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Research Article

Characterization of word-representable graphs using modular decomposition

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Abstract: In this work, we characterize the class of word-representable graphs with respect to the modular decomposition. Consequently, we determine the representation number of a word-representable graph in terms of the permutation-representation numbers of the subgraphs induced by modules and the representation number of the associated quotient graph. In this context, we also obtain a complete answer to the open problem posed by Kitaev and Lozin on the word-representability of the lexicographical product of graphs.

Keywords: word-representable graphs, representation number, comparability graphs, lexicographical product, modular decomposition

AMS Subject classification: 68R10, 68R15, 05C76, 06A07

1. Introduction

The class of word-representable graphs (see Definition 3) is a well-studied graph class in the literature having importance in algebra and graph theory. The class of word-representable graphs includes several fundamental classes of graphs such as comparability graphs, circle graphs, parity graphs, and 3-colorable graphs. For a survey on word-representable graphs and their connections to other contexts, one may refer to the monograph by Kitaev and Lozin [9]. The class of comparability graphs, i.e., the graphs which admit transitive orientations, is precisely the class of permutationally representable graphs [11] and these graphs can be represented by a concatenation of

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permutations on their vertices. Every comparability graph induces a poset based on one of its transitive orientations.

A word-representable graph is said to be k-word-representable (for some positive integer k) if it is represented by a k-uniform word - a word in which every letter appears exactly k times. It is known that a graph is word-representable if and only if it is k-word-representable for some k [10]. The representation number of a word-representable graph is the smallest k such that the graph is k-word-representable. Similarly, the permutation-representation number (in short, *prn*) of a comparability graph is the smallest k such that the comparability graph is represented by a concatenation of k permutations on its vertices. Moreover, the *prn* of a comparability graph is exactly the dimension of the induced poset [16]. The class of word-representable graphs with representation number at most two is characterized as the class of circle graphs [7] and the class of comparability graphs with *prn* at most two is the class of permutation graphs (cf. [17]). In general, determining the representation number of a word-representable graph, and the permutation-representation number of a comparability graph are computationally hard [7, 20].

For a graph $G = (V, E)$, a set $A \subseteq V$ is said to be a module of G if for any $b \in V \setminus A$, either no vertex of A is adjacent to b or all vertices of A are adjacent to b. A modular partition of G is a partition of the vertex set of G into modules. There is a quotient graph associated to every modular partition of G. A modular partition of G and its quotient graph constitute a modular decomposition of G. The modular decomposition has a wide range of applications, including in the theory of posets and scheduling problems. For detailed information on this topic, one may refer to the survey papers [6, 15].

The concept of modular decomposition of a graph was first introduced by Gallai [4] and used to characterize the comparability graphs. Further, in [14], a recognition algorithm for comparability graphs was obtained using modular decomposition. Moreover, the modular decomposition was used to characterize a number of graph classes such as cographs, permutation graphs, and interval graphs (cf. [5]). We now state the following problem for word-representable graphs.

Problem 1. Characterize word-representable graphs with respect to the modular decomposition.

In this paper, we address Problem 1 and, we also determine the representation number of a word-representable graph and the *prn* of a comparability graph in terms of the respective modules and the quotient graph. Additionally, we provide a necessary and sufficient condition under which the lexicographical product of two word-representable graphs is word-representable, thereby offering a complete answer to the open problem posed in [9, Chapter 7].

2. Preliminaries

In this section, we define the concepts that are used in this paper along with the relevant results.

2.1. Word-Representable Graphs

In this subsection, we provide necessary background material on word-representable graphs and partially ordered sets. For more details on these topics, one may refer to [9, 19].

Definition 1. Let X be a finite set (of letters). A *word* over X is a finite sequence of letters of X written by juxtaposing them. A *subword* u of a word w is a subsequence of the sequence w and it is denoted by $u \ll w$.

