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Research Article

Some results on domination coloring and total domination coloring in graphs

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Abstract: A (proper) coloring of G is said to be domination coloring if each vertex dominates at least one color class and each color class dominated by some vertex. The minimum number of colors required for a domination coloring of G is called the domination chromatic number of G, and is denoted by $\chi_{dd}(G)$. For the graph G without isolated vertices, domination coloring of G is said to be the total domination coloring if each vertex dominates at least one color class contained in its open neighborhood. The minimum number of colors required for a total domination coloring of G is called the total domination chromatic number, and is denoted by $\chi_{td}(G)$. In this paper, we have constructed graphs G for arbitrary values of $\chi(G)$ and $\chi_{dd}(G)$, as well as $\chi(G)$ and $\chi_{td}(G)$. An upper bound for domination chromatic number (total domination chromatic number) in terms of dominated chromatic number and domination number (total domination number) is obtained. An upper bound for domination chromatic number in terms of maximum degree and order, as well as in terms of diameter of the graph is obtained. We have also characterized graphs of order n with $\chi_{dom} = n - 1$, $\chi_{dd} = 3$ and $\chi_{td} = 3$.

Keywords: domination coloring, domination chromatic number, total domination coloring, total domination chromatic number.

AMS Subject classification: 05C15, 05C69

1. Introduction

Two most significant concepts which laid foundation for modern graph theory are vertex adjacency and partitioning of vertex set. The present paper deals with the second one. Different ideas of partitioning vertex set lead to the concepts of coloring,

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domination, covering and so on of graphs. Each one of them finds lot of applications in many interdisciplinary as well as multidisciplinary areas like operation research, discrete mathematics, computer science, information technology, biology, medicine and so on. All graphs mentioned in this paper are simple, finite and undirected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Cardinality of $V(G)$ is the order of G. Two vertices u and v are adjacent ($u \sim v$) in G if there is an edge between u and v in G. $N(u)$ denotes the set of all adjacent vertices of u called as the *open neighborhood* of u and cardinality of $N(u)$ is the degree of u. The *minimum* and *maximum degree* of G are denoted by $\delta(G)$ and $\Delta(G)$ respectively . The *closed neighborhood* of the vertex u is $N(u) \cup \{u\}$. A subset I of $V(G)$ is said to be an *independent*(or vertex independent) if no two vertices of I are adjacent. The maximum cardinality of an independent set of G is called *independence number* of G, and is denoted by $\alpha(G)$. For more basic terminology and results on basic graph theory, we refer to [3, 9].

A subset D of $V(G)$ is said to be a *dominating set* of G if every vertex in $V(G) \setminus D$ has neighbor (or adjacent vertex) in D. The minimum cardinality of a dominating set of G is called the *domination number* of G, denoted by $\gamma(G)$. A dominating set D_t of G is said to be *total dominating set* if each vertex of D_t has a neighbor in D_t. The minimum cardinality of a total dominating set of G is called the *total domination number* of G, denoted by $\gamma_t(G)$. For more details on domination, refer to [10]. A *(proper) coloring* of G is the assignment of colors to the vertices of G such that no two adjacent vertices receive the same color. The minimum number of colors used in a coloring of G is called the *chromatic number* of G, denoted by $\chi(G)$. Let $\mathcal{C} = (V_1, V_2, \ldots, V_k)$ be a coloring of G. The sets V_i $(1 \leq i \leq k)$ are said to be the color classes with respect to the coloring $\mathcal{C}$. A vertex u is said to dominate the color class V_i if either $V_i = \{u\}$ or u is adjacent to all vertices of V_i. A color class V_i is said to be dominated by the vertex u, if u is adjacent to all the vertices of V_i. The color assigned to the vertex v is denoted by $col(v)$. For more details on graph coloring, refer to [2, 4].

There are different types of graph coloring parameters based on domination properties of graphs in the literature such as dominator coloring [7, 8], dominated coloring [17], total dominator coloring [12], domination coloring [18], total domination coloring [6], to mention a few. A coloring of G is said to be *dominator coloring* if each vertex dominates at least one color class. The minimum number of colors required for a dominator coloring of G is called the *dominator chromatic number* of G, denoted by $\chi_d(G)$. It was first studied in detail by Gera [8]. *Dominated coloring* of graphs was introduced by Merouane et al. [17] defined by a coloring of G in which each color class is dominated by some vertex. The minimum number of colors required for a dominated coloring of G is called the *dominated chromatic number* of G, denoted by $\chi_{dom}(G)$. A coloring of G is said to be a *domination coloring*, if each vertex dominates at least one color class and each color class is dominated by some vertex. The minimum number of colors required for a domination coloring of G is called the *domination chromatic number* of G, denoted by $\chi_{dd}(G)$. It was introduced by Yangyang Zhou et al. [18]. For a graph G without isolated vertices, a domination coloring of G is said to be a *total domination coloring* if each vertex dominates at least

one color class contained in its open neighborhood. The minimum number of colors required for a total domination coloring of G is called the *total domination chromatic number*, denoted by $\chi_{td}(G)$. It was introduced by Chithra and Joseph [6]. One can also refer to [11, 13–16] for recent results on dominator and dominated coloring of graphs.
The following results are used to prove some of our main results.

Theorem 1. *[5] $\chi_d(G) = 3$ if and only if G is in $\mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6$.*

We will give the definitions of the families $\mathcal{F}_i$ $(0 \leq i \leq 6)$ of graphs introduced by Chellali and Maffray [5], since we will be using some families of graphs in our results.

