Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-13
https://doi.org/10.22049/cco.2025.30729.2595

CCO
Commun. Comb. Optim.

Research Article

Eternal domination stability in graphs

P. Roushini Leely Pushpam

Department of Mathematics, D.B. Jain College, Chennai 600 097, Tamil Nadu, India
roushinip@yahoo.com

Received: 18 June 2025; Accepted: 1 November 2025
Published Online: 7 November 2025

Abstract: The concept of domination stability in graphs was introduced in 1983 by Bauer, Harary, Nieminen and Suffel and has been further studied by Nader Jafari Rad, Elahe Sharifi, Marcin Krzywkowski. The γ^+-stability of G, denoted by $\gamma^+(G)$, is the minimum number of vertices whose removal from G increases the domination number. The γ^--*stability* of G, denoted by $\gamma^-(G)$, is the minimum number of vertices whose removal from G decreases the domination number. The *domination stability* of G, denoted by $st_\gamma(G)$, is the minimum number of vertices whose removal changes the domination number. In this paper the concept of domination stability is extended to m-eternal domination. *Eternal domination* of a graph requires the vertices of a graph to be protected, against infinitely long sequences of attacks, by guards located at vertices (at most one guard in each vertex), and a guard must move from a neighboring vertex to an attacked vertex with the requirement that the configuration of guards induces a dominating set at all times. Two models of the problem, one in which only one guard moves at a time and one in which more than one guard may move simultaneously are studied in the literature. The model of eternal domination in which more than one guard move simultaneously is called the m-eternal domination. The m-eternal domination number, $\gamma_m^\infty(G)$ of a graph G is the minimum number of guards needed to defend G against any such sequence of attacks.

Keywords: domination, eternal domination, stability.

AMS Subject classification: 05C69

1. Introduction

By a graph $G = (V, E)$ we mean a fininite, connected, undirected and a simple graph. The order of G is denoted by n. For graph theoretic terminology we in general follow [10]. One of the fastest growing areas within graph theory is the study of domination and related problems. A set $S \subseteq V$ is called a *dominating set* of G if every vertex in $V - S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set in G is called the *domination number* of G and is denoted by $\gamma(G)$. A comprehensive introduction to domination in graphs, has been given in the book by Haynes et al.

[11]. A dominating set S is a *connected dominating set* if the subgraph $G[S]$ induced by S is connected. The minimum cardinality amongst all connected dominating sets is the *connected domination number* $\gamma_c(G)$.

The concept of domination stablity in graphs was introduced in 1983 by Bauer et al. [2] and further has been studied by Nader Jafari Rad et al. [12]. The γ^--*stability* of G, denoted by $\gamma^-(G)$, is the minimum number of vertices whose removal decreases the domination number. The γ^+-*stability* of G, denoted by $\gamma^+(G)$, is the minimum number of vertices whose removal increases the domination number. The *domination stability* of G, denoted by $st_\gamma(G)$, is the minimum number of vertices whose removal changes the domination number. Thus, $st_\gamma(G) = \min\{\gamma^-(G), \gamma^+(G)\}$. Much work has been done in domination stability and related parameters in [2, 3, 7, 12, 19, 20].

In *eternal domination*, mobile guards on the vertices of a graph are used to defend the graph against an infinite sequence of attacks on vertices. A guard must move from a neighboring vertex to an attacked vertex (we assume attacks happen only at vertices containing no guard and that each vertex contains at most one guard) and the configuration of guards induce a dominating set at all times. Two models of the problem, one in which only one guard moves at a time and one in which more than one guard may move simultaneously are studied in the literature. The model of eternal domination in which more than one guard move simultaneously is called the *m-eternal domination*. The *m-eternal domination number* $\gamma_m^\infty(G)$, of a graph G is the minimum number of guards needed to defend G against any such sequence of attacks, where more than one guard is allowed to move in response to an attack. One can refer to [1, 4–6, 8, 9, 13–18] for an elaborate study of eternal domination.

In this paper the concept of domination stability is extended to m-eternal domination as follows: The ${\gamma_m^\infty}^-$ *stability* of G, denoted by $st^-_{\gamma_m^\infty}(G)$, is defined to be the minimum number of vertices whose removal decreases the m-eternal domination number. The ${\gamma_m^\infty}^+$-*stability* of G, denoted by $st^+_{\gamma_m^\infty}(G)$, is defined to be the minimum number of vertices whose removal increases the m-eternal domination number. If no such set of vertices exists, whose removal increases the eternal domination number, we define $st^+_{\gamma_m^\infty}(G) = \infty$. As a trivial example, $st^-_{\gamma_m^\infty}(P_5) = 1$ and $st^+_{\gamma_m^\infty}(P_5) = \infty$. The *m-eternal domination stability of* G, denoted by $st_{\gamma_m^\infty}(G)$, is the minimum number of vertices whose removal changes the m-eternal domination number of G. Thus, $st_{\gamma_m^\infty}(G) = \min\{st^-_{\gamma_m^\infty}(G), st^+_{\gamma_m^\infty}(G)\}$.

