrightarrow A$ and $\Gamma_3 : B \leftrightarrow B \amalg C \leftrightarrow B \amalg D \leftrightarrow B$ are two maximal cliques induced by $X = \{B\}$. Notice that $\Gamma_2 : B \amalg C \leftrightarrow B$ is not a maximal clique induced by $X = \{B\}$ as it is a subgraph of Γ_3.

Theorem 13. *If Γ is a clique of $GS(A)$, then there is a unique nonempty set X of simple subacts of A inducing Γ.*

Proof. Suppose that Γ is a clique of $GS(A)$. Let X be the set of all possible intersections of the vertices of Γ. Then Γ is induced by X. The uniqueness of X follows from Definition 5. □

Remark 5. In $GS(A)$, we have

(1) $\omega(GS(A)) = \sup\{|V(\Gamma(X))| : X \text{ is a non empty set of simple subacts of } A\}$.

(2) If the order of $GS(A)$ is finite, then $2 \leq \omega(GS(A)) \leq |V(GS(A))|$.

8. Domination and Domination number of $GS(A)$

In this section, we consider the dominating sets of $GS(A)$ and we give a formula to compute the domination number of $GS(A)$.

Recall that a *dominating set* $\mathcal{D}$ of a graph G is a nonempty subset of $V(G)$ such that every vertex not in $\mathcal{D}$ is adjacent to at least one vertex in $\mathcal{D}$. The infimum of the set $\{|\mathcal{D}| : \mathcal{D}$ is a dominating set of $G\}$ is called the *domination number* of G, denoted $\gamma(G)$. And recall that a dominating set $\mathcal{D}$ is called *non-shrinkable* if removing a vertex from $\mathcal{D}$ makes $\mathcal{D}$ a non-dominating set (see [1, Definition 3.47]).

Theorem 14. *Let $\mathcal{M} = \{simple\ subacts\ of\ A\} \cup \{non-simple\ subacts\ of\ A\ with\ no\ simple\ subacts\}$. Then $\mathcal{M}$ is a non-shrinkable dominating set of $GS(A)$.*

Proof. If B is a vertex not in $\mathcal{M}$, then B has a simple subact B_1, and so we have $B \leftrightarrow B_1$. Since by assumption $\mathcal{M}$ is a null graph, it follows that $\mathcal{M}$ is non-shrinkable. □

The above theorem implies directly the following

Corollary 7. *Let $\mathcal{M}$ be the set consisting of all simple subacts of A and all non-simple subacts empty of simple subacts. Then*

(1) $\gamma(GS(A)) \leq |\mathcal{M}|$.

(2) *If $GS(A)$ is connected, then $\mathcal{M}$ consists solely of simple subacts of A.*

Notice that $\mathcal{M}$ is not the only type of dominating sets of $GS(A)$. However, we shall show, in the next work, that every dominating set can be reconstructed to a non-shrinkable set whose vertices are the isolated vertices of $GS(A)$ plus some semisimple subacts. The following notation will be useful in future work.

Notation 1. Let $\mathcal{D}$ be a dominating set of $GS(A)$. Denote by $\tilde{\mathcal{D}}$ the set consisting of the following vertices:

(i) all non-simple subacts empty of simple subacts,

(ii) B such that $B = Soc(B)$ for each $B \in \mathcal{D}$,

(iii) B such that $B \neq Soc(B)$ and $Soc(B) \in \mathcal{D}$ for each $B \in \mathcal{D}$, and

(iv) $Soc(B)$ such that $B \neq Soc(B)$ and $Soc(B) \notin \mathcal{D}$ for each $B \in \mathcal{D}$.

Lemma 4. *If $\mathcal{D}$ is a dominating set of $GS(A)$, then $\tilde{\mathcal{D}}$ is a dominating set. Moreover, $|\tilde{\mathcal{D}}| \leq |\mathcal{D}|$.*

Proof. Assume $\mathcal{D}$ is a dominating set of $GS(A)$. Then $\mathcal{D}$ contains all non-simple subacts empty of simple subacts. Let B be a vertex of $GS(A)$. Without loss of generality, assume B is not a non-simple subact empty of simple subacts. Consider the two possible cases:

Case 1. $B \notin \mathcal{D}$. Then we have $B \leftrightarrow C$ for some vertex $C \in \mathcal{D}$ since $\mathcal{D}$ is a dominating set. Thus $B \leftrightarrow Soc(C)$.