- $\mathcal{F}_0 = \{K_3\}$.
- $\mathcal{F}_1 = \{K_{r,s} \cup K_1 / r \geq 1, s \geq 1\}$.
- A graph G is in class $\mathcal{F}_2$ if it has two non-adjacent vertices a, b such that $V(G) \backslash \{a, b\}$ is an independent set and each of the two sets $N(a) \backslash N(b)$ and $N(b) \backslash N(a)$ is non-empty (the set $N(a) \cap N(b)$ may be empty or not).
- A graph G is in class $\mathcal{F}_3$ if $G = H \bigoplus S$ where H is any bipartite graph with non-empty edge set and $S = K_p$ for any $p \geq 1$.
- A graph G is in class $\mathcal{F}_4$ if its vertex set can be partitioned into five stable (independent) sets X_1, X_2, X_3, X_4, Z such that: each of $X_1, ..., X_4$ is not empty, every vertex of X_i is adjacent to every vertex of X_{i+1} for $i = 1, \cdots, 3$, every vertex of Z is adjacent to every vertex of $X_2 \cup X_3$, $Z \cup X_1 \cup X_4$ is a stable set, every vertex of X_1 has a non-neighbor in X_3, and every vertex of X_4 has a non-neighbor in X_2.
- A graph G is in class $\mathcal{F}_5$ if its vertex set can be partitioned into five non-empty sets $\{x\}, X_1, X_2, X_3, X_4$ such that: for $i = 1, \ldots, 3$, every vertex of X_i is adjacent to every vertex of X_{i+1}, $X_1 \cup X_3$ and $X_2 \cup X_4$ are stable sets, and x is adjacent to every vertex of $X_1 \cup X_4$ (there may be edges between x and $X_2 \cup X_3$ and between X_1 and X_4).
- A graph G is in class $\mathcal{F}_6$ if its vertex set can be partitioned into six non-empty sets $A_1, A_2, A_3, B_1, B_2, B_3$ such that: for $i = 1, \ldots, 3$, $A_i \cup B_i$ is a stable set, every vertex of A_i is adjacent to all of $A_{i+1} \cup B_{i+1} (modulo\ 3)$, every vertex of B_i is adjacent to all of $A_{i+2} \cup B_{i+2} (modulo\ 3)$, and every vertex of A_i has a non-neighbor in $B_{i+2} (modulo\ 3)$.

Some of the other results are used as follows.

Theorem 2. *[1] Let G be a connected graph of order n. Then $\chi_d(G) = n - 1$ if and only if one of the following holds:*
(i) $G \neq K_n$ and K_{n-1} is a subgraph of G.

(ii) $V(G) = V_1 \cup \{u, v\}$, where $\langle V_1 \rangle = K_2$, $deg(u) = 1$, $uv \in E(G)$ and v is nonadjacent to at least one vertex of V_1.

Theorem 3. *[7] Let G be connected graph. Then $\chi_d(G) = 2$ if and only if G is complete bipartite.*

Theorem 4. *[18] Let G be connected graph. Then $\chi_{dd}(G) = 2$ if and only if G is complete bipartite.*

Theorem 5. *[6] Let G be connected graph. Then $\chi_{td}(G) = 2$ if and only if G is complete bipartite.*

Theorem 6. *[7] Let G be connected graph of order n. Then $\chi_d(G) = n$ if and only if G is K_n.*

Theorem 7. *[17] Let G be connected graph of order n. Then $\chi_{dom}(G) = n$ if and only if G is K_n.*

Theorem 8. *[6] For any graph G, $\chi_d(G) \leq \chi_{dd}(G) \leq \chi_{td}(G)$*

In section 2, we construct graphs G for arbitrary values of $\chi(G)$ and $\chi_{dd}(G)$, as well as $\chi(G)$ and $\chi_{td}(G)$. In section 3, we obtain an upper bound for domination chromatic number (total domination number) in terms of dominated chromatic number and domination number (total domination number), and an upper bound for domination chromatic number in terms of maximum degree and order, as well as in terms of diameter of a graph is obtained. In section 4, we characterize graphs of order n with $\chi_{dd} = 3$ and $\chi_{td} = 3$ and $\chi_{dom} = n - 1$.

2. Realization results

In this section, we construct graphs G for arbitrary values of $\chi(G)$ and $\chi_{dd}(G)$, as well as $\chi(G)$ and $\chi_{td}(G)$.

Theorem 9. *For any integers a and b with $a \geq b \geq 3$, there exists a graph with domination chromatic number a and chromatic number b.*

Proof. Let a and b be integers with $a \geq b \geq 3$. Then by the division algorithm, there exists a positive integer q and non-negative integer r such that $a = q \cdot b + r$, where $0 \leq r < b$. Now we shall construct a graph G with $\chi_{dd}(G) = a$ and $\chi(G) = b$. The required graph constructed as follows in two cases.

Case 1. when b divides a $(r = 0)$.
Consider q vertex disjoint copies of complete graph K_b. For $1 \leq i \leq q$, let the i^{th} copy of K_b be denoted by K_b^i. Let $V(K_b^i) = \{v_{(i-1)b+1}, v_{(i-1)b+2}, \ldots, v_{(i-1)b+b}\}$. Now

consider $q-1$ vertex disjoint copies of $K_2 = x_iy_i$ and join x_i to $v_{(i-1)b+2}$ and join y_i to v_{ib+1}, for all $1 \leq i \leq q-1$. The resulting graph is the graph G_1. An example of a graph G_1 for $a = 12$ and $b = 4$ as shown in Figure 1. It is very clear that $\chi(G_1) \geq b$ since G_1 has a clique of order b. We can see that the $\mathcal{C}$ be a coloring of G_1 such that $col(v_{(i-1)b+k}) = k$ (for $1 \leq k \leq b$), $col(x_i) = 1$ and $col(y_i) = 2$ is a proper coloring of G_1 with b number of colors. Thus, $\chi(G_1) = b$. We shall now establish a domination coloring of G_1.