2. Notation

The *degree* of a vertex v in a graph G is the number of edges of G incident with v and it is denoted by $deg(v)$. The minimum degree of G is the minimum degree among the vertices of G and it is denoted by $\delta(G)$. The maximum degree of G is the maximum degree among the vertices of G and it is denoted by $\Delta(G)$. A vertex of degree zero in G is called an *isolated* vertex. A vertex of degree one in G is called a *leaf vertex*. A *path* in a graph G, is an alternating sequence of vertices and edges beginning and ending with vertices, such that all the vertices are distinct. A path on n vertices is denoted by P_n. A *cycle* on n vertices is denoted by C_n. A *wheel* graph is a graph in which one vertex is adjacent to every vertex on a cycle and a wheel on n vertices is denoted by W_n. A graph G is *connected* if every pair of vertices are joined by a path. A *subgraph* H of a graph G, is a graph where its vertex set and edge set are subsets of the original graph G. A *spanning subgraph* is a subgraph containing all the vertices of G. A connected acyclic graph is called a *tree.* A *forest* is a disjoint union of trees. A *rooted tree* is a tree in which one vertex has been designated the root. Every node in a rooted tree (except the root itself) has a unique parent. Nodes that are children, or children-of-children, or further down the tree are *descendants.* An edge incident with a leaf vertex is called a *pendant edge.* A *support vertex* is a vertex adjacent to a leaf vertex. A support vertex is called a *strong support* if it is adjacent to at least two leaf vertices and a support vertex is called a weak support if it is adjacent to exactly one leaf. An *end support* is a support vertex adjacent to exactly one non leaf vertex. A vertex v in G is said to be a *junction vertex* if $deg(v) \geq 3$. An (x, y)-path in a graph is called a *pendant path* if x is a junction vertex and y is a leaf vertex and the internal vertices of the path are of degree 2. An (x, y)-path is called a *non-pendant path* if both x and y are junction vertices and the internal vertices of the path are of degree 2. A *maximal pendant path* in a graph G is a pendant path P in G that is not contained in a longer pendant path.

A *complete graph* on n vertices is a graph in which every pair of vertices are adjacent and is denoted by K_n. A subset S of V is called an *independent set* of G if no two vertices of S are adjacent in G. The maximum cardinality of an independent set is called the *independence number* of G and is denoted by $\beta(G)$. A maximum independent set is called a *β-set* of G. A *k-partite graph* is a graph whose vertices are partitioned into k different independent sets. A *complete k-partite graph* is a k-partite graph in which there is an edge between every pair of vertices from different independent sets. A *complete multipartite graph* is a graph that is complete k-partite for some k and it is denoted by $K_{r_1,r_2,\ldots,r_k}$, where $r_1, r_2, \ldots, r_k$ are the sizes of the k-independent sets respectively. A *clique* of a graph is an induced subgraph of the graph that is a clique.

The *distance* between two points u and v in G is the length of a shortest path joining them. It is denoted by $d(u, v)$. The greatest distance between any two vertices in a graph G is called the *diameter* of G and it is denoted by $diam(G)$. A *diametral path* of a graph is a shortest path between a pair of vertices whose length is equal to the diameter of the graph. In a graph $G = (V, E)$, the *open neighbourhood* of a vertex $v \in V$ is $N(v) = \{u \in V : uv \in E\}$. The *closed neighbourhood* of a vertex V is $N[v] = N(v) \cup \{v\}$.

3. Bounds

Theorem 1. *For any graph G, $st_{\gamma_m^\infty}(G) \leq \delta(G) + 1$.*

Proof. Let v be a vertex of degree $\delta(G)$. Remove all the vertices in $N(v)$ to obtain

G'. If $\gamma_m^\infty(G') \neq \gamma_m(G)$, then $st_{\gamma_m^\infty}(G) \leq \delta(G)$.
If $\gamma_m^\infty(G') = \gamma_m(G)$, then remove v from G' to obtain G''. Clearly

$$\begin{aligned}\gamma_m^\infty(G'') &= \gamma_m(G') - 1\\ &< \gamma_m(G)\end{aligned}$$

Thus, $st_{\gamma_m^\infty}(G) \leq \delta(G) + 1$. □

It is clear that for complete graphs K_n, $st_{\gamma_m^\infty}(K_n) = n$. In the following theorem we establish an upper bound for $st_{\gamma_m^\infty}(G)$ for non complete graphs G in terms of the order of the graph.

Theorem 2. *For any non complete graph G, $st_{\gamma_m^\infty}(G) \leq n - 2$.*

Proof. For any non complete graph G, $\gamma_m^\infty(G) \geq 2$. Let H be a maximum clique in G. Remove all the vertices in $V(G)\backslash V(H)$ from G. We observe that $\gamma_m^\infty(H) = 1$. Thus $st_{\gamma_m^\infty}(G) \leq n - |V(H)|$ and $|V(H)| \geq 2$. Thus $st_{\gamma_m^\infty}(G) \leq n - 2$. □

In the next theorem we characterize graphs G with $st_{\gamma_m^\infty}(G) = n - 2$.

