Communications in Combinatorics and Optimization
Vol. 11, No. 4 (2026), pp. 1093-1106
https://doi.org/10.22049/cco.2025.30952.2676

CCO
Commun. Comb. Optim.

Research Article

Total domination versus triad domination

Teresa W. Haynes[1,2,*], Michael A. Henning[2]

[1]Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN 37614-0002 USA
haynes@etsu.edu

[2]Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park, 2006 South Africa
mahenning@uj.ac.za

Received: 26 August 2025; Accepted: 1 November 2025
Published Online: 7 November 2025

Dedicated to Odile Favaron

Abstract: A dominating set in a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. A total dominating set in G is a dominating set S with the additional property that the subgraph $G[S]$ induced by S is isolate-free. A triad dominating set S (also called a 3-component dominating set in the literature) is a dominating set in which every component in $G[S]$ has order at least 3. The triad domination number, denoted $\gamma_{\rm td}(G)$, of G is the minimum cardinality among all triad dominating sets of G. We observe that $\gamma(G) \le \gamma_t(G) \le \gamma_{\rm td}(G)$, where $\gamma(G)$ is the domination number of G and $\gamma_t(G)$ is the total domination number of G. We show that the ratio $\frac{\gamma_{\rm td}(G)}{\gamma_t(G)}$ is at most $\frac{3}{2}$. We establish properties of the graphs G satisfying $\gamma_{\rm td}(G) = \frac{3}{2}\gamma_t(G)$ and characterize the trees achieving this equality.

Keywords: domination, total domination, triad domination, 3-component domination.

AMS Subject Classification: 05C69

1. Introduction

Triad domination, also called 3-component domination in the literature, is a robust form of domination that requires the components induced by a dominating set to have order at least 3. Dominating sets inducing large components have been studied in [2, 4, 5, 9, 11, 12], for example. We begin with some basic terminology.

For a set S of vertices in a graph G, we denote the *subgraph induced by* S by $G[S]$. A *component* of a graph G is a maximal connected subgraph of G. A *dominating set* in

* *Corresponding Author*

a graph G is a set S of vertices of G such that every vertex in $V(G) \setminus S$ has a neighbor in S, where two vertices are neighbors if they are adjacent. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum cardinality among all dominating sets of G. A *total dominating set*, abbreviated *TD-set*, in G is a dominating set S of G with the additional property that $G[S]$ is isolate-free, that is, the components of $G[S]$ have cardinality at least 2. The *total domination number* of G, denoted $\gamma_t(G)$, is the minimum cardinality among all total dominating sets of G, and a total dominating set of cardinality $\gamma_t(G)$ is called a γ_t*-set* of G. A thorough treatment of domination in graphs and its variants can be found in the books [6–8, 10].
For $k \geq 1$ an integer, a dominating set S is called a k*-component dominating set* if every component in $G[S]$ has order at least k. The k*-component domination number*, denoted $\gamma_k(G)$, is the minimum cardinality among all k-component dominating sets of G, and such a set with minimum cardinality is called a γ_k*-set* of G. Since the vertex set of any connected graph G of order $n \geq k$ is a k-component dominating set of G, the k-component domination number is well-defined for such graphs and $k \leq \gamma_k(G) \leq n$. This concept of component domination was introduced in 2016 by Alvarado, Dantas, and Rautenbach [1]. We note that for $k = 1$ and $k = 2$, the k-component domination numbers are the domination number and the total domination number, respectively, and we state this formally as follows.

Observation 1. The following properties hold in a connected graph G of order $n \geq k$.
(a) $\gamma_1(G) = \gamma(G)$.

(b) If $n \geq 2$, then $\gamma_2(G) = \gamma_t(G)$.

(c) If $n \geq k \geq 2$, then $\gamma_{k-1}(G) \leq \gamma_k(G) \leq n$.

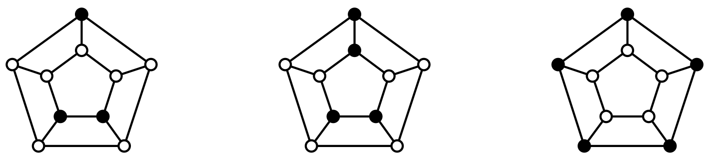
1.1. Triad domination

Among the k-component domination parameters, the 3-component domination number stands out as especially compelling due to the foundational roles of the 1-component domination and 2-component domination numbers. Hence, we find 3-component domination interesting in its own right and in this special case we coin the term *triad dominating set* for a 3-component dominating set and we denote the *triad domination number* of G by $\gamma_{\rm td}(G)$ rather than $\gamma_3(G)$.
The shift from "k-component domination" to "triad domination" for the special case of $k = 3$ sets this parameter apart and aligns the notation more closely with the standard notation used for the total domination number. We remark that the notation $\gamma_k(G)$ for the k-component domination number is also used in the literature for multiple domination (where a vertex not in the dominating set S is dominated by at least k vertices in S) and is used for distance domination (where a vertex not in the dominating set S is within distance k from at least one vertex in S). Thus, another motivation for the proposed notation change is to avoid confusion with other parameters. By Observation 1, we have the following inequality chain.

