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Research Article

Independent domination, order, size, and maximum degree

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Dedicated to Odile Favaron

A Waissekorn

A Waisskorn hán i in dr Hánd:
Wás sett i mit'm denn mácha?

Sett i's im Vogel z'frasse gah?
Do war mi Kernle vrlore!

Sett i's in Gold ifàsse lo?
Ja, tat dás mim Kerle o gfálle?

I waiss jetz, wás i mit'm mách:

Dr fichte Bode sett's Kernle há
Un dert sett's kime un triwe
Un im Summer
Hundert junge Kernle mir bringe!

(Georges Zink)

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Abstract: For a connected graph G with $n > 2\Delta$ vertices, m edges, and maximum degree at most $\Delta \geq 3$, we show $i(G) \leq \left(1 - \Omega\left(\frac{1}{\Delta^4}\right)\right) n - \frac{m}{\Delta} + O\left(\frac{1}{\Delta^2}\right)$ and discuss related problems.

Keywords: independent domination, regular graph, bound

AMS Subject classification: 05C69

1. Introduction

Odile Favaron's first mathematical paper [6] contained results on independent domination in graphs [19], a fruitful topic in graph theory. While the grains of domination in graphs were seeded in the 19th century with problems from recreational mathematics, its formal study was initiated by Berge [1] and Ore [21] in 1962. In [6] Odile and her coauthors proved a result that implies the following upper bound on the *independent domination number* $i(G)$ of a (finite, simple, and undirected) graph G of order n and minimum degree δ:

$$i(G) \leq n + \delta - \sqrt{n\delta}.$$

The independent domination number $i(G)$ of a graph G was formally defined in [7] as the minimum cardinality of a set I of vertices of G that is

- *independent*, meaning that no two vertices from I are adjacent in G, and
- *dominating*, meaning that every vertex of G that is not in I has a neighbor in I.

As detailed in [19], this notion received considerable and continuing attention with deep and fundamental results in recent years [2–4]. Also Odile returned regularly to this and closely related topics [5, 10–16].
A recurrent feature of domination in graphs is that very simple observations lead to first results that motivate challenging mathematical problems. The present paper starts from the following simple observation and illustrates the mentioned feature quite well:

Let Δ be an integer at least 3. Let G be a graph with n vertices, m edges, and maximum degree at most Δ. If I is a minimum independent dominating set of G, then all edges of G are incident with vertices in $V(G) \setminus I$, which implies $m \leq \Delta(n - i(G))$ or, equivalently,

$$i(G) \leq n - \frac{m}{\Delta}. \tag{1.1}$$

It is easy to deduce that (1.1) is satisfied with equality if and only if every component of G is isomorphic to K_1 or $K_{\Delta,\Delta}$; for the sake of completeness, we give a brief self-contained argument in Section 2.
Our simple observation immediately motivates the following challenging question:

> *How much can (1.1) be improved provided that G is connected and $n > 2\Delta$, that is, once the two connected extremal graphs are excluded.*

Note that next to the order n, the upper bound in (1.1) involves the size m as well as a bound Δ on the maximum degree. There are related known results. Dankelmann et al. [8] determined the largest size m of a graph of given order n and independent domination number, which implies a best possible upper bound on the independent domination number in terms of n and m alone. Haviland [18] refined this result for graphs not containing generalized claws. Concerning upper bounds in terms of the order and the maximum degree, regular graphs received special attention. Rosenfeld [22] showed that $i(G) \leq \frac{n}{2}$ for every connected Δ-regular graph G of order n with equality if and only if G is $K_{\Delta,\Delta}$. Similarly as above, it is natural to ask how much Rosenfeld's bound can be improved once the extremal graph is excluded. For $\Delta = 3$, Lam et al. [20] gave the best-possible answer by showing that $i(G) \leq \frac{2}{5}n$ for every connected cubic graph G of order n at least 8. Answering a question posed by Goddard and Henning [17], Cho et al. [2] recently showed

$$i(G) \leq \frac{\Delta - 1}{2\Delta - 1}n \tag{1.2}$$

for every connected Δ-regular graph G of order n that is distinct from $K_{\Delta,\Delta}$. Further related deep results for non-regular graphs were obtained by Cho et al. [3].
We derive our following main result as a consequence of (1.2).