Theorem 3. *For any graph G, $st_{\gamma_m^\infty}(G) = n - 2$ if and only if $\beta(G) = 2$ and G does not have triangles.*

Proof. Suppose that $st_{\gamma_m^\infty}(G) = n-2$. First,, we claim that $\gamma_m^\infty(G) = 2$. If $\gamma_m^\infty(G) = 1$, then G is complete and $st_{\gamma_m^\infty}(G) = n$. Hence $\gamma_m^\infty(G) \geq 2$. Suppose that $\gamma_m^\infty(G) \geq 3$. Consider two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ (the possibility that the edges are adjacent is not ruled out). Remove all the vertices in $V(G)\backslash\{u_1, u_2, v_1, v_2\}$ from G to obtain G' so that $\gamma_m^\infty(G') \leq 2$. Thus, $st_{\gamma_m^\infty}(G) \leq n - 3$, a contradiction. Hence, $\gamma_m^\infty(G) = 2$. Next, we claim that $\beta(G) = 2$. Let S be a β-set of G. If $\beta(G) = 1$, then G is complete and $st_{\gamma_m^\infty}(G) = n$, a contradiction. If $\beta(G) \geq 3$, then remove all the vertices in $V(G)\backslash S$ from G to obtain a graph G' so that $\gamma_m^\infty(G') = \beta(G') \geq 3$. Thus, $st_{\gamma_m^\infty}(G) \leq n - 3$, a contradiction. Hence, $\beta(G) = 2$.

Suppose G has a triangle induced by u_1, u_2, u_3. Then removing all the vertices of G except u_1, u_2, u_3, we see that $st_{\gamma_m^\infty}(G) \leq n - 3$, a contradiction. Thus, G does not contain a triangle.

Conversely, suppose that $\beta(G) = 2$ and G does not have a triangle. We claim that $st_{\gamma_m^\infty}(G) = n - 2$. Since $\beta(G) = 2$, $\gamma_m^\infty(G) \geq 2$. Further, in [9], it has been proved that $\gamma_m^\infty(G) \leq \beta(G) = 2$. Thus, $\gamma_m^\infty(G) = 2$. Removing any number of vertices from G, will leave a graph with independence number 2. Hence, there is no possibility that $\gamma_m^\infty(G)$ will increase. Thus, to reduce $\gamma_m^\infty(G)$ by 1, one has to remove $n - 2$ vertices from G which leaves a graph with exactly one edge so that γ_m^∞ of the resulting graph is 1. Thus $st_{\gamma_m^\infty}(G) = n - 2$. □

4. Some Standard Graphs

Wayne Goddard et al. [9] have proved that for Paths P_n and Cycles C_n, $\gamma_m^\infty(P_n) = \lceil n/2 \rceil$ and $\gamma_m^\infty(C_n) = \lceil n/3 \rceil$. It is clear that for wheels W_n, $\gamma_m^\infty(W_n) = 2$. It is also clear that the removal of any set of vertices from a path P_n will not increase $\gamma_m^\infty(P_n)$. Hence $st^+_{\gamma_m^\infty}(P_n) = \infty$.

Theorem 4. *For paths P_n, $n \geq 3$, $st^-_{\gamma_m^\infty}(P_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even.} \end{cases}$*

Proof. By Theorem 1, $st^-_{\gamma_m^\infty}(P_n) \leq 2$. Let $P_n = (v_1, v_2, \ldots, v_n)$.

Case (i): n is odd.
Clearly $st^-_{\gamma_m^\infty}(P_3) = 1$.
For $n \geq 5$, removing v_5 from P_n, we see that

$$\begin{aligned} \gamma_m^\infty(P_n - v_5) &= \gamma_m^\infty(P_4 \cup P_{n-5}) \\ &= 2 + \frac{n-5}{2} = \frac{n-1}{2} < \frac{n}{2}. \end{aligned}$$

Thus $\gamma_m^\infty(P_n - v_5) < \gamma_m^\infty(P_n)$.
Thus, $st^-_{\gamma_m^\infty}(P_n) = 1$.
Case (ii): n is even, $n \geq 4$.
For $1 \leq i \leq n$,

$$\gamma_m^\infty(P_n - v_i) = \gamma_m^\infty(P_{i-1}) + \gamma_m^\infty(P_{n-i}) = \left\lceil \frac{i-1}{2} \right\rceil + \left\lceil \frac{n-i}{2} \right\rceil = \frac{n}{2} = \gamma_m^\infty(P_n).$$

Hence, $st^-_{\gamma_m^\infty}(P_n) \geq 2$.

Removing v_1 and v_2 from v_n, we see that

$$\gamma_m^\infty(P_n - \{v_1, v_2\}) = \left\lceil \frac{n-2}{2} \right\rceil < \frac{n}{2}.$$

Thus, $st^-_{\gamma_m^\infty}(P_n) = 2$. □

By Theorem 4 and the fact that $st^+_{\gamma_m^\infty}(P_n) = \infty$, we have the following theorem.

Theorem 5. *For paths P_n, $n \geq 3$, $st_{\gamma_m^\infty}(P_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$.*

Theorem 6. *For cycles C_n, $n \geq 3$,* $st^{+}_{\gamma_m^\infty}(C_n) = \begin{cases} \infty & \text{if } n = 3,4,5,7 \\ 1 & \text{otherwise} \end{cases}$.