Observation 2. If G is a connected graph of order at least 3, then

$$\gamma(G) \le \gamma_t(G) \le \gamma_{\rm td}(G). \tag{1.1}$$

We note that the inequalities in Inequality (1) may be strict. For example, if G is the 5-prism $G = C_5 \,\square\, K_2$ illustrated in Figure 1, then $\gamma(G) = 3$, $\gamma_t(G) = 4$, and $\gamma_{\rm td}(G) = 5$, where a γ-set is indicated by the shaded vertices in Figure 1(a), a γ_t-set is indicated by the shaded vertices in Figure 1(b), and a $\gamma_{\rm td}$-set is indicated by the shaded vertices in Figure 1(c).



(a) $\gamma(G) = 3$ (b) $\gamma_t(G) = 4$ (c) $\gamma_{\rm td}(G) = 5$

Figure 1. **The 5-prism** $G = C_5 \,\square\, K_2$

1.2. Graph theory notation

For graph theory notation and terminology, we generally follow [8]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n = |V(G)|$. The *open neighborhood* of a vertex v in G is $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. We denote the *degree* of a vertex v in G by $\deg_G(v)$. An *isolated vertex* is a vertex of degree 0, while a vertex of degree 1 is called a *leaf.* The (unique) neighbor of a leaf is a *support vertex.* We denote a cycle on n vertices by C_n. For a set S of vertices in a graph G and a vertex $v \in S$, the set

$$\mathrm{epn}_G(v, S) = \{u \in V(G) \setminus S : N(u) \cap S = \{v\}\}$$

is called the set of S-*external private neighbors* of v with respect to S. Thus, a vertex $u \in V(G)$ is an S-external private neighbor of $v \in S$ if $u \in V(G) \setminus S$ and the only neighbor of u in S is v. For an integer $k \ge 1$, we use the standard notation $i \in [k]$ to mean that i is an integer and $1 \le i \le k$.

By a *partition* of a set S, we mean a family $\pi = \{S_1, \dots, S_q\}$ of nonempty pairwise disjoint sets whose union equals S, that is, for all i and j with $1 \le i < j \le q$, we have $S_i \cap S_j = \emptyset$ and the union of the sets S_i over all $i \in [q]$ is the set S, that is,

$$S = \bigcup_{i=1}^{q} S_i.$$

The *distance* $d_G(u,v)$, between two vertices u and v in a connected graph G is the minimum length among all u,v-paths in G. A *packing* in a graph G is a set P of vertices whose closed neighborhoods are pairwise disjoint; that is, $d_G(u,v) \geq 3$ for every two distinct vertices $u,v \in P$.
If S is a set of vertices in a graph G and $v \in V(G)$, then we define the *distance* from v to S, denoted $d_G(v,S)$, as the minimum distance in G from v to a vertex in S, that is,

$$d_G(v,S) = \min_{u \in S} d_G(u,v).$$

Moreover, if X and Y are two subsets of vertices in G, then we define the *distance* between X and Y, denoted $d_G(X,Y)$, as the minimum distance between a vertex in X and a vertex in Y, that is,

$$d_G(X,Y) = \min_{x \in X, y \in Y} d_G(x,y).$$

We say that a component of a graph is *k-large* if it has order at least k and *k-small* otherwise. We simply say large and small, dropping the k, if k is understood from the context. A *connected dominating set* in G is a dominating set S of G with the additional property that $G[S]$ is connected, and the *connected domination number* of a graph G, denoted $\gamma_c(G)$, is the minimum cardinality among all connected dominating sets of G.

2. Background and discussion of results

In [9], the authors established an upper bound on the connected domination number of a graph in terms of its k-component domination number for all $k \geq 1$.

Theorem 3. ([9]) *For $k \geq 1$ if G is a connected graph of order at least k, then*

$$\gamma_c(G) \leq \left(\frac{k+2}{k}\right) \gamma_k(G) - 2.$$

The following bound, due to Favaron and Kratsch [3] in 1991, is a corollary to Theorem 3.

Theorem 4. ([3]) *If G is a connected graph of order at least 2, then $\gamma_c(G) \leq 2\gamma_t(G) - 2$.*

Using Theorem 4, we have the following upper bound on the k-component domination number in terms of the total domination number.

Theorem 5. ([9]) *For $k \geq 3$ if G is a connected graph of order at least k, then*

$$\gamma_k(G) \leq \max\{2\gamma_t(G) - 2, k\}.$$

In Section 3, we will prove the following for a connected graph G of order $n \geq k \geq 3$.