Theorem 1. *Let Δ be an integer at least 3. If G is a connected graph with $n > 2\Delta$ vertices, m edges, and maximum degree at most Δ, then*

$$i(G) \leq \left(1 - \Omega\left(\frac{1}{\Delta^4}\right)\right)n - \frac{m}{\Delta} + O\left(\frac{1}{\Delta^2}\right). \tag{1.3}$$

All proofs are postponed to Section 2. For the two asymptotic terms in (1.3), our proof actually yields detailed specific forms. In view of Theorem 1, we propose the following problem:

> *Let Δ be an integer at least 3. Determine all a_Δ, b_Δ, and c_Δ such that*
>
> $$i(G) \leq a_\Delta n - b_\Delta m + c_\Delta$$
>
> *holds for every connected graph G with $n > 2\Delta$ vertices, m edges, and maximum degree at most Δ.*

In order to prove Theorem 1, we embed the graph G in a connected Δ-regular graph G' and exploit (1.2). The embedding relies on the following result, which we obtain as a corollary of a result of Erdős and Kelly [9]. Recall that $d_G(u)$ denotes the degree of the vertex u in the graph G.

Corollary 1. *Let Δ be an integer at least 3. If G is a connected graph of order $n > 2\Delta$ and maximum degree at most Δ with $\sum\limits_{u \in V(G)} (\Delta - d_G(u)) = k$, then there is a connected Δ-regular graph G' of order at most*

$$n + \left\lfloor \frac{k}{\Delta} \right\rfloor + \Delta + 2$$

that contains G as an induced subgraph.

We proceed to the proofs.

2. Proofs

Firstly, we present a brief self-contained argument that K_1 and $K_{\Delta,\Delta}$ are the only connected extremal graphs for (1.1). Clearly, both graphs satisfy (1.1) with equality. Now, suppose that G is a connected graph with n vertices, m edges, maximum degree at most Δ, and independent domination number i such that $m = \Delta(n-i)$. If $m = 0$, then, trivially, the graph G is K_1. Now, suppose that $m > 0$. For a contradiction, we suppose that G is not $K_{\Delta,\Delta}$. Let I be a minimum independent dominating set of G, and let $R = V(G) \setminus I$. Since every vertex in R is incident with at most Δ edges, the equality $m = \Delta(n-i) = \Delta|R|$ implies that all vertices in R have degree Δ and that R is independent, that is, the graph G is bipartite with partite sets I and R. Since every vertex in I is incident with at most Δ edges, counting the edges between I and R yields $|I| \geq |R|$. Since R is an independent dominating set of G, we obtain $|I| \leq |R|$. Altogether, it follows that $|I| = |R|$ and that G is Δ-regular. Since G is not $K_{\Delta,\Delta}$, there are non-adjacent vertices $x \in I$ and $y \in R$, and the set Z of vertices z in R with $N_G(y) = N_G(z)$ contains less than Δ vertices. Now, the set $(I \setminus N_G(y)) \cup Z$ is an independent dominating set containing less than $|I|$ vertices, which is a contradiction. Hence, the only connected extremal graphs for (1.1) are K_1 and $K_{\Delta,\Delta}$.

Secondly, we prove the embedding statement.

Proof of Corollary 1. Let Δ and G be as in the statement. Let p be such that $n+p$ is the minimum order of a connected Δ-regular graph G' that contains G as an induced subgraph. Erdős and Kelly [9] showed that p is the smallest integer such that

(a) $p\Delta \geq k$,

(b) $p^2 - (\Delta+1)p + k \geq 0$,

(c) $p \geq \max\{\Delta - d_G(u) : u \in V(G)\}$, and

(d) $(p+n)\Delta$ is even.

Let $q \in \left\{\left\lfloor \frac{k}{\Delta} \right\rfloor + \Delta + 1, \left\lfloor \frac{k}{\Delta} \right\rfloor + \Delta + 2\right\}$ be such that $(q+n)\Delta$ is even. Clearly,

$$\begin{aligned} q\Delta &\geq k, \\ q^2 - (\Delta+1)q + k &= q(q-(\Delta+1)) + 1 \geq \left(\left\lfloor \frac{k}{\Delta} \right\rfloor + \Delta + 1\right) \left\lfloor \frac{k}{\Delta} \right\rfloor + k \geq 0, \text{ and} \\ q &\geq \Delta + 1 \geq \max\{\Delta - d_G(u) : u \in V(G)\}. \end{aligned}$$

Hence q satisfies (a), (b), (c), and (d). It follows that $p \leq q \leq \left\lfloor \frac{k}{\Delta} \right\rfloor + \Delta + 2$, which completes the proof. □

Thirdly, we prove (a detailed version) of our main result.