Proof. One can easily verify that $st^{+}_{\gamma_m^\infty}(C_n) = \infty$, for $n = 3,4,5,7$.
For $n = 6$ and $n \geq 8$,

$$\gamma_m^\infty(C_n - v) = \gamma_m^\infty(P_{n-1}) = \left\lceil \frac{n-1}{2} \right\rceil > \left\lceil \frac{n}{3} \right\rceil.$$

Thus $\gamma_m^\infty(C_n - v) > \gamma_m^\infty(C_n)$, where v is a vertex in C_n. Thus, $st^{+}_{\gamma_m^\infty}(C_n) = 1$. □

Theorem 7. *For cycles $G = C_n$, $n \geq 5$,*

$$st^{-}_{\gamma_m^\infty}(C_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 2 & \text{if } n \equiv 0 \ (mod\ 3) \\ \lceil \frac{n}{3} \rceil + 1 & \text{if } n \equiv 2 \ (mod\ 3) \\ \lceil \frac{n}{3} \rceil & \text{if } n \equiv 1 \ (mod\ 3). \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Suppose that $n \equiv 0 \pmod 3$. Then $n = 3k$ for some $k \geq 2$. In this case $\gamma_m^\infty(G) = k$. Let S be a set of vertices of G such that $\gamma_m^\infty(G - S) \leq k - 1$. Let D be a γ_m^∞-set of $G - S$ and so $|D| = \gamma_m^\infty(G - S) \leq k - 1$. Since each component of the subgraph induced by D is a path of order at least 2 and $\gamma_m^\infty(P_n) = \lceil \frac{n}{2} \rceil$, one can easily observe that, $G - S$ contains at most $2|D|$ vertices. Also order of $G - S$ is $3k - |S|$. Therefore,

$$3k - |S| \leq |D| + |D| \leq 2(k-1)$$

which implies that $|S| \geq k + 2$. This is true for every subset S of $V(G)$. Conversely, if we take $S^* = \{v_1, v_2, \ldots, v_{k+2}\}$, then $\gamma_m^\infty(G - S^*) = \gamma_m^\infty(P_{3k-k-2}) = \gamma_m^\infty(P_{2k-2}) = k - 1$. Hence, $st^{-}_{\gamma_m^\infty}(C_n) \leq |S^*| = k + 2$. Thus, $st^{-}_{\gamma_m^\infty}(C_n) = k + 2$.
Suppose that $n \equiv 1 \pmod 3$. Then $n = 3k + 1$ for some $k \geq 1$. In this case $\gamma_m^\infty(G) = k+1$. Let S be a set of vertices of G such that $\gamma_m^\infty(G - S) \leq k$. Proceeding analogously as in the previous case, we deduce that $|S| \geq k + 1$ and thus $st^{-}_{\gamma_m^\infty}(G) \geq k + 1$. Since $\gamma_m^\infty(P_n) = \lceil \frac{n}{2} \rceil$, one can easily observe that $G - S$ contains at most $2|D|$ vertices.
Conversely if $S^* = \{v_1, v_2, \ldots, v_{k+1}\}$, then

$$\gamma_m^\infty(G - S^*) = \gamma_m^\infty(P_{3k+1-k-1}) = \gamma_m^\infty(P_{2k}) = k.$$

Hence $st^{-}_{\gamma_m^\infty}(G) \leq |S^*| = k + 1$. Therefore, $st^{-}_{\gamma_m^\infty}(G) = k + 1$. A similar argument holds for the case $n \equiv 2 \pmod 3$. □

As an immediate consequence of Theorems 6 and 7, we have the following theorem.

Theorem 8. *For cycles C_n, $n \geq 4$*

$$st_{\gamma_m^\infty}(C_n) = \begin{cases} 3 & \textit{if } n = 5,7 \\ 2 & \textit{if } n = 4 \\ 1 & \textit{otherwise} \end{cases}.$$

For wheels W_n, $\gamma_m^\infty(W_n) = 2$. Hence, to decrease the value to 1, one has to delete $n-3$ vertices from W_n so that a cycle C_3 is obtained and $\gamma_m^\infty(C_3) = 1$. Thus we have the following observation.

Observation 9. For wheels W_n, $n \geq 5$, $st^-_{\gamma_m^\infty}(W_n) = n - 3$.

Theorem 10. *For wheels W_n, $n \geq 5$*

$$st^+_{\gamma_m^\infty}(W_n) = \begin{cases} \infty & \textit{if } n = 5,6 \\ 2 & \textit{if } n = 7 \\ 1 & \textit{if } n \geq 8. \end{cases}$$

Proof. One can easily verify that when $n = 5,6$, $st^+_{\gamma_m^\infty}(W_n) = \infty$. When $n = 7$, removing the central vertex and a vertex on the rim of W_7, we have P_5 with $\gamma_m^\infty(P_5) = 3$. Hence $st^+_{\gamma_m^\infty}(W_7) = 2$. When $n \geq 8$, removing the central vertex from W_n, we get C_{n-1} and $\gamma_m^\infty(C_{n-1}) = \lceil \frac{n-1}{3} \rceil \geq 3$. Thus $st^+_{\gamma_m^\infty}(W_n) = 1$. □

From Observation 9 and Theorem 10 the following theorem is immediate.