For example, $aacbabca$ is a word over the set $\{a, b, c\}$ and $acbaca \ll aacbabca$. Let w be a word over a set X and $Y \subseteq X$. We write $w|_Y$ to denote the subword of w that is obtained by deleting all the letters belonging to $X \setminus Y$ from w. For example, if $w = aacbabca$, $w|_{\{a,b\}} = aababa$. Let w be a word over a set containing the letters a, b. We say a and b *alternate* in w if $w|_{\{a,b\}}$ is in one of the following forms: $(ab)^k$, $(ba)^k$, $a(ba)^k$ or $b(ab)^k$ for some positive integer $k \geq 1$, where, for a word x, x^k denotes the word $xx \cdots x$ for k times. A word w is called a *k-uniform word* if every letter appears exactly k times in w. Note that a 1-uniform word w is a permutation on the set of letters of w.

Definition 2. A *graph* G is a pair (V, E), where V is a finite set and E is a set of 2-element subsets of V. The elements of V and E are called *vertices* and *edges* of the graph G, respectively. If $\{a, b\} \in E$, then we say a and b are *adjacent*. The *neighborhood* of a vertex a in G, denoted by $N_G(a)$, is the set of all vertices adjacent to a.

For $A \subseteq V$, the *subgraph of G induced by A*, denoted by $G[A]$, is the graph whose vertex set is A and the edge set consists of all 2-element subsets of A that are in E. For $a, b \in V$, we say a is *reachable* from b (or vice versa) if there is a sequence of vertices $\langle a = a_0, a_1, \ldots, a_k = b\rangle$ such that $\{a_{i-1}, a_i\} \in E$ for all $1 \leq i \leq k$. A graph G is said to be *connected* if any two distinct vertices are reachable from one another. In this paper, we consider only connected graphs.

Definition 3. A graph $G = (V, E)$ is said to be a *word-representable graph* if there exists a word w over the vertex set V of G such that for all $a, b \in V$, a and b are adjacent in G if and only if a and b alternate in w. In this case, we say w represents G. A graph G is said to be *k-word-representable* if there exists a k-uniform word representing G. The *representation number* of a word-representable graph G, denoted by $\mathcal{R}(G)$, is the smallest k such that G is k-word-representable.

Definition 4. For some positive integer k, let $p_1, p_2, \ldots, p_k$ be permutations on the vertex set of a graph G such that $p_1 p_2 \cdots p_k$ represents G, then G is said to be a *permutationally k-representable graph*, or simply, a *permutationally representable graph*. The *permutation-representation number* (in short, *prn*) of a permutationally representable graph G, denoted by $\mathcal{R}^p(G)$, is the smallest k such that G is permutationally k-representable.

We now recall the concepts of comparability graphs, posets and their connection to word-representable graphs.

Definition 5. An *orientation* of a graph is an assignment of direction to each edge so that the resultant graph is a directed graph, in which the edges are ordered pairs of vertices. An orientation is *transitive* if (a, b) and (b, c) are edges, then (a, c) is also an edge in the resultant directed graph, for all vertices a, b, c. A graph is said to be a *comparability graph* if it admits a transitive orientation.

Definition 6. A *partially ordered set* (in short, *poset*) is a pair $(P, \prec)$, where P is a nonempty set and $\prec$ is a partial order on P, i.e., $\prec$ is a transitive relation on P such that $a \not\prec a$ for all $a \in P$. Two elements a, b in a poset are said to be *comparable* if $a \prec b$ or $b \prec a$; otherwise, we say they are *incomparable*.

A poset $(P, \prec)$ is often represented by its underlying set P, when the partial order is clear in a given context. A partial order on a set P is said to be a *linear order* if any two elements of P are comparable.

Definition 7. Let $(P, \prec)$ be a poset. A *realizer* of the poset P is a collection of linear orders $\{\prec_1, \prec_2, \ldots, \prec_k\}$ (for some positive integer k) on P such that for every $a, b \in P$, $a \prec b$ if and only if $a \prec_i b$, for all $1 \leq i \leq k$. The *dimension* of a poset P, denoted by $\dim(P)$, is the smallest positive integer k such that P has a realizer of size k.