Theorem 2. *Let Δ be an integer at least 3. If G is a connected graph with $n > 2\Delta$ vertices, m edges, independent domination number i, and maximum degree at most Δ, then*

$$i \leq \left(1 - \frac{1}{4\Delta^4 - 10\Delta^3 + 11\Delta^2 + 2\Delta - 4}\right) n - \frac{m}{\Delta} + \frac{2(\Delta+2)(\Delta-1)}{4\Delta^4 - 10\Delta^3 + 11\Delta^2 + 2\Delta - 4}. \quad (2.1)$$

Proof. Let I be a minimum independent dominating set and let $R = V(G) \setminus I$. Let $\text{Def}(G) = \{u \in V(G) : d_G(u) < \Delta\}$. By (1.1), the graph G contains at most $\Delta(n-i)$ edges. Let $x \geq 0$ be such that $m = \Delta(n-i) - x$. Let G contain exactly y edges with both endpoints in R. Since the degree sum of the vertices in R is $m+y$, it follows that

$$0 \leq \sum_{u \in R} (\Delta - d_G(u)) = \Delta(n-i) - (m+y) = x - y,$$

which also implies that

$$x \geq y. \quad (2.2)$$

Since there are $m-y$ edges between I and R, it follows that $i \geq \frac{m-y}{\Delta}$. Let $z \geq 0$ be such that $i = \frac{m-y+z}{\Delta}$. Since the degree sum of the vertices in I equals $m-y$, it follows that

$$\sum_{u \in I} (\Delta - d_G(u)) = \Delta i - (m-y) = z.$$

Altogether, we have $\sum\limits_{u \in V(G)} (\Delta - d_G(u)) = x - y + z$. Let G' be as in Corollary 1 for $k = x - y + z$. For $p = n(G') - n$, Corollary 1 implies $p \leq \frac{x-y+z}{\Delta} + \Delta + 2$.
If I' is an independent dominating set of G', then $I' \cap V(G)$ dominates all vertices in $V(G) \setminus \text{Def}(G)$ and can be extended to an independent dominating set of G by adding

at most $|\mathrm{Def}(G)| \leq k = x - y + z$ vertices. Using (1.2), we obtain

$$i \;\leq\; \frac{\Delta-1}{2\Delta-1}\left(n + \frac{x-y+z}{\Delta} + \Delta + 2\right) + x - y + z. \tag{2.3}$$

Let I_R be a maximal independent set of $G[R]$. Since $G[R]$ has exactly y edges, we obtain $|I_R| = n - i - y'$ for some $0 \leq y' \leq y$. Since each of the y' vertices in $R \setminus I_R$ has a neighbor in R, the set I_R dominates all but at most $(\Delta-1)y'$ vertices from I and can be extended to an independent dominating set of G by adding at most $(\Delta-1)y'$ vertices from I. This implies $i \leq n - i - y' + (\Delta-1)y' = n - i + (\Delta-2)y' \leq n - i + (\Delta-2)y$. Combining this with $i = \frac{m-y+z}{\Delta} = n - i - \frac{(x+y-z)}{\Delta}$ yields

$$z \leq x + (\Delta-1)^2 y. \tag{2.4}$$

Since $i = \frac{m-y+z}{\Delta} = n - i - \frac{1}{\Delta}(x+y-z)$, we have

$$i = \frac{n}{2} - \frac{1}{2\Delta}(x+y-z). \tag{2.5}$$

Combining (2.3) and (2.5), we obtain

$$\begin{aligned}
0 &\leq 2\Delta(2\Delta-1)\left(\left(\frac{\Delta-1}{2\Delta-1}\left(n + \frac{x-y+z}{\Delta} + \Delta + 2\right) + x - y + z\right) - \left(\frac{n}{2} - \frac{1}{2\Delta}(x+y-z)\right)\right) \\
&= -\Delta n + (4\Delta^2 + 2\Delta - 3)x - (4\Delta^2 - 2\Delta - 1)(y-z) + 2\Delta(\Delta-1)(\Delta+2) \\
&\overset{(2.4)}{\leq} -\Delta n + (8\Delta^2 - 4)x + (4\Delta^4 - 10\Delta^3 + 3\Delta^2 + 2\Delta)y + 2\Delta(\Delta-1)(\Delta+2) \\
&\overset{(2.2)}{\leq} -\Delta n + (4\Delta^4 - 10\Delta^3 + 11\Delta^2 + 2\Delta - 4)x + 2\Delta(\Delta-1)(\Delta+2),
\end{aligned}$$

which yields a lower bound on x. Together with $m = \Delta(n-i) - x$, this implies the stated bound. □

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Statements and Declarations

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