Claim 1. *In any domination coloring of G_1, the color assigned to a vertex in K_b^i cannot be assigned to any vertex in the other copy of K_b.*

Proof of the claim. Suppose there is vertex c in K_b^1 and a vertex d in K_b^2 such that they receive same color in a domination coloring $\mathcal{C}_1$ of G_1. Then the color class containing c and d is not dominated by any vertex in G_1, which will contradict that $\mathcal{C}_1$ is a domination coloring of G_1. Hence the claim.

Thus by Claim 1, in any domination coloring of G_1, every vertex of K_b^i has to receive a distinct color. Thus, $\chi_{dd}(G_1) \geq qb$. Consider the coloring $\mathcal{C}_2$ of G_1 such that $col(v_{(i-1)b+k}) = (i-1)b+k$ (for $1 \leq k \leq b$ and $1 \leq i \leq q$), $col(x_i) = col(v_{(i-1)b+3})$ and $col(y_i) = col(v_{(i-1)b+2})$. It can be easily checked that the coloring $\mathcal{C}_2$ is a domination coloring of G_1 with qb number of colors used. Thus, $\chi_{dd}(G_1) = qb = a$ and G_1 is the required graph.

Case 2. when b does not divides a ($r \neq 0$).

Consider the graphs G_1 (constructed in Case 1), K_r and $K_2 = xy$. Let $V(K_r) = \{u_1, u_2, \ldots, u_r\}$. Now the graph G_2 formed by the union of G_1, K_r, $K_2 = xy$ and joining x to $v_{(q-1)b+2}$ and y to u_1. Since $r < b$ and using similar arguments that we used to prove $\chi(G_1) = b$ and $\chi_{dd}(G_1) \geq qb$, we can see that $\chi(G_2) = b$ and $\chi_{dd}(G_2) \geq qb + r$ (can be proved using Claim 1). Now consider the coloring $\mathcal{C}_2$ of G_2 in which the vertices of G_1 are assigned colors as defined in the coloring $\mathcal{C}_2$ (in case 1), $col(u_j) = qb + j$ (for $1 \leq j \leq r$), $col(x) = col(v_{(q-1)b+3})$ and $col(y) = qb + r$. It can be checked that $\mathcal{C}_3$ is a domination coloring of G_2 with $qb + r$ number of colors used. Thus $\chi_{dd}(G_2) = qb + r = a$ and G_2 is the required graph. □

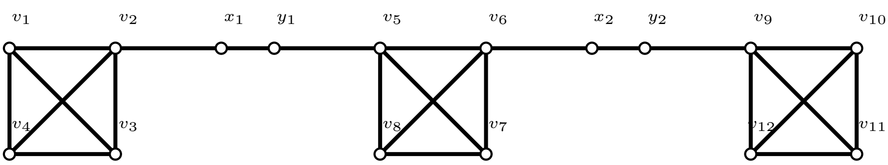

Figure 1. **The graph G_1 with $a = 12$ and $b = 4$**

Theorem 10. *For any integers a and b with $a \geq b \geq 4$, there exists a graph with total domination chromatic number a and chromatic number b.*

Proof. Let a and b be integers with $a \geq b \geq 3$. Then by the division algorithm, there exists a positive integer q and non-negative integer r such that $a = q \cdot b + r$, where $0 \leq r < b$. Now we shall construct a graph G with $\chi_{dd}(G) = a$ and $\chi(G) = b$. The required graph is discussed in cases as follows.

Case 1. When $r = 0$.
Then the graph G_1 constructed in Theorem 9 is the required graph since $\chi(G_1) = b$ and the domination coloring $\mathcal{C}_2$ (described in Theorem 9) of G_1 is also a total domination coloring of G_1. Thus, $\chi_{td}(G_1) = a$ (since by Theorem 8, $\chi_{dd}(G_1) \leq \chi_{td}(G_1)$).

Case 2. When $r = 1$.
Consider the graph G_1 constructed in Theorem 9. Now the graph G_3 is such that $V(G_3) = V(G_1) \cup \{u, v\}$ and $V(G_3) = E(G_1) \cup \{uv, uv_{(q-1)b+2}\}$. By Claim 1 of Theorem 9 and Theorem 8, we have $\chi_{td}(G_3) \geq \chi_{dd}(G_3) \geq qb$. Since v is pendent vertex of G_3, in any total domination coloring of G_3, vertex u has to be given a unique distinct color. Thus, $\chi_{td}(G_3) \geq qb + 1$. Now we shall describe a total domination coloring of G_3 using $qb + 1$ number of colors. Consider the coloring $\mathcal{C}_4$ of G_3 in which the vertices of G_1 receive the colors as described in coloring $\mathcal{C}_2$ (in Theorem 9), $col(u) = qb + 1$ and $col(v) = col(v_{(q-1)b+2})$. It can checked that $\mathcal{C}_4$ is a total domination coloring of G_3. Thus, $\chi_{td}(G_3) = qb + 1 = a$.

Case 3. When $r > 1$.
Then the graph G_2 constructed in Theorem 9 is the required graph since $\chi(G_2) = b$ and the domination coloring $\mathcal{C}_3$ (described in Theorem 9) of G_2 is also a total domination coloring of G_2. Thus, $\chi_{td}(G_2) = a$. □

3. Bounds

In this section, we obtain an upper bound for domination chromatic number (total domination number) in terms of dominated chromatic number and domination number (total domination number), and an upper bound for domination chromatic number in terms of maximum degree and order, as well as in terms of diameter of a graph is obtained.
A graph G is a bipartite graph with $V(G) = V_1 \cup V_2$ such that $V_1 = \{u_1, u_2, u_3, ..., u_r\}$ and $V_2 = \{v_1, v_2, v_3, ..., v_s\}$ where $r \leq s$ is said to belong to the family $\Im$ if the following conditions are satisfied.