Let $G = (V, E)$ be a comparability graph with a transitive orientation D. Then, G induces a poset, denoted by $P_G = (V, \prec)$, such that $a \prec b$ in P_G if and only if (a, b) is an edge in the resultant directed graph with respect to D. The following results relating comparability graphs with word-representable graphs are useful in this paper.

Theorem 1 ([11]). *A graph is a comparability graph if and only if it is a permutationally representable graph.*

Theorem 2 ([10]). *If G is a word-representable graph, then the subgraph induced by $N_G(a)$ is a comparability graph, for every vertex a of G.*

Theorem 3 ([7, 16]). *For a comparability graph G, $\mathcal{R}^p(G) = \dim(P_G)$.*

2.2. Modular Decomposition

In this subsection, we present the preliminaries of modular decomposition. For more details, one may refer to the surveys [6, 15], while it is worth noting that [4] is the seminal paper on modular decomposition.

Definition 8. Let $G = (V, E)$ be a graph. A set of vertices $A \subseteq V$ is a *module* of G if for any $b \in V \setminus A$, either $N_G(b) \cap A = \varnothing$ or $A \subseteq N_G(b)$. A graph is said to be a *prime graph* if it contains only trivial modules, viz., the singletons and the whole set V. Otherwise, the graph is said to be *decomposable*.

Definition 9. A *modular partition* of a graph $G = (V, E)$ is a partition of V into modules of G. For some positive integer k, let $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$ be a modular partition of G, i.e., A_i is a module of G for all $1 \le i \le k$. Choose exactly one vertex a_i from each A_i and let $V' = \{a_1, a_2, \ldots, a_k\}$. The *quotient graph* associated to $\mathcal{P}$ of G, denoted by ${}^{G}/_{\mathcal{P}}$, is the graph (V', E'), where $\{a_i, a_j\} \in E'$ if and only if $\{a_i, a_j\} \in E$, for all $1 \le i, j \le k$. A modular partition $\mathcal{P}$ of G and its quotient graph ${}^{G}/_{\mathcal{P}}$ constitute a *modular decomposition* of G.

Remark 1. For some positive integer k, let $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$ be a modular partition of a graph G. It is evident that ${}^{G}/_{\mathcal{P}}$ is isomorphic to an induced subgraph of G. Moreover, the original graph G can be reconstructed from ${}^{G}/_{\mathcal{P}}$ by replacing each vertex of ${}^{G}/_{\mathcal{P}}$ with the corresponding induced subgraph $G[A_i]$, for $1 \le i \le k$.

A module A is said to be *strong* if A does not intersect with any other module or whenever $A \cap A' \neq \varnothing$ for some module A' then one is contained in the other. Note that the whole set V and the singletons are strong modules of $G = (V, E)$. Further, a module A is called *maximal* if there is no module A' of G such that $A \subset A' \subset V$. It is known that every graph has a unique maximal modular partition (a partition consisting of only maximal strong modules). All modules of a graph G can be obtained through the notion called modular decomposition tree of G, whose nodes are the strong modules of G. The modular decomposition tree of G has V as the root and the children of a node A in the tree are the parts of the maximal modular partition of the induced subgraph $G[A]$. The modular decomposition tree of G and modular decomposition of G can be computed in $O(n + m)$ time, where n and m are the number of vertices and the number of edges of G [13]. We now present the structural characterization of comparability graphs with respect to a modular decomposition of a graph.

Theorem 4 ([5, 14]). *For some positive integer k, let $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$ be a modular partition of a decomposable graph G. Then G is a comparability graph if and only if ${}^{G}/_{\mathcal{P}}$ and each of the induced subgraphs $G[A_i]$ are comparability graphs.*

3. Characterization

In the present context, to study modular decomposition, first we consider the following operation which replaces a module in place of a vertex in a graph.