Case 2. $B \in \mathcal{D}$. If $B = Soc(B)$, then by Notation 1(ii) we have $B \in \tilde{\mathcal{D}}$. Otherwise we have two subcases. In the first subcase, assume $B \neq Soc(B)$ and $Soc(B) \in \mathcal{D}$.

By Notation 1(iii) we have $B \in \tilde{\mathcal{D}}$. In the second subcase, assume $B \neq Soc(B)$ and $Soc(B) \notin \mathcal{D}$. Then in the subcase when $Soc(B)$ is simple, this implies that $B \leftrightarrow Soc(B)$. In this subcase when $Soc(B)$ is not simple, using the fact that $\mathcal{D}$ is a dominating set, there exists $D \in \mathcal{D}$ such that $Soc(B) \leftrightarrow D$ which implies $Soc(B) \leftrightarrow Soc(D)$ which in turn implies $B \leftrightarrow Soc(D)$. From the discussion above, we conclude that $\tilde{\mathcal{D}}$ is a dominating set. Further, define the function $f:\ \mathcal{D} \to \tilde{\mathcal{D}}$ by

$$f(I) = \begin{cases} B, & \text{if } B = Soc(B), \\ B, & \text{if } B \neq Soc(B) \text{ and } Soc(B) \in \mathcal{D}, \\ Soc(B), & \text{if } B \neq Soc(B) \text{ and } Soc(B) \notin \mathcal{D}, \\ B, & \text{if } B \text{ is a non-simple subact empty of simple subacts.} \end{cases}$$

Then f is an onto function. Therefore, $|\tilde{\mathcal{D}}| \leq |\mathcal{D}|$. □

Lemma 5. *Suppose that A is not a simple S-act and $\mathcal{D}$ a non-shrinkable dominating set of $GS(A)$. Then A and $Soc(A)$ cannot be inside $\mathcal{D}$ at once. Moreover, if $A \notin \mathcal{D}$, then $\mathcal{D}$ contains a simple subact.*

Proof. Assume that $A \neq Soc(A)$, and both A and $Soc(A)$ are inside $\mathcal{D}$. Then obviously $\mathcal{D}$ is shrinkable, which is a contradiction. The rest of the proof follows from the fact that A is only adjacent to simple subacts of A. □

Notice that $\tilde{\mathcal{D}}$ may be shrinkable as indicated in the following example.

Example 9. Let $A = B \amalg C$ be a semisimple S-act. Consider the dominating sets $\mathcal{D}_1 = \{B, A\}$ and $\mathcal{D}_2 = \{B, C\}$. We have $\tilde{\mathcal{D}_1} = \mathcal{D}_1$ and $\tilde{\mathcal{D}_2} = \mathcal{D}_2$. Also, $\mathcal{D}_1$ is shrinkable to $\{A\}$ while $\mathcal{D}_2$ is not shrinkable.

Remark 6. In $GS(A)$, if $B \leftrightarrow C$, then $B \leftrightarrow Soc(C)$, $Soc(B) \leftrightarrow C$, and $Soc(B) \leftrightarrow Soc(C)$. Conversely, If $Soc(B) \leftrightarrow Soc(C)$, then $B \leftrightarrow Soc(C)$ and $Soc(B) \leftrightarrow C$, but it is not necessary that $B \leftrightarrow C$.

In the next theorem, we show that any dominating set of $GS(A)$ can be replaced by a non-shrinkable dominating set consisting of semisimple subacts of A.