1. $N(u_1) = V_2$ and
2. $N(u_i) = \{v_i\}$ for $i = 2, 3, ..., r$.

An example of a graph belong to the family $\Im$ is shown in Figure 2.

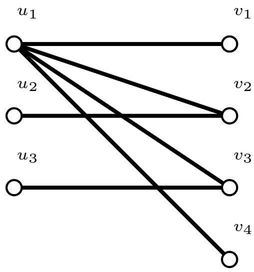

Figure 2. **A graph** $G \in \Im$

Theorem 11. *Let $G = (V_1 \cup V_2, E)$ be any bipartite graph of order n with maximum degree Δ. Then $\chi_{dd}(G) \leq n + 1 - \Delta$ and the bound is sharp.*

Proof. Let v be the vertex of G with $deg(v) = \Delta$. Now assigning a common color to all vertices of $N(v)$ and a distinct new color to the vertices of $V(G) \backslash N(v)$ gives rise to a domination coloring of G using $n + 1 - \Delta$ number of colors. This implies that $\chi_{dd}(G) \leq n + 1 - \Delta$.
This bound is sharp for the graphs belonging to the family $\Im$. Since $V_1 \backslash \{u_1\}$ is the set of $r - 1$ pendent vertices and $N(u_1) = V_2$. This implies at least $(r - 1) + 2$ colors are required for any domination coloring of G. Now the coloring $(\{u_1\}, \{u_2\}, ..., \{u_r\}, V_2)$ is a domination coloring of G using $r + 1$ (where $r + 1 = n + 1 - \Delta = r + s + 1 - s$) number of colors. □

Proposition 1. *For a bi-star $S_{p,q}$,* $\chi_{dd}(S_{p,q}) = \begin{cases} 3 & \text{if } \ p = 1 \ \textit{ or } \ q = 1 \\ 4 & \text{otherwise.} \end{cases}$

Proof. Let $S_{p,q}$ be a bi-star with vertex set $V(S_{p,q}) = \{a, b, u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_q\}$, where $N(a) = \{u_1, u_2, \ldots, u_p, b\}$, $N(b) = \{v_1, v_2, \ldots, v_q, a\}$ and $deg(u_i) = deg(v_j) = 1$ for $1 \leq i \leq p$ and $1 \leq j \leq q$. We have $\chi_{dd}(G) = 2$ if and only if G is complete bipartite (From Theorem 4). Hence, $\chi_{dd}(S_{p,q}) \geq 3$.
Suppose $p = 1$ or $q = 1$, then either $(\{b\}, \{u_1\}, \{a, v_1, v_2, \ldots, v_q\})$ or $(\{a\}, \{v_1\}, \{b, u_1, u_2, \ldots, u_p\})$ is domination coloring of G. So $\chi_{dd}(S_{p,q}) = 3$. Otherwise, $(\{a\}, \{b\}, \{u_1, \ldots, u_p\}, \{v_1, \ldots, v_q\})$ is a domination coloring of G with four number of colors. Thus, we have $\chi_{dd}(S_{p,q}) = 4$. □

Theorem 12. *For any graph G, $\chi_{dd}(G) \leq \chi_{dom}(G) + \gamma(G)$ and the bound is sharp. Further, if $\chi_{dd}(G) = \chi_{dom}(G) + \gamma(G)$, then in any χ_{dom}-coloring $\mathcal{C}$ of G, no color class in $\mathcal{C}$ is contained in any γ-set of G.*

Proof. Let G be any graph with a dominated coloring $\mathcal{C}$ and $S = \{x_1, x_2, ..., x_\gamma\}$ be a γ-set of G. We obtain a domination coloring of G by assigning new distinct color (not used in $\mathcal{C}$) to each vertex of S and the vertices of $V(G)\backslash S$ are colored as colored in the coloring $\mathcal{C}$. Hence, $\chi_{dd}(G) \leq \chi_{dom}(G) + \gamma(G)$. The bound is sharp for bi-stars $S_{p,q}$ with $p, q \geq 2$ since $\gamma(S_{p,q}) = 2$, $\chi_{dom}(S_{p,q}) = 2$ and $\chi_{dd}(S_{p,q}) = 4$.
Let $\chi_{dd}(G) = \chi_{dom}(G) + \gamma(G)$. On contrary assume that there is a χ_{dom}-coloring $\mathcal{C}_1$= $(V_1, V_2, \ldots, V_{\chi_{dom}})$ of G such that $V_1 \subseteq S$, for some γ-set S of G. Let $v \in V_1$. Consider the new coloring $\mathcal{C}_2$ such that all the vertices of $S\backslash\{v\}$ are assigned distinct new colors (not used in the coloring $\mathcal{C}_1$) and the remaining vertices are assigned colors as in the coloring $\mathcal{C}_1$. So the number of colors used in $\mathcal{C}_2$ is $\chi_{dom}(G) + \gamma(G) - 1$. We shall prove that $\mathcal{C}_2$ is a domination coloring of G. Clearly in the coloring $\mathcal{C}_2$, a vertex of S dominates itself and a vertex of $V(G)\backslash S$ dominates a vertex of S to which it is adjacent to. Also $\mathcal{C}_2$ is a dominated coloring of G. Thus, $\mathcal{C}_2$ is a domination coloring of G that uses less than $\chi_{dom}(G) + \gamma(G)$ number of colors, a contradiction. □

By following the same arguments used in Theorem 12 for a γ_t-set of G, we obtain the following result.