Theorem 11. *For wheels W_n, $n \geq 5$*

$$st_{\gamma_m^\infty}(W_n) = \begin{cases} 3 & \textit{if } n = 6 \\ 2 & \textit{if } n = 5,7 \\ 1 & \textit{if } n \geq 8. \end{cases}$$

Theorem 12. *Let $G = K_{r_1,r_2,\ldots,r_s}$ be a complete multipartite graph with $s, r_s \geq 2$ and $r_1 \leq r_2 \leq \ldots \leq r_s$. Then $st_{\gamma_m^\infty}(G) = n - r_s$, where $n = \sum_{i=1}^{s} r_i$.*

Proof. It is clear that $\gamma_m^\infty(G) = 2$. First, we prove that

$$st^-_{\text{ffl}\gamma_m^\infty}(G) = \begin{cases} n-3 & \text{if } s > 2, \\ n-2 & \text{if } s = 2, \end{cases} \tag{1}$$

and

$$st^+_{\text{ffl}\gamma_m^\infty}(G) = \begin{cases} n - r_s & \text{if } r_s > 2, \\ \infty & \text{if } r_s = 2. \end{cases} \tag{2}$$

Suppose $s > 2$. Then G contains a triangle. Thus, to reduce $\gamma_m^\infty(G)$ by 1, we need to remove $n-3$ vertices from G to get K_3, and $\gamma_m^\infty(K_3) = 1$. Thus, when $s > 2$, we have $st^-_{\gamma_m^\infty}(G) = n-3$. Clearly, when $s = 2$, we need to remove $n-2$ vertices from G to obtain a K_2 and $\gamma_m^\infty(K_2) = 1$. Thus, when $s = 2$, $st^-_{\gamma_m^\infty}(G) = n-2$. Hence, (1) is proved.
In order to find $st^+_{\gamma_m^\infty}(G)$, suppose that $r_s > 2$. Then we need to remove $n - r_s$ vertices from G to obtain r_sK_1, and $\gamma_m^\infty(r_sK_1) = r_s > 2$. Thus, when $r_s > 2$, we have $st^+_{\gamma_m^\infty}(G) = n - r_s$.
If $r_s = 2$, then one can observe that there exists no subset S of $V(G)$ such that the removal of S from G will increase $\gamma_m^\infty(G)$. Hence $st^+_{\gamma_m^\infty}(G) = \infty$. Hence, (2) is proved.
By (1) and (2), we see that $st_{\gamma_m^\infty}(G) = n - r_s$. □

5. Trees

By Theorem 1, for any tree T, $st_{\gamma_m^\infty}(T) \leq 2$. In this section first we characterize trees T with $st_{\gamma_m^\infty}(T) = 2$. For this purpose we define a family $\mathscr{T}_1$ of trees as follows.

Definition 1 (Family $\mathscr{T}_1$). Let T be a tree with no strong supports. Perform the following operation $\mathscr{O}_1$ on T.

Operation $\mathscr{O}_1$ Identify a maximal even pendant path in T. Let v be an end of this path such that $\deg(v) = 2$ in T. Remove the edge between v and its parent vertex from T. If the resulting graph does not contain a strong support, then stop the process. Otherwise, repeat the operation $\mathscr{O}_1$ until no junction vertices remain. $T \in \mathscr{T}_1$ if each component of the final graph is a P_k, k is even.

Theorem 13. *For any tree T of order $n \geq 2$, $st_{\gamma_m^\infty}(T) = 2$ if and only if $T \in \mathscr{T}_1$.*

Proof. Suppose that $T \in \mathscr{T}_1$. Since at every stage even paths are removed and the final graph is also an even path, the removal of any single vertex will not affect $\gamma_m^\infty(T)$. Hence $st_{\gamma_m^\infty}(T) \geq 2$ which implies that $st_{\gamma_m^\infty}(T) = 2$.
Conversely suppose that $st_{\gamma_m^\infty}(T) = 2$. We claim that $T \in \mathscr{T}_1$. If T has a strong support v, then 2 guards are required to safeguard the vertices in $N[v]$. If the two guards safeguard more than two vertices in $N(v)$, then the removal of v in T will increase $\gamma_m^\infty(T)$. Thus $st_{\gamma_m^\infty}(T) = 1$, a contradiction. If the two guards safeguard exactly two vertices in $N(v)$, then the removal of a leaf neighbor of v will decrease $\gamma_m^\infty(T)$ by 1. Thus $st_{\gamma_m^\infty}(T) = 1$, a contradiction. Thus, T does not contain a strong support. Now perform operation $\mathscr{O}_1$ in T. If the resulting graph has a strong support, then as discussed earlier $st_{\gamma_m^\infty}(T) = 1$, a contradiction. Otherwise repeat the process until no junction vertices remain. Suppose the final graph is an odd path, then by Theorem 4, $st_{\gamma_m^\infty}(T) = 1$, a contradiction. Thus, the final graph is an even path. Hence, $T \in \mathscr{T}_1$. □

Remark 1. It is clear that by the definition of $\mathscr{T}_1$, for any tree $T \in \mathscr{T}_1$ there exists no set of vertices in T whose removal will increase $\gamma_m^\infty(T)$. Thus for any tree $T \in \mathscr{T}_1$, $st^+_{\gamma_m^\infty}(T) = \infty$.