Theorem 15. *Let $\mathcal{D}$ be a dominating set of $GS(A)$. Then there exists a non-shrinkable dominating set $\mathcal{Y}$ satisfying $|\mathcal{Y}| \leq |\mathcal{D}|$, where $\mathcal{Y}$ consists of semisimple subacts of A and the isolated vertices of $GS(A)$. Moreover, if $A \notin \mathcal{Y}$, then $\mathcal{Y}$ has to contain at least one simple subact.*

Proof. By Lemma 4, we can replace $\mathcal{D}$ with $\tilde{\mathcal{D}}$. Then $|\tilde{\mathcal{D}}| \leq |\mathcal{D}|$, and all elements of $\tilde{\mathcal{D}}$ are semisimple subacts or the isolated vertices except possibly vertices $B \in \mathcal{D}$ where $B \neq Soc(B)$ and $Soc(B) \in \mathcal{D}$. In view of Notation 1, such the vertice B and $Soc(B)$

are elements in $\tilde{\mathcal{D}}$. Now, we replace such a B with any simple subact (we denote it by T_B) contained in B. We call the new set $\mathcal{D}'$. It's easy to see that $\mathcal{D}'$ consists of only semisimple subacts and $|\mathcal{D}'| \leq |\tilde{\mathcal{D}}| \leq |\mathcal{D}|$. Next to prove that $\mathcal{D}'$ is a dominating set, let C be a vertex not in $\mathcal{D}'$. If $C \notin \tilde{\mathcal{D}}$, then C is adjacent to some vertex in $\tilde{\mathcal{D}}$. We need only to worry about the case when $C \leftrightarrow B$, where $B \neq Soc(B)$ and both B, $Soc(B) \in \mathcal{D}$. Applying Remark 6, we have $C \leftrightarrow Soc(B)$ and $Soc(B) \in \mathcal{D}'$. If $C \in \tilde{\mathcal{D}}$, then in this case we need only to worry about the case when $C = B$ where $B \neq Soc(B)$ and both B, $Soc(B) \in \mathcal{D}$. By the definition of $\mathcal{D}'$, we get that $B \leftrightarrow T_B$ and $T_B \in \mathcal{D}'$. At the end, we obtain that $\mathcal{D}'$ is a dominating set. Since $\mathcal{D}'$ may be shrinkable, then $\mathcal{D}'$ contains an non-shrinkable dominating set $\mathcal{Y}$. The rest follows from Lemma 5. □

Since A is adjacent only to simple subacts, the non-shrinkable dominating set $\mathcal{Y}$ mentioned in the previous theorem contains either A or at least a simple subact of A. Thus, we have the following corollary.

Corollary 8. *If A is semisimple, then*

$$\gamma(GS(A)) = \inf\{|\mathcal{Y}| : \mathcal{Y} \textit{ is a non-shrinkable dominating set consisting of the isolated vertices and semisimple subacts of } A \textit{ such that } A \in \mathcal{Y} \textit{ or } \mathcal{Y} \textit{ contains at least one simple subact}\}.$$

If A is not semisimple, then

$$\gamma(GS(A)) = \inf\{|\mathcal{Y}| : \mathcal{Y} \textit{ is a non-shrinkable dominating set consisting of the isolated vertices and semisimple subacts of } A \textit{ and at least one simple subact}\}.$$

Example 10. Let $A = B \amalg C$ be a semisimple S-act. Then $\mathcal{Y} = \{A\}$ is a non-shrinkable dominating set of $GS(A)$ with the least cardinality. Thus, $\gamma(GS(A)) = 1$.

Example 11. Let $A = B \amalg C \amalg D$ be a semisimple S-act. Then $\mathcal{Y} = \{B \amalg C, D\}$ is an non-shrinkable dominating set of $GS(A)$ with the least cardinality. Thus, $\gamma(GS_S(A)) = 2$.

Taking note of the above two examples, we can't help but ask the question: *If A is a semisimple S-act with n components where $n > 1$, does $\gamma(GS_S(A))$ equal $n - 1$?* At the end of this paper, we will answer this question.