Corollary 1. *For any graph G, $\chi_{td}(G) \leq \chi_{dom}(G) + \gamma_t(G)$ and the bound is sharp. Further, if $\chi_{td}(G) = \chi_{dom}(G) + \gamma_t(G)$, then in any χ_{dom}-coloring $\mathcal{C}$ of G, no color class in $\mathcal{C}$ is contained in any γ_t-set of G.*

The bound $\chi_{td}(G) = \chi_{dom}(G) + \gamma_t(G)$ is attained for all bi-stars $S_{p,q}$ with $p, q \geq 2$.

Theorem 13. *Let G be a graph of order n with diameter k (where $k \geq 2$). Then $\chi_{dd}(G) \leq n - \lfloor \frac{k+1}{3} \rfloor$.*

Proof. Let G be a graph with diam$(G) = k$. Then there exists two vertices of G, say x and y such that the distance between x and y is k. For the ease of notation, let $x = u_1$ and $y = u_{k+1}$. Let $u_1u_2u_3 \ldots u_ku_{k+1}$ be a path $P : u_1u_{k+1}$ between u_1 and u_{k+1} of length k in G. We construct a sequence of sets A_i as follows. Let A_1 be the set containing first three vertices of P (i.e $A_1 = \{u_1, u_2, u_3\}$). Let A_2 be the set containing first three vertices of $P - A_1$ (i.e $A_2 = \{u_4, u_5, u_6\}$). Let A_3 be the set containing first three vertices of $P - (A_1 \cup A_2)$ (i.e $A_3 = \{u_7, u_8, u_9\}$). Continuing in this manner, we construct $\lfloor \frac{k+1}{3} \rfloor$ number of sets from the path $P : u_1u_{k+1}$. Let the sets be $A_1, A_2, \ldots, A_{\lfloor \frac{k+1}{3} \rfloor}$. Now we describe a domination coloring $\mathcal{C}$ of G as follows. Let two non-adjacent vertices of A_i (for $1 \leq i \leq \lfloor \frac{k+1}{3} \rfloor$) receive the color i. So number of colors used to color the non-adjacent vertices of A_i are $\lfloor \frac{k+1}{3} \rfloor$. Color all the remaining vertices of G (which are $n - 2\lfloor \frac{k+1}{3} \rfloor$ in number) by distinct new color. So the total number of colors used in $\mathcal{C}$ is $n - 2\lfloor \frac{k+1}{3} \rfloor + \lfloor \frac{k+1}{3} \rfloor = n - \lfloor \frac{k+1}{3} \rfloor$. Also it is very clear to see that $\mathcal{C}$ is a domination coloring of G. Thus, $\chi_{dd}(G) \leq n - \lfloor \frac{k+1}{3} \rfloor$. This bound is sharp for paths. In [18], the author's proved that, for paths of length $n \geq 2$, $\chi_{dd}(P_n) = 2 \cdot \lfloor \frac{n}{3} \rfloor + mod(n, 3)$. Since, $diam(P_n) = n - 1$, then by above upper bound, $\chi_{dd}(P_n) \leq n - \lfloor \frac{n}{3} \rfloor$. It is easy to verify, for any $n \geq 2$, $2 \cdot \lfloor \frac{n}{3} \rfloor + mod(n, 3) = n - \lfloor \frac{n}{3} \rfloor$. □

4. Characterization

4.1. Characterization of graphs with $\chi_{dd} = 3$ and $\chi_{td} = 3$

Lemma 1. *Let G be a graph of order n. Then $\chi_{dd}(G) = 3$ if and only if $\chi_d(G) = 3$ and there is a domination coloring of G with three colors.*

Proof. Let G be a graph with $\chi_{dd}(G) = 3$. This implies, there is a domination coloring of G with three colors and $\chi_d(G) \leq 3$. If $\chi_d(G) = 2$, then $G = K_{r,s}$ (by Theorem 3). Also, we have $\chi_{dd}(K_{r,s}) = 2$ (by Theorem 4). Therefore, $\chi_d(G) = 3$. Conversely, Suppose $\chi_d(G) = 3$ and there is a domination coloring of G with three colors (which imply that $\chi_{dd}(G) \leq 3$). Since $\chi_{dd}(G) \geq \chi_d(G) = 3$ (by Theorem 8), we have $\chi_{dd}(G) = 3$. □

Now in addition to the graph classes $\mathcal{F}_i$ $(0 \leq i \leq 6)$ described in Section 1, we define some sub-classes of the class $\mathcal{F}_4$ and also recall the classes $\mathcal{F}_2$, $\mathcal{F}_4$, as they will be used in our next result.

- $\mathcal{F}_4$ - class :A graph G is in class $\mathcal{F}_4$ if its vertex set can be partitioned into five independent sets V_1, V_2, V_3, V_4, Z such that: each of $V_1, ..., V_4$ is not empty, every vertex of V_i is adjacent to every vertex of V_{i+1} for $i = 1, \cdots, 3$, every vertex of Z is adjacent to every vertex of $V_2 \cup V_3$, $Z \cup V_1 \cup V_4$ is an independent set, every vertex of V_1 has a non-neighbor in V_3, and every vertex of V_4 has a non-neighbor in V_2.

- $\mathcal{F}_2$ - class :A graph G is in class $\mathcal{F}_2$ if it has two non-adjacent vertices a, b such that $V(G)\backslash\{a, b\}$ is an independent set and each of the two sets $N(a)\backslash N(b)$ and $N(b)\backslash N(a)$ is non-empty (the set $N(a) \cap N(b)$ may be empty or not).