Definition 10. Let $G = (V \cup \{a\}, E)$, where $a \notin V$, and $M = (V', E')$ be two graphs. The graph $G_a[M] = (V'', E'')$ is defined by $V'' = V \cup V'$ and the edge set E'' consists of the edges of $G[V]$, edges of E' and $\Big\{\{b, c\} \mid b \in V', c \in N_G(a)\Big\}$. We say $G_a[M]$ is obtained from G by replacing the vertex a of G with the module M.

Remark 2. In Definition 10, note that V' is a module in the graph $G_a[M]$. For instance, let $b \in V'' \setminus V' = V$. Thus, both a and b are vertices of G, and we have either $b \notin N_G(a)$ or $b \in N_G(a)$. Thus, by definition of $G_a[M]$, either $N_{G_a[M]}(b) \cap V' = \varnothing$ or $V' \subseteq N_{G_a[M]}(b)$.

Remark 3. For two word-representable graphs G and M, $G_a[M]$ is not always a word-representable graph. For instance, consider the graph obtained by replacing one of the two vertices of the complete graph K_2 with the cycle C_5. This graph is nothing but the wheel graph W_5 (see Fig. 1), which is not a word-representable graph (cf. [9, Chapter 3]).

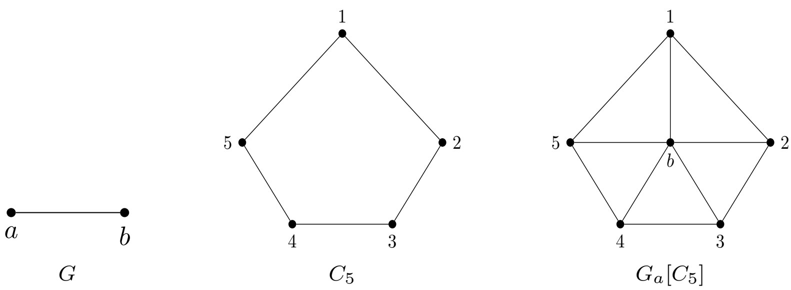

Figure 1. **Replacing the vertex a of G with the module C_5**

In what follows, G and M denote the graphs $G = (V \cup \{a\}, E)$ with $a \notin V$ and $M = (V', E')$, unless stated otherwise. The following theorem gives us a sufficient condition for the word-representability of $G_a[M]$.

Theorem 5 ([8]). *Suppose G is a word-representable graph. If M is a comparability graph, then $G_a[M]$ is word-representable.*

Further, on the representation number of $G_a[M]$, it was stated in [8] that if $\mathcal{R}(G) = k$ and $\mathcal{R}(M) = k'$, then $\mathcal{R}(G_a[M]) = \max\{k, k'\}$. Although it was stated that k' is the representation number of the graph M, the concept of *prn* of M was used in the proof given in [8]. If $k' = \mathcal{R}(M)$, then the statement does not hold in some cases, as shown in the following counterexample.

Example 1. Consider the graph obtained by replacing a vertex, say a, of $G = K_2$ with the cycle C_6. Note that $\mathcal{R}(G) = 1$ and $\mathcal{R}(C_6) = 2$ [9, Chapter 3]. But $\mathcal{R}(G_a[C_6]) \neq 2$. In fact, $G_a[C_6]$ is the wheel W_6 and its representation number is 3 (by [7, Lemma 3]), as $\mathcal{R}^p(C_6) = 3$ (cf. [17]).

In the following, we state the correct version for giving representation number of $G_a[M]$.