Theorem 6. *If G is a connected graph of order $n \geq k \geq 3$, then $\gamma_k(G) \leq \frac{k}{2}\gamma_t(G)$.*

Comparing the upper bounds of $2\gamma_t(G) - 2$ and $\frac{k}{2}\gamma_t(G)$, we note that the bound of Theorem 5 is better than the bound Theorem 6 for $k \geq 4$; while for $k = 3$ and $\gamma_t(G) \geq 4$, the bound in Theorem 6 is the better choice.
We show that the only possibilities for sharpness of the bound in Theorem 6 are if $\gamma_t(G) = 2$ or if $k = 3$. In fact, if G is a graph of order $n \geq k$ and $\gamma_t(G) = 2$, then $\gamma_k(G) = k$, achieving the bound of $\frac{k}{2}\gamma_t(G)$ for all $k \geq 3$. Thus, we turn our attention to $k = 3$ (triad domination) and consider the ratio

$$\frac{\gamma_{\rm td}(G)}{\gamma_t(G)} \leq \frac{3}{2}$$

in Section 4, where we present properties of graphs achieving this ratio. Finally, in Section 5, we characterize the extremal trees.

3. Proof of Theorem 6

We now present a proof to Theorem 6. Recall its statement.

Theorem 6 *If G is a connected graph of order $n \geq k \geq 3$, then*

$$\gamma_k(G) \leq \frac{k}{2}\gamma_t(G).$$

Proof. Let G be a connected graph of order $n \geq k \geq 3$, and let S be a γ_t-set of G. If every component of $G[S]$ is a k-large component, then S is a k-component dominating set and $\gamma_t(G) \leq \gamma_k(G) \leq |S| = \gamma_t(G)$, implying that $k \leq \gamma_k(G) = \gamma_t(G) < \frac{k}{2}\gamma_t(G)$. Hence, we may assume that $G[S]$ has at least one k-small component.
If $G[S]$ is connected, that is, $G[S]$ has exactly one component, then $G[S]$ is a small component and adding exactly $k - |S|$ vertices of $V(G) \setminus S$ to S creates a k-component dominating set of cardinality k. Thus, $k \leq \gamma_k(G) \leq |S| + (k - |S|) = k \leq \frac{k}{2}\gamma_t(G)$ since $\gamma_t(G) \geq 2$. Therefore, we assume that $G[S]$ is not connected. Let $G[S]$ have $q \geq 2$ components, and let $G_1, G_2, \ldots, G_q$ be the components of $G[S]$. Let $S_i = V(G_i)$ for $i \in [q]$, and so

$$S = \bigcup_{i=1}^{q} S_i.$$

We note that $|S_i| \geq 2$ for $i \in [q]$ since S is a TD-set of G. It follows that $\gamma_t(G) \geq 4$ and $q \leq \left\lfloor \frac{1}{2}|S| \right\rfloor$. Relabeling the components of $G[S]$ if necessary, we assume that G_1 is a k-small component, that is, $2 \leq |S_1| < k$. Since G is connected and S is a

TD-set, we note that $d_G(S_1, S \setminus S_1) \le 3$. Hence, there exists a component, say G_2, in $G[S]$ such that $d_G(S_1, S_2) \le 3$. Let u be a vertex of G_1 and v be a vertex of G_2 for which $d_G(u, v) \le 3$. Note that G_2 may be a k-large component. Let I be the internal vertices on a shortest u, v-path. Thus, $I \subseteq V(G) \setminus S$ and $1 \le |I| \le 2$.

We first add the vertices of I to S creating a component G^* having $r \ge |S_1| + |S_2| + |I| \ge 5$ vertices. If $k \le r$, then no additional vertices are needed for G^* to be a k-large component. In this case, we have added at most two vertices. If $k > r$, then after the addition of the vertices of I, at most $k - r \le k - 5$ additional vertices are needed to build a k-large component containing G^* as a subgraph. Hence, we can create a k-large component supergraph of $G_1 \cup G_2$ by adding at most two vertices for $3 \le k \le 5$ and adding at most $2 + k - 5 = k - 3$ vertices to S for $k \ge 6$.