- $\mathcal{F}'_4$ - class : A graph G is in class $\mathcal{F}'_4$ if the vertex set of G can be partitioned into independent sets V_1, V_2, V_3, V_4 and Z such that each V_i is non-empty for i=1,2,3 and 4. Every vertex of V_i is adjacent to every vertex of V_{i+1} for i=1,2 and 3. The set $V_1 \cup V_4 \cup Z$ is an independent set, every vertex of Z is adjacent to every vertex of $V_2 \cup V_3$, each vertex of V_1 has a non-neighbor in V_3 and each vertex of V_4 has non-neighbor in V_2 and there is a vertex in $V_2 \cup V_3$ adjacent to every vertex of $V_1 \cup V_4 \cup Z$.

- $\mathcal{F}''_4$ - class : A graph G is in class $\mathcal{F}''_4$ if the vertex set of G can be partitioned into independent sets V_1, V_2, V_3 and V_4 such that each V_i is non-empty for i = 1,2,3 and 4 with $\mid V_1 \mid = 1$ or $\mid V_4 \mid = 1$. Every vertex of V_i is adjacent to every vertex of V_{i+1} for i = 1, 2 and 3. The sets $V_1 \cup V_4$, $V_1 \cup V_3$ and $V_2 \cup V_4$ are independent sets.

Theorem 14. *$\chi_{dd}(G) = 3$ if and only if $G \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_3 \cup \mathcal{F}_4' \cup \mathcal{F}_4'' \cup \mathcal{F}_5 \cup \mathcal{F}_6$.*

Proof. Let G be a graph with $\chi_{dd}(G) = 3$. By Lemma 1, $\chi_d(G) = 3$ and there is a domination coloring of G with three colors. Chellali and Maffray [5] proved that $\chi_d(G) = 3$ if and only if G is in $\mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6$. It can be checked that the χ_d-colorings defined in [5] for the classes $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_3, \mathcal{F}_5, \mathcal{F}_6$ also a domination coloring with three colors. Thus, $G \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_3 \cup \mathcal{F}_5 \cup \mathcal{F}_6$. Now we are left with the graphs of $\mathcal{F}_2$ and $\mathcal{F}_4$. Let $G \in \mathcal{F}_2$. Let $a, b \in V(G)$ satisfying the properties of $\mathcal{F}_2$ and let $x \in N(a)\backslash N(b)$ and $y \in N(b)\backslash N(a)$ be arbitrary. Now we shall prove that if $G \in \mathcal{F}_2$, then $\chi_{dd}(G) \geq 4$ by the following two cases.

Case 1. $N(a) \cap N(b)$ is empty.
Then in any domination coloring of G, at least four colors are required to color the vertices a, x, b and y. Because if any two vertices of the set $\{a, x, b, y\}$ are given the same color, then the definition of dominated coloring fails. Thus, $\chi_{dd}(G) \geq 4$.

Case 2. $N(a) \cap N(b)$ is non-empty.
Let $z \in N(a) \cap N(b)$. Again in any domination coloring of G, at least four colors are required to color the vertices a, x, z, b and y. Thus, $\chi_{dd}(G) \geq 4$.
We shall now check with the graphs of $\mathcal{F}_4$. Let $G \in \mathcal{F}_4$ and $V(G)$ be partitioned into independent sets V_1, V_2, V_3, V_4 and Z satisfying the properties of $\mathcal{F}_4$. We have to find all the graphs in $\mathcal{F}_4$-class that have a domination coloring with three colors. If there is a vertex in $V_2 \cup V_3$ which is adjacent to every vertex of $V_1 \cup V_4$, then the coloring $(V_1 \cup V_4 \cup Z, V_2, V_3)$ is a domination coloring of G with three colors. This shows $G \in \mathcal{F}_4'$. So we shall now assume that there is no vertex in $V_2 \cup V_3$ which is adjacent to every vertex of $V_1 \cup V_4$(in other words, every vertex in V_2 has non-neighbor in V_4 and every vertex in V_3 has non-neighbor in V_1). Hence, let $G \in \mathcal{F}_4 - \mathcal{F}_4'$ be any graph such that it has a domination coloring $\mathcal{C}$ with three colors. We shall discuss the following two cases depending on the cardinality of Z.

Case 3. Z is non-empty.
If two vertices of Z receive different colors, then in any domination coloring of G at least two more new colors are required to color the vertices of $V_2 \cup V_3$ (since Z is adjacent to every vertex in $V_2 \cup V_3$), which is contradiction to the fact that $\chi_{dd}(G) = 3$. Hence, every vertex of Z receive same color. Again if two vertices of V_2 receive different colors, then in domination coloring of G at least two more new colors are required to color the vertices of $V_3 \cup Z$. Hence, every vertex of V_2 receive same color. By symmetry, every vertex of V_3 receive same color. Let us assign color 1 to all vertices of Z, color 2 to all vertices of V_2 and color 3 to all vertices of V_3. Since there is no vertex in G that dominates $V_1 \cup V_4$. This implies that there exists a vertex in $V_1 \cup V_4$ which has color 2 or 3. Suppose there is a vertex $u \in V_1$ which has color 3. This shows u must dominates color class 2, therefore every vertex of V_4 must have color 1. But no vertex of V_4 dominates color classes 1, 2 or 3. This shows there is no domination coloring with three colors. In this case $\chi_{dd}(G) > 3$.