Theorem 6. *Let G be a word-representable graph and M be a comparability graph. If $\mathcal{R}(G) = k$ and $\mathcal{R}^p(M) = k'$, then $\mathcal{R}(G_a[M]) = \max\{k, k'\}$.*

Corollary 1. *Suppose a word-representable graph H' is obtained from a comparability graph H by adding an all-adjacent vertex, i.e., a vertex which is adjacent to all vertices of H. Then, $\mathcal{R}(H') = \mathcal{R}^p(H)$.*

Proof. It can be observed that the graph H' is obtained from the complete graph K_2 by replacing one of its two vertices with the module H. Since $\mathcal{R}(K_2) = 1$, by Theorem 6, we have $\mathcal{R}(H') = \mathcal{R}^p(H)$. □

We now prove that the comparability of M is necessary for $G_a[M]$ to be a word-representable graph.

Theorem 7. *The graph $G_a[M]$ is word-representable if and only if G is a word-representable graph and M is a comparability graph.*

Proof. If G is a word-representable graph and M is a comparability graph, then by Theorem 5, $G_a[M]$ is a word-representable graph. Conversely, suppose $G_a[M]$ is a word-representable graph. Then, G is word-representable as G is isomorphic to an induced subgraph of $G_a[M]$. Since G is connected, there exists a vertex, say b, of G such that $b \in N_G(a)$. Then, from the construction of $G_a[M]$ it is evident that b is adjacent to all vertices of M. Consider the induced subgraph $G_1 = (G_a[M])[V' \cup \{b\}]$. Clearly, G_1 is a word-representable graph. Further, note that the subgraph induced by $N_{G_1}(b)$ in G_1 is M. Then, in view of Theorem 2, M is a comparability graph. □

In the following theorem, we provide a characterization for $G_a[M]$ to be a comparability graph. We also present the *prn* of $G_a[M]$.

Theorem 8. *The graph $G_a[M]$ is a comparability graph if and only if G and M are comparability graphs. Moreover, if $\mathcal{R}^p(G) = k$ and $\mathcal{R}^p(M) = k'$, then $\mathcal{R}^p(G_a[M]) = \max\{k, k'\}$.*

Proof. Suppose $G_a[M]$ is a comparability graph. Since G and M are isomorphic to certain induced subgraphs of $G_a[M]$, G and M are comparability graphs. Conversely, suppose G and M are comparability graphs. Let the words $p_1p_2 \cdots p_k$ and $p'_1p'_2 \cdots p'_{k'}$

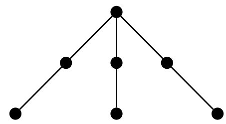

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Figure 2. **A tree with** ***prn*** **3.**

represent the graphs G and M respectively, where each p_i $(1 \le i \le k)$ is a permutation on the vertices of G and each p'_i $(1 \le i \le k')$ is a permutation on the vertices of M. Suppose $\max\{k, k'\} = t$. If $k' < k$, then set $p'_j = p'_{k'}$ for all $k'+1 \le j \le t$ and note that $p'_1 p'_2 \cdots p'_t$ represents M. Similarly, if $k < k'$, then set $p_j = p_k$ for all $k+1 \le j \le t$ and note that $p_1 p_2 \cdots p_t$ represents G. In any case, the words $w = p_1 p_2 \cdots p_t$ and $w' = p'_1 p'_2 \cdots p'_t$ represent the graphs G and M, respectively. For $1 \le i \le t$, let $p_i = r_i a s_i$ so that $p_{i|_V} = r_i s_i$.
Let $v_i = r_i p'_i s_i$, for all $1 \le i \le t$. Note that each v_i is a permutation on the vertices of $G_a[M]$. We show that the word $v = v_1 v_2 \cdots v_t$ represents the graph $G_a[M]$.
Note that $G[V]$ and M are induced subgraphs of $G_a[M]$. Further, since $v|_V = p_{1|_V} p_{2|_V} \cdots p_{t|_V} = w|_V$ and $v|_{V'} = w'$, the subwords $v|_V$ and $v|_{V'}$ of v represent the graphs $G[V]$ and M, respectively. Thus, any two vertices of $G[V]$ (or any two vertices of M) are adjacent if and only if they alternate in the word v.
Let b, b' be two vertices of $G_a[M]$ such that $b \in V$ and $b' \in V'$. Then b and b' are adjacent in $G_a[M]$ if and only if $b \in N_G(a)$. Further, note that each v_i is constructed from p_i by replacing a with p'_i. Then, it is easy to see that for every $b' \in V'$, b' alternates with $b \in V$ in v if and only if b alternates with a in w. Thus, for every $b' \in V'$ and $b \in V$, b' alternates with b in v if and only if $b \in N_G(a)$. Hence, v represents the graph $G_a[M]$ permutationally.
Therefore, we have $\mathcal{R}^p(G_a[M]) \le t = \max\{k, k'\}$. Further, note that G and M are isomorphic to certain induced subgraphs of $G_a[M]$ so that $\mathcal{R}^p(G_a[M]) \ge t$. Hence, $\mathcal{R}^p(G_a[M]) = \max\{k, k'\}$. □