Now we consider the remaining, if any, k-small components of $G[S]$. Since the newly created k-large component contains $G_1 \cup G_2$ as a subgraph and $|S_1| + |S_2| \ge 4$, there are at most $\frac{1}{2}(|S| - 4)$ such remaining components. We note that since G is connected, S is a TD-set, and $n \ge k$, it is possible to create a k-large component containing any remaining k-small component G_i in $G[S]$ as a subgraph by adding to S at most $k - |S_i| \le k - 2$ vertices of $V(G) \setminus S$. By our previous comments, there are at most $\frac{1}{2}(|S| - 4)$ of these components. Thus, for $3 \le k \le 5$, we have

$$\begin{aligned}
\gamma_k(G) &\le |S| + 2 + \tfrac{1}{2}(|S| - 4)(k - 2) \\
&= \tfrac{k}{2}|S| - 2k + 6 \\
&= \tfrac{k}{2}\gamma_t(G) - 2k + 6 \\
&\le \tfrac{k}{2}\gamma_t(G).
\end{aligned}$$

For $k \ge 6$, we have

$$\begin{aligned}
\gamma_k(G) &\le |S| + (k - 3) + \tfrac{1}{2}(|S| - 4)(k - 2) \\
&= \tfrac{k}{2}|S| - k + 1 \\
&= \tfrac{k}{2}\gamma_t(G) - k + 1 \\
&< \tfrac{k}{2}\gamma_t(G).
\end{aligned}$$

This concludes the proof of Theorem 6. □

By Theorem 6, if G is a connected graph of order $n \ge k \ge 3$, the following ratio holds:

$$\frac{\gamma_k(G)}{\gamma_t(G)} \le \frac{k}{2}. \tag{3.1}$$

From the proof of Theorem 6, we deduce that equality is only possible if $\gamma_t(G) = 2$ or $k = 3$. As we noted in Section 2, tightness occurs for all k if $\gamma_t(G) = 2$. Henceforth, we restrict our attention to the case when $k = 3$, that is, we consider the ratio of the triad domination number and the total domination number.

4. Properties of extremal graphs

Let G be a connected graph of order $n \geq 3$. In the case of $k = 3$, Theorem 6 states that

$$\gamma_{\rm td}(G) \leq \frac{3}{2}\gamma_t(G). \tag{4.1}$$

Next we consider properties of graphs attaining the upper bound in Inequality (4.1). We note that if $\gamma_t(G) = 3$, then every γ_t-set of G is also a $\gamma_{\rm td}$-set of G and $\gamma_{\rm td}(G) = \gamma_t(G) = 3 = k$.

Theorem 7. *Let G be a connected graph of order $n \geq 3$ with $\gamma_t(G) \geq 3$. If $\gamma_{\rm td}(G) = \frac{3}{2}\gamma_t(G)$, then for every γ_t-set S of G, the following properties hold:*

(a) *$G[S] = qK_2$, where $q = \frac{1}{2}\gamma_t(G)$.*

(b) *Every independent subset of S is a packing in G.*

(c) *Every vertex in S has an S-external private neighbor.*

Proof. Let G be a connected graph of order $n \geq 3$ with $\gamma_t(G) \geq 3$ satisfying $\gamma_{\rm td}(G) = \frac{3}{2}\gamma_t(G)$. Let S be an arbitrary γ_t-set S of G. If $G[S]$ is connected, then S is a triad dominating set, and so $\gamma_{\rm td}(G) \leq |S| = \gamma_t(G) < \frac{3}{2}\gamma_t(G)$, a contradiction. Hence, $G[S]$ has at least two components. Note that since S is a TD-set of G, every component of $G[S]$ has cardinality at least 2, implying that $\gamma_t(G) \geq 4$ and every small component of $G[S]$ is a K_2-component. Let $G_1, G_2, \ldots, G_q$ denote the components of $G[S]$, where $q \geq 2$. Let $S_i = V(G_i)$ for $i \in [q]$.
We proceed further by proving three claims, the first of which shows that every component of $G[S]$ is a small component. Since we are considering the triad domination number, in what follows we simply refer to a 3-large component of $G[S]$ (of order at least 3) as a large component and a 3-small component of $G[S]$ (of order 2) as a small component.

Claim 1. *$G[S] = qK_2$, where $q = \frac{1}{2}\gamma_t(G)$.*

Proof. Suppose, to the contrary, that $G[S]$ has at least one large component, say G_q, with vertex set S_q. Thus, $|S_q| \geq 3$. If every component of $G[S]$ is a large component, then S is a triad dominating set, implying that $\gamma_{\rm td}(G) = \gamma_t(G) < \frac{3}{2}\gamma_t(G)$, a contradiction. Hence, $G[S]$ has at least one small component. Renaming components if necessary, let $G_1, \ldots, G_r$ denote the small components of $G[S]$ where $r \geq 1$, and so $G_{r+1}, \ldots, G_q$ denote the large components of $G[S]$. Thus, $G_i = K_2$ for all $i \in [r]$. Let $S_i = \{u_i, v_i\}$ for $i \in [q]$. We note that

$$r \leq \frac{1}{2}(|S| - |S_q|) \leq \frac{1}{2}(|S| - 3) = \frac{1}{2}(\gamma_t(G) - 3). \tag{4.2}$$