Case 4. Z is empty.
Suppose there are two vertices x and y in V_2 such that $col(x) = 1$ and $col(y) = 2$. This

shows that every vertex of $V_1 \cup V_3$ has color 3. Every vertex of V_1 must dominates color class 1 or 2. Let $u_1 \in V_1$ dominates color class 2. This shows every vertex of V_4 has color 1 and must dominates color class 2. This implies that y is adjacent to every vertex of $V_1 \cup V_4$ (one of the property of graph class $\mathcal{F}_4'$). This contradicts that $G \in \mathcal{F}_4 - \mathcal{F}_4'$. Thus, every vertex of V_2 have same color. Similarly, by symmetry every vertex of V_3 have same color. So let us assign color 1 to V_2 and color 2 to V_3. Suppose $\mid V_1 \mid, \mid V_4 \mid \geq 2$. Since there is no vertex in G that dominates $V_1 \cup V_4$, there is a vertex in $V_1 \cup V_4$ that has color 1 or 2. Suppose there exists a vertex $v_1 \in V_1$ that has color 2. Let x,$y \in V_4$ be arbitrary. Both x and y does not dominates color classes 1 or 2. This show x and y dominates its own color classes and hence domination coloring requires more than three colors. This is a contradiction to that fact that $\chi_{dd}(G) = 3$. So we shall assume that $\mid V_1 \mid= 1$ or $\mid V_4 \mid= 1$ and discuss the following two cases. Let $\mid V_1 \mid= 1$ and $V_1 = \{u\}$. Clearly $V_1 \cup V_3$ is an independent set.

Case 5. $\{u\}$ is a color class.
This implies that $V_2 \cup V_4$ must be a color class. This shows $V_2 \cup V_4$ is an independent set. Thus, $G \in \mathcal{F}_4''$.

Case 6. $\{u\}$ is not a color class.
Assign color 1 to V_2 and color 2 to V_3. Then u must dominates color class 1. No vertex of V_4 has color 1. This shows every vertex of V_4 has color 3 and must dominates its own color class. This shows $\mid V_4 \mid = 1$ and hence, $V_2 \cup V_4$ is an independent set. Thus, $G \in \mathcal{F}_4''$.
]smallskip Conversely, if $G \in \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_3 \cup \mathcal{F}_5 \cup \mathcal{F}_6$, then the χ_d-coloring of G defined in [5] is also a domination coloring of G with three colors. If $G \in \mathcal{F}_4'$, then $(V_1 \cup V_4 \cup Z, V_2, V_3)$ is a domination coloring of G with three colors. If $G \in \mathcal{F}_4''$, then $(V_1, V_2 \cup V_4, V_3)$ is a domination coloring of G with three colors. Thus, by all these cases and by Theorem 4, $\chi_{dd}(G) = 3$. □

Lemma 2. *Let G be a graph without isolates. Then $\chi_{td}(G) = 3$ if and only if $\chi_{dd}(G) = 3$ and there is a total domination coloring of G with three colors.*

Proof. Let G be a graph with $\chi_{td}(G) = 3$. This implies there is a total domination coloring of G with three colors and $\chi_{dd}(G) \leq 3$. Suppose $\chi_{dd}(G) = 2$, then $G = K_{r,s}$ (by Theorem 4). But we have $\chi_{td}(K_{r,s}) = 2$ (by Theorem 5). This proves $\chi_{dd}(G) = 3$. Conversely, let $\chi_{dd}(G) = 3$ and there is a total domination coloring of G with three colors. Clearly $\chi_{td}(G) \leq 3$. Since $\chi_{td}(G) \geq \chi_{dd}(G)$, this implies $\chi_{td}(G) = 3$. □

We shall now define two subclasses $\mathcal{F}_5'$ and $\mathcal{F}_5''$ of $\mathcal{F}_5$.

- $\mathcal{F}_5'$ - class: A graph $G \in \mathcal{F}_5$ is in class $\mathcal{F}_5'$ if x is adjacent to every vertex of V_2 or x is adjacent to every vertex of V_3.
- $\mathcal{F}_5''$ - class: A graph $G \in \mathcal{F}_5 - \mathcal{F}_5'$ is in class $\mathcal{F}_5''$ if every vertex of V_1 adjacent to every vertex of V_4.

Theorem 15. *Let G be a graph without isolates. Then $\chi_{td}(G) = 3$ if and only if G is in $\mathcal{F}_3 \cup \mathcal{F}'_4 \cup \mathcal{F}'_5 \cup \mathcal{F}''_5 \cup \mathcal{F}_6$.*

Proof. Let G be a graph with $\chi_{td}(G) = 3$. By Lemma 2, $\chi_{dd}(G) = 3$ and there is a total domination coloring of G with three colors. We have already proved in Theorem 14 that $\chi_{dd}(G) = 3$ if and only if G is in $\mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_3 \cup \mathcal{F}'_4 \cup \mathcal{F}''_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6$. The total domination coloring is not defined for graphs in $\mathcal{F}_0 \cup \mathcal{F}_1$ (Since those graphs have isolated vertices). Also the χ_{dd}-colorings defined for the classes $\mathcal{F}_3, \mathcal{F}'_4$ and $\mathcal{F}_6$ in Theorem 14 is also a total domination coloring with three colors. Now we are left for which graphs in $\mathcal{F}''_4$ and $\mathcal{F}_5$, there is total domination coloring with three colors. Let $G \in \mathcal{F}''_4$ be arbitrary and $V_1 = \{u_1\}$. Let u_2, u_3, u_4 are vertices of V_2 , V_3 and V_4 respectively. Any total domination coloring of G requires four colors to color the vertices u_1, u_2, u_3 and u_4 (because of the properties of graph class $\mathcal{F}''_4$). This shows $\chi_{td}(G) > 3$. We shall discuss about the graphs $G \in \mathcal{F}_5$ in the following cases.
Case 1. x is adjacent to every vertex of V_2 or x is adjacent to every vertex of V_3.
In this case, $(\{x\}, V_1 \cup V_3, V_2 \cup V_4)$ is a total domination coloring of G with three colors. Thus, $G \in \mathcal{F}'_5$.
Case 2. x has non-neighbor in V_2 and x has non-neighbor in V_3.
Subcase 2.1. Every vertex of V_1 adjacent to every vertex of V_4.
Let $K_1 = V_2 \cap N(x)$ and $K_2 = V_2 \backslash N(x)$. In this case, $(K_1 \cup V_4, \{x\} \cup K_2, V_1 \cup V_3)$ is a total domination coloring of G with three colors. Thus, $G \in \mathcal{F}''_5$.
Subcase 2.2. There is a vertex in V_1 which has a non-neighbor in V_4.
Let u_1, u_2, u_3 and u_4 be vertices of V_1, V_2, V_3 and V_4 respectively such that u_1 and u_4 are non-adjacent and x is not adjacent to both u_2 and u_3. Assign color 1 to u_2 and color 2 to u_3. This shows x must dominates color class 3, this implies that $col(x) \neq 3$. Let $col(x) = 1$. This shows u_3 must dominates color class 3 and u_4 must dominates color class 2. This shows color of $u_1 \neq 2, 3$. Hence we have to assign a new color to u_1. This shows $\chi_{td}(G) > 3$.
Converse is obvious. □