In the next theorem we establish an upper bound for $st^-_{\gamma_m^\infty}(T)$, where T is a tree, in terms of its maximum degree.

Theorem 14. *For any tree T with $\Delta(T) \geq 3$, $st^-_{\gamma_m^\infty}(T) \leq \Delta(T) - 1$.*

Proof. If T is a star, then clearly $\gamma_m^\infty(T) = 2$ and the removal of $n-2$ vertices will reduce $\gamma_m^\infty(T)$ by 1. Hence, $st^-_{\gamma_m^\infty}(T) \leq \Delta(T) - 1$. Suppose T is not a star. Let P be a diametral path in T. Let v_1, v_2, v_3 be the last three vertices in P such that v_3 is a leaf vertex. If v_2 is of degree 2, then clearly the removal of v_2 and v_3 from T will reduce $\gamma_m^\infty(T)$ by 1. Hence $st^-_{\gamma_m^\infty}(T) \leq 2 \leq \Delta(T) - 1$. If v_2 is of degree at least three, then at least 2 guards are required to safeguard the vertices in $N[v_2]$. Thus the removal of all the leaf neighbors of v_2 will decrease $\gamma_m^\infty(T)$ by 1. Thus $st^-_{\gamma_m^\infty}(T) \leq \Delta(T) - 1$. □

Now we proceed to characterize trees T with $st^-_{\gamma_m^\infty}(T) = \Delta(T) - 1$.
Klostermeyer et al. [14] have proved that for any graph G, $\gamma_m^\infty(G) \leq \gamma_c(G) + 1$, where $\gamma_c(G)$ is the connected domination number of G. A Theorem and a Corollary proved by Klostermeyer et al. [14] which characterizes trees T with $\gamma_m^\infty(T) < \gamma_c(T) + 1$ are stated below.

Theorem 15. *[14] A tree T has $\gamma_m^\infty(T) < \gamma_c(T) + 1$ if and only if T has a spanning forest consisting of rK_2's and trees $T_1, T_2, \ldots, T_k$ on at least three vertices such that at least k loners of T are leaves of the $k + r$ trees in the collection.*
By a loner they mean a vertex which is not adjacent to a leaf vertex.

Corollary 1. *[14] Let T be a tree. Then $\gamma_m^\infty(T) < 1 + \gamma_c(T)$ if and only if there exists a set of edges whose deletion creates a spanning forest consisting of r K_2's and trees $T_1, T_2, \ldots, T_k$ on at least three vertices such that at least k loners of T are leaves of the $k + r$ trees in the collection.*

The following lemma is proved using Theorem 15 and Corollary 1.

Lemma 1. *Let T be a tree with no vertex of degree 2. Then $\gamma_m^\infty(T) = m + 1$, where m is the number of junction vertices of T.*

Proof. Since T does not have vertices of degree 2, it is clear that there exists no set of edges such that the removal of these edges will leave a forest as specified in Corollary 1. Hence by Corollary 1, $\gamma_m^\infty(T) = \gamma_c(T) + 1 = m + 1$, where m is the number of junction vertices of T. □

Two families of trees $\mathscr{T}_2$ and $\mathscr{T}_3$ are defined as follows.

Definition 2 (Family $\mathscr{T}_2$). Let T be a tree with $\Delta(T) \geq 4$, rooted at a vertex w with $deg(w) \geq 3$ and all its end supports are of degree $\Delta(T)$. Perform the following operation on T.

Operation $\mathscr{O}_2$: Remove an edge between a vertex v and its parent in T such that $deg(v) = 2$ and all its descendents are of degree at least 3 (other than the leaf vertices). Similarly remove all such edges and let T' be the resulting graph.
Check whether T' contains an end support of degree less than $\Delta(T)$. If so, stop the process. Otherwise, perform operation $\mathscr{O}_2$ successively until no vertex of degree 2 remains. Let T^* be the final graph. Then $T \in \mathscr{T}_2$ if each component of T^* satisfy the following conditions:

1. All its end supports are of degree $\Delta(T)$
2. There exist no two adjacent supports that have exactly two non leaf neighbors respectively and the sum of their leaf neighbors is at most $\Delta(T) - 2$.

Definition 3 (Family $\mathscr{T}_3$). Let T be a tree with $\Delta(T) = 3$. Root the tree at a vertex w with $deg(w) = 3$. Perform operations $\mathscr{O}_1$ and $\mathscr{O}_2$ (as stated earlier). Again perform the operations $\mathscr{O}_1$ and $\mathscr{O}_2$ in the resulting graph. Repeat the process until none of the operations $\mathscr{O}_1$ and $\mathscr{O}_2$ can be performed. Let T^* be the final graph. Then $T \in \mathscr{T}_3$ if no component of T^* is an odd path.