The connectivity of G implies that for each $i \in [r]$, at least one of u_i and v_i has a neighbor x_i in $V(G) \setminus S$. Letting

$$X = \bigcup_{i=1}^{r} \{x_i\},$$

the set $S \cup X$ is a triad dominating set of G, implying by Inequality (4.2) that

$$\gamma_{\rm td}(G) \leq |S| + |X| \leq \gamma_t(G) + r \leq \gamma_t(G) + \frac{1}{2}(\gamma_t(G) - 3) < \frac{3}{2}\gamma_t(G),$$

a contradiction. We conclude that every component of $G[S]$ is small, that is, $G[S] = qK_2$, where $q = \frac{1}{2}\gamma_t(G)$. □

Claim 2. *If $S' \subseteq S$ is an independent set, then S' is a packing in G.*

Proof. Let S' be an independent subset of S. Claim 1 implies that every two vertices in S' belong to different small components of $G[S]$. Suppose, to the contrary, that S' is not a packing in G. Then there exists vertices u and v in S', such that $d_G(u, v) = 2$. Let u' and v' be the neighbors of u and v, respectively, in $G[S]$. Let $w \in V(G) \setminus S$ be a common neighbor of u and v in G. Now, $G[S \cup \{w\}]$ contains a large component containing u, u', v, and v'. As before, at most one vertex needs to be added to every remaining small component to create a triad dominating set from S, and there are at most $\frac{1}{2}(|S| - 4)$ such components. Thus,

$$\gamma_{\rm td}(G) \leq |S| + 1 + \frac{1}{2}(|S| - 4) = \frac{3}{2}|S| - 1 = \frac{3}{2}\gamma_t(G) - 1 < \frac{3}{2}\gamma_t(G),$$

a contradiction. Hence, S' is a packing in G. □

Claim 3. *Every vertex in S has an S-external private neighbor.*

Proof. Since $\gamma_t(G) \geq 4$, by Claim 1, we have that $G[S]$ consists of $q \geq 2$ components and each component is a K_2-component. Recall that $G_1, G_2, \ldots, G_q$ denote the components of $G[S]$ and recall that $S_i = V(G_i)$ for $i \in [q]$. Let $S_i = \{u_i, v_i\}$ for each $i \in [q]$. Since G is connected and S is a TD-set, we note that $d_G(S_i, S \setminus S_i) \leq 3$ for each $i \in [q]$. By Claim 2, $d_G(S_i, S \setminus S_i) \geq 3$ for each $i \in [q]$. Thus, for each $i \in [q]$, there exists some $j \in [q]$ where $i \neq j$ such that $d_G(S_i, S_j) = 3$. Relabeling the vertices if necessary, we may assume that $d_G(u_i, u_j) = 3$. Let $u_i x y u_j$ be a shortest u_i, u_j-path in G. We note that $x, y \in V(G) \setminus S$.

To complete the proof, it suffices to show that $\text{epn}(u_i, S) \neq \emptyset$ and $\text{epn}(v_i, S) \neq \emptyset$. If $x \in \text{epn}(u_i, S)$, then u_i has an S-external private neighbor, as desired. Thus, assume that x is not an S-external private neighbor of u_i, that is, x has a neighbor in $S \setminus \{u_i\}$. Claim 2 implies that the only possible neighbor of x in S is v_i (since no pair of nonadjacent vertices in S have a common neighbor). It follows that if u_i has no S-external private neighbor, then $N[u_i] \subseteq N[v_i]$. In particular, $v_i x$ is an edge of G. But then we can create a large component containing u_i, v_i, u_j, and v_j by adding x and y to S. Further, after the addition of these two vertices to S, at most $q-2$ small components remain. Thus, we need to add at most $q - 2 = \frac{1}{2}(|S| - 4)$ additional vertices to $S \cup \{x, y\}$ to create a triad dominating set S'. But now the set $S' \setminus \{u_i\}$ is also a triad dominating set, implying that $\gamma_{\rm td}(G) \le |S' \setminus \{u_i\}| = |S'| - 1 \le |S| - 1 + 2 + \frac{1}{2}(|S| - 4) = \frac{3}{2}|S| - 1 = \frac{3}{2}\gamma_t(G) - 1 < \frac{3}{2}\gamma_t(G)$, a contradiction. It follows that $\text{epn}(u_i, S) \neq \emptyset$.