4.2. Characterization of graphs with $\chi_{dom} = n - 1$

In Theorem 1, Arumugam et al. [1] characterized graphs with $\chi_d(G) = n - 1$. In this section, we will characterize graphs G with $\chi_{dom}(G) = n - 1$ in the next result.

Theorem 16. *If G be a connected graph of order n with vertex set $V(G)$. Then $\chi_{dom}(G) = n - 1$ if and only if $V(G) = H \cup \{v\}$, where $\langle H \rangle = K_{n-1}$ and $1 \leq deg(v) \leq n - 2$.*

Proof. Let G be a connected graph of order n with $\chi_{dom}(G) = n - 1$. Let $\mathcal{C} = (V_1, V_2, \ldots, V_{n-1})$ be any χ_{dom}-coloring of G. Exactly one color class has two vertices and remaining classes are singleton sets. Let $| V_i |= 1$ for $i = 1, 2, \ldots, n - 2$ and $V_{n-1} = \{u, v\}$, this implies u and v are non-adjacent. Let x dominates V_{n-1} and let G_1 be the graph induced on $V(G) - \{u, v\}$ i.e. $G_1 = \langle V(G) - \{u, v\} \rangle$. Suppose $\chi_{dom}(G_1) = m < n - 2$. Let $\mathcal{C}_1 = (U_1, U_2, \ldots, U_m)$ be any χ_{dom}-coloring of G_1. This implies $\mathcal{C}_2 = (U_1, U_2, \ldots, U_m, V_{n-1})$ is a dominated coloring of G with at most $n-1$ colors number of colors. This contradicts $\chi_{dom}(G) = n-1$. Hence, $\chi_{dom}(G_1) = n-2$.

Claim 2. *G_1 is connected.*

Proof of the claim. Suppose G_1 is disconnected. Let C_1 and C_2 are two components of G_1. Since G is connected, u or v has neighbors in both C_1 and C_2 (otherwise if u has neighbors only in C_1 and v has neighbors only in C_2 or vice-versa, we arrive at a contradiction to the fact G is connected). Let u have neighbors x in C_1 and y in C_2. Consider a coloring $\mathcal{C}_3 = (W_1, W_2, \ldots, W_{n-2})$, where $W_1 = \{x, y\}$, $W_2 = \{u, v\}$ and all remaining vertices spread over $W_3, W_4, \ldots, W_{n-2}$ such that each contains a single vertex. Obviously $\mathcal{C}_3$ is dominated coloring of G with at most $n-2$ number of colors. This is contradiction.
Thus, by Claim 2 and the fact that $\chi_{dom}(G_1) = n-2$, we conclude that $G_1 = K_{n-2}$ (by Theorem 7). Therefore, $G_1 = K_{n-2}$ and $x \in V(G_1)$. Now suppose both u and v have non-neighbors in $V(G_1)$. Let u_1 and v_1 be non neighbors of u and v respectively in G_1. Consider the coloring $\mathcal{C}' = (X_1, X_2, \ldots, X_{n-2})$, where $X_1 = \{u, u_1\}, X_2 = \{v, v_1\}$ and all remaining vertices spread over $X_3, X_4, \ldots, X_{n-2}$ such that each contains a single vertex. Clearly $\mathcal{C}'$ is a dominated coloring of G with at most $n-2$ number of colors (Since X_1 and X_2 is dominated by x and other classes are dominated by the vertex adjacent to it), a contradiction to the fact that $\chi_{dom}(G) = n-1$. This shows u or v is adjacent to all vertices of G_1. Let u be adjacent to every vertex of $V(G_1)$ and $H = V(G) - \{v\}$. Obviously $\langle H \rangle = K_{n-1}$. Since v is adjacent to x and non-adjacent to u, $1 \leq deg(v) \leq n-2$.
Converse is obvious. □

5. Open problems and discussions

In order to find necessary and sufficient conditions for graphs attaining the upper bound of Theorem 12 and Corollary 1, we pose the following conjectures.

Conjecture 1. If every χ_{dom}-coloring of G is such that no color class is contained in any γ-set of G, then $\chi_{dd}(G) = \chi_{dom}(G) + \gamma(G)$.

Conjecture 2. If every χ_{dom}-coloring of G is such that no color class is contained in any γ_t-set of G, then $\chi_{td}(G) = \chi_{dom}(G) + \gamma_t(G)$.

We have characterized graphs with $\chi_{dom}(G) = n-1$ in Theorem 16. Also, the characterization of graphs with $\chi_d(G) = n-1$ is proved in Theorem 1 by Arumugam et al. [1]. This paves a new way to study the following problems.

Problem 1. Characterize the graphs G of order n with $\chi_{dd}(G) = n-1$.

Problem 2. Characterize the graphs G of order n with $\chi_{td}(G) = n - 1$.

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