Theorem 16. *Let T be a tree in $\mathscr{T}_2$. Then $st^-_{\gamma_m^\infty}(T) = \Delta(T) - 1$.*

Proof. Let $T \in \mathscr{T}_2$. Perform operation $\mathscr{O}_2$ successively and let T^* be the final graph. Let $H_1, H_2, \ldots, H_k$ be the components of T^*. By the definition of $\mathscr{T}_2$, each H_i for $1 \leq i \leq k$ satisfy the conditions given in the definition of $\mathscr{T}_2$. Hence, by Lemma 1, $\gamma_m^\infty(H_i) = m_i + 1$, where m_i is the number of junction vertices of $H_i, 1 \leq i \leq k$. But these $m_i + 1$ guards cannot defend any vertex in T not in H_i, $1 \leq i \leq k$. Thus $\gamma_m^\infty(T) = \sum_{i=1}^k m_i + k$, where $\sum_{i=1}^k m_i$ is the number of junction vertices in T. Now the removal of any junction vertex in T will increase $\gamma_m^\infty(T)$. Further, since every end support is of degree $\Delta(T)$, the removal of any number of leaves less than $\Delta(T) - 1$ will not alter $\gamma_m^\infty(T)$. Hence, $st^-_{\gamma_m^\infty}(T) \geq \Delta(T) - 1$. If all the leaves adjacent to an end support are removed, then $\gamma_m^\infty(T)$ decreases by 1. Hence $st^-_{\gamma_m^\infty}(T) \leq \Delta(T) - 1$ and thus, $st^-_{\gamma_m^\infty}(T) = \Delta(T) - 1$. □

Theorem 17. *Let T be a tree with $\Delta(T) \geq 4$. Then $st^-_{\gamma_m^\infty}(T) = \Delta(T) - 1$ if and only if $T \in \mathscr{T}_2$.*

Proof. Supose that $st^-_{\gamma_m^\infty}(T) = \Delta(T) - 1$. First we claim that all its end supports are of degree $\Delta(T)$. Suppose an end support is of degree at most $\Delta(T) - 1$. Then removing all the leaf neighbors in case of a strong support and the leaf and the

corresponding support vertex in case of a weak support will decrease $\gamma_m^\infty(T)$ by 1. Hence $st^-_{\gamma_m^\infty} < \Delta(T) - 1$, a contradiction.
Now remove the edges as described in operation $\mathscr{O}_2$ to obtain a graph T'. Again in T' check whether there is an end support of degree $\Delta(T) - 1$. If so, as before we get a contradiction. Repeat the process until no vertices of degree two remain. Let T^* be the final graph. Let $H_1, H_2, \ldots, H_k$ be the components of T^*. Now we claim that each component of T^* satisfy the conditions given in the definiton of $\mathscr{T}_2$. By Lemma 1, for $1 \leq i \leq k$, $\gamma_m^\infty(H_i) = m_i + 1$, where m_i is the number of junction vertices of H_i. If some end support of H_i is not of degree $\Delta(T)$, then as before we get a contradiction. Further, the $m_i + 1$ guards in H_i cannot defend any vertex of T not in H_i. Hence, $\gamma_m^\infty(T) = \sum_{i=1}^{k} m_i + k$.
Now if there exists two adjacent supports in H_i such that they have exactly 2 non leaf neighbors respectively and the sum of their leaf neighbors is less than $\Delta(T) - 1$, then the removal of all these leaf vertices will result in a subgraph H' of H_i such that there are 2 induced subgraphs G_1 and G_2 in H' with $|V(H')| = |V(G_1)| + |V(G_2)|$ and $G_1, G_2 \in \mathscr{T}_2$. Let t_1 and t_2 be the number of junction vertices of G_1 and G_2 respectively. Thus,

$$\gamma_m^\infty(H') = \gamma_m^\infty(G_1) + \gamma_m^\infty(G_2) = t_1 + t_2 + 2 = m_i$$

Hence,

$$\gamma_m^\infty(H_i) = m_i + 1 = \gamma_m^\infty(H') + 1$$

which implies that $\gamma_m^\infty(H') < \gamma_m^\infty(H_i)$. Thus, $\gamma_m^\infty(T^*) < \gamma_m^\infty(T)$.
Hence, $st^-_{\gamma_m^\infty}(T) < \Delta(T) - 1$, a contradiction. Thus, each $H_i \in \mathscr{T}_2$ and so $T \in \mathscr{T}_2$.
Converse holds by Theorem 16. □

Theorem 18. *Let T be a tree with $\Delta(T) = 3$. Then $st^-_{\gamma_m^\infty}(T) = \Delta(T) - 1$ if and only if $T \in \mathscr{T}_3$.*

Proof. Suppose $st^-_{\gamma_m^\infty}(T) = \Delta(T) - 1 = 2$. Perform operations $\mathscr{O}_1$ and $\mathscr{O}_2$ in T successively till none of the operations can be performed. Let T^* be the final graph. We claim that none of the components of T^* is an odd path. Suppose some component of T^* say Q is an odd path, then by Theorem 4, $st^-_{\gamma_m^\infty}(Q) = 1$ which implies that $st^-_{\gamma_m^\infty}(T) = 1$, a contradiction. Hence, $T \in \mathscr{T}_3$.
Conversely supopse $T \in \mathscr{T}_3$. Then by definition, each component of T^*, the final graph which is obtained by performing operations $\mathscr{O}_1$ and $\mathscr{O}_2$ successively on T is either an even path or a tree which does not contain a vertex of degree 2. Let H be a component of T^*. If H is not a path, then by Lemma 1, $\gamma_m^\infty(H) = m + 1$, where m is the number of junction vertices of H and these guards cannot safeguard any vertex in T which is not in H and the removal of any single vertex will increase $\gamma_m^\infty(T)$. If