Finally, suppose that v_i has no S-external private neighbor. But then adding x and y to $S \setminus \{v_i\}$ creates a large component containing u_i, u_j, and v_j. As before at most $q-2 = \frac{1}{2}(|S|-4)$ vertices in addition to x and y are needed to build a triad dominating from $S \setminus \{v_i\}$. Again, $\gamma_{\rm td}(G) \le |S \setminus \{v_i\}| + 2 + \frac{1}{2}(|S| - 4) = \frac{3}{2}\gamma_t(G) - 1 < \frac{3}{2}\gamma_t(G)$, a contradiction. Thus, $\text{epn}(v_i, S) \neq \emptyset$. This property holds for all $i \in [q]$. We conclude that every vertex in S has an S-external private neighbor. □

This completes the proof of Theorem 7. □

We note that the only cycles achieving the bound of Theorem 6 are the cycles C_3, C_4, and C_8. To construct an example of an infinite family of graphs that attain the bound, let H be an arbitrary connected graph, and let G be obtained from H as follows: for each vertex $v \in V(H)$, add a vertex disjoint 4-cycle C_v and add the edge joining v to exactly one vertex of C_v. We call the subgraph of G induced by the set $V(C_v) \cup \{v\}$ a *unit* of G, denoted G_v, and we call the graph H the *base graph* of G. Let $\mathcal{G}$ denote the family of all such graphs G.

Every γ_t-set of a graph G in the family $\mathcal{G}$ contains two vertices from the unit G_v for each $v \in V(H)$, and every $\gamma_{\rm td}$-set of G contains three vertices from the unit G_v for each $v \in V(H)$. Thus, $\gamma_{\rm td}(G) = \frac{3}{2}\gamma_t(G)$. An example of a graph G that belongs to the infinite family $\mathcal{G}$ is illustrated in Figure 2, where the shaded vertices in Figure 2(a) indicate a γ_t-set of G and the shaded vertices in Figure 2(b) indicate a $\gamma_{\rm td}$-set of G.

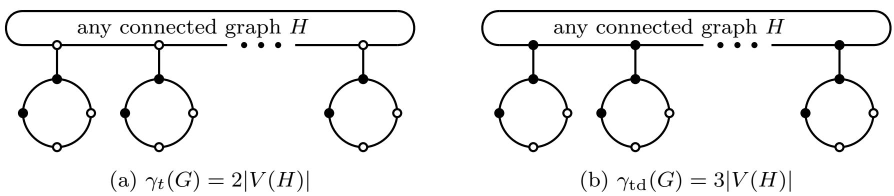


Figure 2. **A graph G in the family $\mathcal{G}$ with associated base graph H**

5. Extremal trees

To give a characterization of the trees T for which $\gamma_{\rm td}(T) = \frac{3}{2}\gamma_t(T)$, we need some additional terminology. A double star $S(r,s)$, for $1 \le r \le s$, is a tree with exactly two (adjacent) vertices that are not leaves, with one of these vertices having r leaf neighbors and the other s leaf neighbors. A support vertex with exactly one nonleaf neighbor is called a *terminal support vertex*.

For a positive integer q, let H_q be any forest consisting of the union of q double stars with centers labeled u_i and v_i for $i \in [q]$, and let L be the set of leaves of H_q. We note that two H_q forests need not be isomorphic, in particular, although both have q double stars, the number of leaves in L as well as the double star subgraphs may vary.

We define a family $\mathcal{T}$ of trees as follows. A tree T_q is in $\mathcal{T}$ if it can be obtained from a forest H_q by adding edges between vertices in L in such a way to ensure that T_q is connected, no cycle is formed, and at least one of the following holds for each u_i and v_i where $i \in [q]$.

(a) both u_i and v_i are support vertices,

(b) at least one of u_i and v_i is a terminal support vertex.

We refer to H_q as the *underlying forest* of T_q. Note that if $T = T_q \in \mathcal{T}$, then T has order at least 4. See Figure 3 for an example of a tree T in family $\mathcal{T}$, where the underlying forest H_6 is given by the six double stars indicated by the dashed boxes and where the set L of leaves of H_6 are indicated by the white vertices.

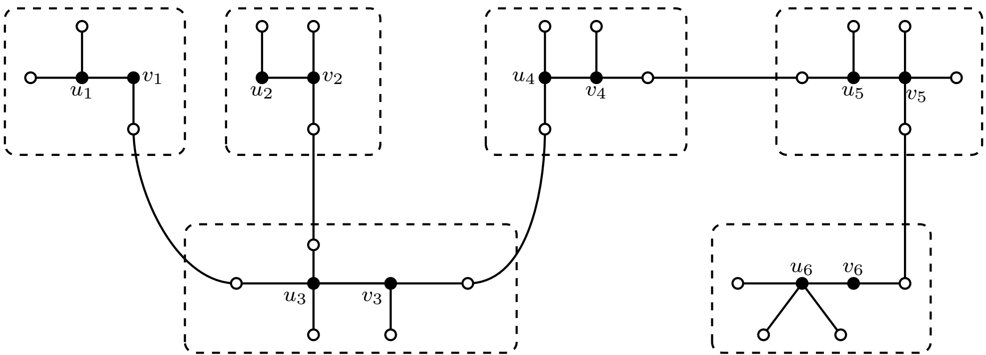


Figure 3. A tree in the family $\mathcal{T}$

Lemma 1. *If $T \in \mathcal{T}$, then $\gamma_{\rm td}(T) = \frac{3}{2}\gamma_t(T)$.*

Proof. Let $T \in \mathcal{T}$. Then, using the notation of the construction, $T = T_q$ is obtained from an underlying forest H_q of $q \geq 1$ double stars with its centers labeled u_i and v_i, respectively, for $i \in [q]$. Recall that L is the set of leaves of H_q. We prove two claims.