Lemma 6. *Let A be a semisimple S-act with n components where $n > 1$. Then $\gamma(GS(A)) \leq n - 1$.*

Proof. Suppose that $A = A_1 \amalg A_2 \amalg \cdots \amalg A_n$ is a semisimple S-act. If $n = 2$, then $\mathcal{D} = \{A\}$ is a non-shrinkable dominating set of $GS(A)$ with the least cardinality, and so $\gamma(GS_S(A)) = 1$. If $n > 2$, let $\mathcal{D} = \{A_1, A_2, \cdots, A_{n-2}, A_{n-1} \amalg A_n\}$. Assume B is a vertex out of $\mathcal{D}$. We consider two cases. In the first case, if $B = A$ or B includes a component A_j for $1 \leq j \leq n-2$, then B is adjacent to $A_j \in \mathcal{D}$. In the second case, if $B \neq A$ and B does not contain any of the components A_j, where $j = 1, 2, \cdots, n-2$, then B can only be one of A_{n-1} and A_n. These two possibilities imply that B is adjacent to $A_{n-1} \amalg A_n$ which is a vertex in $\mathcal{D}$. Therefore, we obtain that $\mathcal{D}$ is a non-shrinkable dominating set of $GS(A)$ with cardinality equal to $n-1$. Consequently, $\gamma(GS(A)) \leq n-1$. □

Theorem 16. *Let A be a semisimple S-act with n components where $n > 1$. Then $\gamma(GS(A)) = n-1$.*

Proof. We shall do this by induction on n. For $n = 2$, $\{A\}$ is a non-shrinkable dominating set of $GS(A)$ with the least cardinality. So $\gamma(GS(A)) = 1$, as desired.
Assume for n components our statement is true.
Now let A be a semisimple S-act with $n+1$ components. Using Lemma 6, we have $\gamma(GS(A)) \leq n$. Suppose that $\gamma(GS(A)) < n$. Then there exists a non-shrinkable dominating set $\mathcal{D}$ such that $|\mathcal{D}| = \gamma(GS(A)) < n$. By Lemma 4, we can assume, without lose of generality, that $\mathcal{D}$ contains a simple subact A_1 of A. Let B be a vertex outside $\mathcal{D}$ and $A_1 \not\subset B$. Then B is not adjacent to A_1. But there exists a vertex $C \in \mathcal{D}$ such that B is adjacent to C through a simple subact $A_2 \neq A_1$ since $\mathcal{D}$ is non-shrinkable. Further, let $\mathcal{D}'$ be the same set as $\mathcal{D}$ but with the component A_1 is removed from each vertex of $\mathcal{D}$ containing it. Then we have $|\mathcal{D}'| < |\mathcal{D}| < n$ which implies $|\mathcal{D}'| \leq n-2$. Moreover, $\mathcal{D}'$ is a dominating set for $GS(A')$ where A' is the semisimple S-act with n components which has the same components of A except A_1. Since $|\mathcal{D}'| < n-1$, we obtain $\gamma(GS(A')) < n-1$ which is a contradiction to the induction hypothesis. Thus, we obtain that $\gamma(GS(A)) = n$. □

Remark 7. Theorem 16 solves the following coloring optimization problem: Given n distinct colors (where n is an even natural number) and the ability to select $1, 2, \cdots,$ or n colors without replacement and without order (resulting in $2^n - 1$ possibilities), what is the minimum number of possibilities such that any other possibility that is not among the ones chosen has one color in common with at least one of the ones chosen? The answer is $n-1$ possibilities. Moreover, this set of possibilities contains $n-2$ single colors in addition to the pair of the two remaining colors.

In conclusion, the simple-intersection graph $GS(A)$ stands out as a natural and significant type of intersection graphs for an S-act A. This paper has presented several results (e.g., Theorems 5, 9, 11 and 15) that highlight the impact of simplicity on the structure of A. From an algebraic perspective, an S-act over a semigroup S serves as a nonadditive generalization of a R-module over a ring R. Consequently, this paper extends the work of [1]. Additionally, the proof of Theorem 8 allows us to enhance [1, Theorem 3.23]. Specifically, for the simple-intersection graph $GS_R(M)$ of a left module M, we establish that $diam(GS_R(M)) \leq 2$. Furthermore, since a semigroup S

can be regarded as an S-act through its own operation, setting A as S in the current paper yields corresponding results for the semigroup S.

Statements and Declarations

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