H is an even path, then $\gamma_m(H) = \frac{|V(H)|}{2}$ and the removal of any single vertex will not alter $\gamma_m^\infty(T)$. Thus $st^-_{\gamma_m^\infty}(T) \geq 2$. Now either removing any support of degree 2 and its leaf neighbor or removing 2 leaf neighbors adjacent to a strong support will reduce $\gamma_m^\infty(T)$ by one. Thus, $st^-_{\gamma_m^\infty}(T) = 2 = \Delta(T) - 1$. □

Statements and Declarations

The author declares that no funds, grants, or other support were received during the preparation of this manuscript. The author has no relevant financial or non- financial interests to disclose. This research did not generate or analyze any datasets.

References

[1] G. Bagan, A. Joffard, and H. Kheddouci, *Eternal dominating sets on digraphs and orientations of graphs*, Discrete Appl. Math. **291** (2021), 99–115. https://doi.org/10.1016/j.dam.2020.10.024.

[2] D. Bauer, F. Harary, J. Nieminen, and C.L. Suffel, *Domination alteration sets in graphs*, Discrete Math. **47** (1983), 153–161. https://doi.org/10.1016/0012-365X(83)90085-7.

[3] R.C. Brigham, P.Z. Chinn, and R.D. Dutton, *Vertex domination-critical graphs*, Networks **18** (1988), no. 3, 173–179. https://doi.org/10.1002/net.3230180304.

[4] A.P. Burger, E.J. Cockayne, W.R. Grundlingh, C.M. Mynhardt, J.H. Van Vuuren, and W. Winterbach, *Finite order domination in graphs*, J. Combin. Math. Combin. Comput. **49** (2004), 159–176.

[5] A.P. Burger, E.J. Cockayne, W.R. Grundlingh, C.M. Mynhardt, J.H. van Vuuren, and W. Winterbach, *Infinite order domination in graphs*, J. Combin. Math. Combin. Comput. **50** (2004), 179–194.

[6] E.J. Cockayne, P.J.P. Grobler, W.R. Grundlingh, J. Munganga, and J.H. Van Vuuren, *Protection of a graph*, Util. Math. **67** (2005), 19–32.

[7] Stephen Finbow and Martin F van Bommel, *The eternal domination number for $3 \times n$ grid graphs.*, Australas. J Comb. **76** (2020), no. 1, 1–23.

[8] A. Gagnon, A. Hassler, J. Huang, A. Krim-Yee, F. Mc Inerney, A. Zacarías, B. Seamone, and V. Virgile, *A method for eternally dominating strong grids*, Discrete Math. Theor. Comput. Sci. **22** (2020), no. 2, #8.

[9] W. Goddard, S.M. Hedetniemi, and S.T. Hedetniemi, *Eternal security in graphs*, J. Combin. Math. Combin. Comput. **52** (2005), no. 1, 169–180.

[10] F. Harary, *Graph Theory*, Addison Wesley, Reading, Mass, 1969.

[11] T.W. Haynes, S. Hedetniemi, and P. Slater, *Fundamentals of Domination in Graphs*, Taylor & Francis, 1998.

[12] N. Jafari Rad, E. Sharifi, and M. Krzywkowski, *Domination stability in graphs*, Discrete Math. **339** (2016), no. 7, 1909–1914. https://doi.org/10.1016/j.disc.2015.12.026.

[13] W.F. Klostermeyer and G. MacGillivray, *Eternal domination in trees*, J. Combin. Math. Combin. Comput. **91** (2014), 31–50.

[14] W.F. Klostermeyer and C.M. Mynhardt, *Eternal total domination in graphs*, Ars Combin. **107** (2012), 473–492.

[15] F. Mc Inerney, N. Nisse, and S. Pérennes, *Eternal domination: D-dimensional cartesian and strong grids and everything in between*, Algorithmica **83** (2021), no. 5, 1459–1492.
https://doi.org/10.1007/s00453-020-00790-8.

[16] P.R.L. Pushpam and G. Navamani, *Eternal m-security in certain classes of graphs*, J. Combin. Math. Combin. Comput. **92** (2015), 25–38.

[17] P.R.L Pushpam and P.A. Shanthi, *Changing and unchanging in m-eternal total domination in graphs*, Electron. Notes Discrete Math. **53** (2016), 181–198.
https://doi.org/10.1016/j.endm.2016.05.017.

[18] ———, *m-eternal total domination in graphs*, Discrete Math. Algorithms Appl. **8** (2016), no. 3, 1650055.
https://doi.org/10.1142/S1793830916500555.

[19] D.P. Sumner, *Critical concepts in domination*, Discrete Math. **86** (1990), no. 1–3, 33–46.
https://doi.org/10.1016/0012-365X(90)90347-K.

[20] D.P. Sumner and P. Blitch, *Domination critical graphs*, J. Combin. Theory Ser. B **34** (1983), no. 1, 65–76.
https://doi.org/10.1016/0095-8956(83)90007-2.