Claim 4. $\gamma_t(T) = 2q$.

Proof. Note that $\{u_i, v_i : i \in [q]\}$ is a TD-set of T, implying that $\gamma_t(T) \leq 2q$. Suppose, to the contrary, that $\gamma_t(T) < 2q$, and let S be a γ_t-set of T. Since $\gamma_t(T) < 2q$, it follows that there exists a double star in the underlying forest H_q having at most one vertex in S. Without loss of generality, assume that $u_1 \notin S$. Recall that every support vertex of T must be in S, and so u_1 is not a support vertex of T. By the definition of $\mathcal{T}$, we have that v_1 must be a terminal support vertex with u_1 as its only nonleaf neighbor, and so $v_1 \in S$. But then v_1 must have a neighbor in S. Thus, a leaf neighbor of v_1 is in S, contradicting the fact that at most one vertex from this underlying double star subgraph is in S. Hence, $\gamma_t(T) \geq 2q$. As observed earlier, $\gamma_t(T) \leq 2q$. Consequently, $\gamma_t(T) = 2q$. □

Claim 5. $\gamma_{\rm td}(T) = 3q$.

Proof. By Theorem 6 and Claim 4, we have $\gamma_{\rm td}(G) \leq \frac{3}{2}\gamma_t(G) = \frac{3}{2} \times 2q = 3q$. Suppose, to the contrary, that $\gamma_{\rm td}(T) < 3q$, and let S be a $\gamma_{\rm td}$-set of T. We note that every support vertex of T must be in S. Since $\gamma_{\rm td}(T) < 3q$, it follows that there exists a double star in the underlying forest H_q having at most two vertices in S. Without loss of generality, assume that the double star in H_q contributing at most two vertices to S has centers labeled u_1 and v_1, that is, $|(N_T[u_1] \cup N_T[v_1]) \cap S| \leq 2$. If both u_1 and v_1 are in S, then by assumption, these are the only two vertices from $N_T[u_1] \cup N_T[v_1]$ in S. But then u_1 and v_1 are in a component of order 2 in $T[S]$, contradicting the fact that S is a triad dominating set of T. Hence, at most one of u_1 and v_1 is in S. Assume, without loss of generality, $u_1 \notin S$. Again since every support vertex of T must be in S, it follows that u_1 is not a support vertex of T. By construction of T, the vertex v_1 must be a terminal support vertex with u_1 as its only nonleaf neighbor, and so $v_1 \in S$. Since v_1 must be in a large component of S, we infer that two leaf neighbors of v_1 are in S, contradicting our earlier supposition that the double star with centers labeled u_1 and v_1 contains at most two vertices in S. Hence, $\gamma_{\rm td}(T) \geq 3q$. As observed earlier, $\gamma_{\rm td}(T) \leq 3q$. Consequently, $\gamma_{\rm td}(T) = 3q$. □

By Claims 4 and 5, we have $\gamma_{\rm td}(T) = 3q = \frac{3}{2} \times 2q = \frac{3}{2}\gamma_t(T)$, completing the proof of Lemma 1. □

Theorem 8. *Let T be a tree with order $n \geq 3$. Then $\gamma_{\rm td}(T) = \frac{3}{2}\gamma_t(T)$ if and only if T is the star $K_{1,n}$ or $T \in \mathcal{T}$.*

Proof. We first note that the result holds for stars $T = K_{1,n-1}$ of order $n \geq 3$, as $\gamma_t(T) = 2$, and $\gamma_{\rm td}(T) = 3 = \frac{3}{2}\gamma_t(T)$. Henceforth, we may assume that T is not a star. Thus, T has order $n \geq 4$ and $\mathrm{diam}(T) \geq 3$. If $T \in \mathcal{T}$, then by Lemma 1, we have $\gamma_{\mathrm{td}}(T) = \frac{3}{2}\gamma_t(T)$, as desired.

Next assume that T is a tree of order $n \geq 4$ with $\gamma_{\mathrm{td}}(T) = \frac{3}{2}\gamma_t(T)$, and let S be a γ_t-set of T. We will show that $T = T_q \in \mathcal{T}$ for some $q \geq 1$. By Theorem 7(a), we have $T[S] = qK_2$, where $q = \frac{1}{2}\gamma_t(G)$. Label the edges of $T[S]$ as u_iv_i for $i \in [q]$. Thus, $\gamma_t(T) = 2q$ for some integer $q \geq 1$, and by assumption, $\gamma_{\mathrm{td}}(T) = \frac{3}{2}\gamma_t(T) = 3q$.

Let $U_i = \mathrm{epn}_T(u_i, S)$ and $V_i = \mathrm{epn}_T(v_i, S)$ for $i \in [q]$. By Theorem 7(c), every vertex in S has an S-external private neighbor, and so $U_i \neq \emptyset$ and $V_i \neq \emptyset$ for all $i \in [q]$. Since T is a tree, no pair of adjacent vertices u_i and v_i share a common neighbor for $i \in [q]$. Moreover, by Theorem 7(b), every independent subset of S is a packing, and so no pair of nonadjacent vertices in S share a common neighbor. Hence, since S is a γ_t-set in T, the set $\{U_i, V_i : i \in [q]\}$ is a partition of $V(T) \setminus S$.

Since T is a tree, there are no edges in the induced subgraph $T[U_i \cup V_i]$ for $i \in [q]$ (else T would have a cycle). Hence, for each $i \in [q]$, the induced subgraph $F_i = T[U_i \cup V_i \cup \{u_i, v_i\}]$ is a double star. Let

$$H_q = \bigcup_{i=1}^{q} F_i \quad \text{and} \quad L = \bigcup_{i=1}^{q} (U_i \cup V_i),$$

that is, H_q is the forest consisting of the union of these q double stars and L is the set of leaves of H_q. If $q = 1$, then $T = F_1$ is a double star, and so $T = T_1 \in \mathcal{T}$, as desired. Hence, we may assume that $q \geq 2$. Since T is a tree, T is connected by exactly $q - 1 \geq 1$ edges in $T[L]$. Thus, T can be formed from the forest H_q by adding edges between the vertices of L in such a way to connect the vertices without forming a cycle.

All that remains to be shown is that at least one of Condition (a) and Condition (b) in the definition of the family $\mathcal{T}$ is satisfied. If every vertex in S is a support vertex, then T satisfies Condition (a). In this case, adopting our notation in the definition of the family $\mathcal{T}$, we have that $T = T_q \in \mathcal{T}$ and H_q is the underlying forest of T_q, yielding the desired result.

Hence, we may assume, without loss of generality, that u_1 is not a support vertex of T. It follows that every vertex in U_1 has a neighbor in L. Since T is a tree, we note that no two vertices in U_1 are adjacent to vertices in the same double star component of H_q (else T has a cycle). Thus, for each vertex u in U_1, we can select a neighbor u' of u such that u' is a leaf in a double star subgraph F_i of H_q for some $i \in [q]$ and $i \neq 1$ and no other vertex of U_1 has a neighbor in F_i. Let X be the set of these selected vertices. We note that $|X| = |U_1|$.

We show next that v_1 is a terminal support vertex in T. Suppose to the contrary, that v_1 is not a terminal support vertex in T. Thus, there exists a vertex, say v, in V_1 such that v has a neighbor $w \in L$ in the tree T and $w \in F_j$ for some $j \in [q]$ and $j \neq 1$. By our previous comments, since T is a tree, we note that $w \notin X$. We now build a triad dominating set of T. For each double star F_i, $i \neq 1$, in the forest H_q that does not have a vertex in $X \cup \{w\}$, we randomly choose a vertex from $U_i \cup V_i$

and label the collection of these vertices as X'. We note that $|X \cup \{w\}| + |X'| = q-1$. The set $D = (S \setminus \{u_1\}) \cup \{v\} \cup (X \cup \{w\}) \cup X'$ is a triad dominating set of T. Hence,

$$\begin{aligned}\gamma_{\rm td}(T) \le |D| &= |(S \setminus \{u_1\}) \cup \{v\}| + |(X \cup \{w\}) \cup X'| \\ &= |S| - 1 + 1 + q - 1 \\ &= 2q + q - 1 \\ &< 3q \\ &= \tfrac{3}{2}\gamma_t(T),\end{aligned}$$

a contradiction. Thus, v_1 is a terminal support vertex, implying that Condition (b) is satisfied. Hence for each vertex in S, at least one of Condition (a) and Condition (b) is satisfied. Therefore adopting our notation in the definition of the family $\mathcal{T}$, we have that $T = T_q \in \mathcal{T}$ and H_q is the underlying forest of T_q, yielding the desired result and completing the proof of Theorem 8. □

Acknowledgements: Research of the second author was supported in part by the University of Johannesburg.

Statements and Declarations

The authors declare that no funds, grants, or other support were received during the preparation of this manuscript. The authors have no relevant financial or non-financial interests to disclose. All authors contributed to the study conception and design and commented on previous versions of the manuscript. All authors read and approved the final manuscript. This research did not generate or analyze any datasets.

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