A graph $G = (V, E)$ is called a 3-leaf power if there is a tree T with V as its leaves such that for all $a, b \in V$, $\{a, b\} \in E$ if and only if their distance in T is at most three. These graphs were introduced in [18], and a forbidden induced subgraph characterization for this class of graphs was obtained in [3]. Further, in [1], it was proved that a connected graph G is a 3-leaf power if and only if it is obtained from a suitable tree (called an associated tree of G), say T_G, by replacing each vertex of T_G with cliques. Moreover, this characterization leads to linear-time (i.e., $O(n+m)$ time, where n is the number of vertices and m is the number of edges of G) recognition of the class of 3-leaf power graphs. Since both trees and cliques are comparability graphs, in view of Theorem 8, it is evident that 3-leaf power graphs are comparability graphs, and hence they are word-representable. Furthermore, we have the following theorem on the representation number as well as on the *prn* of a 3-leaf power graph.

Theorem 9. *If G is a 3-leaf power graph, then $\mathcal{R}(G) \leq 2$ and $\mathcal{R}^p(G) \leq 3$. Moreover, $\mathcal{R}^p(G) = 3$ if and only if the graph S (depicted in Fig. 2) is isomorphic to an induced subgraph of G.*

Proof. Note that a 3-leaf power graph G is obtained from an associated tree T_G by replacing iteratively each vertex of T_G with cliques. Let $a_1, a_2, \ldots, a_n$ be the vertices of T_G and $M_1, M_2, \ldots, M_n$ be the cliques replacing $a_1, a_2, \ldots, a_n$, respectively, to obtain G. Recall that $\mathcal{R}^p(M_i) = 1$, for all $1 \leq i \leq n$ (cf. [17]). Hence, in view of Theorem 6, we have

$$\mathcal{R}(G) = \max\{\mathcal{R}(T_G), \mathcal{R}^p(M_1), \mathcal{R}^p(M_2), \ldots, \mathcal{R}^p(M_n)\} = \mathcal{R}(T_G).$$

Similarly, in view Theorem 8, we have $\mathcal{R}^p(G) = \mathcal{R}^p(T_G)$. It is known that the representation number and the *prn* of trees are at most 2 and 3, respectively (cf. [9, Section 3.1] and [17, Theorem 4]). Hence, $\mathcal{R}(T_G) \leq 2$ and $\mathcal{R}^p(T_G) \leq 3$ so that $\mathcal{R}(G) \leq 2$ and $\mathcal{R}^p(G) \leq 3$.

Suppose $\mathcal{R}^p(G) = 3$ so that we have $\mathcal{R}^p(T_G) = 3$. Hence, in view of [17, Theorem 4], the graph S is isomorphic to an induced subgraph of T_G. Further, since T_G is an induced subgraph of G, we have S is isomorphic to an induced subgraph of G. The converse part holds as $\mathcal{R}^p(S) = 3$ (cf. [17, Section 4]). □

As a consequence of the above results, we now study the word-representability of the lexicographical product of any two graphs.

Definition 11. Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. The lexicographical product of G and G', denoted by $G \circ G'$, defined by $G \circ G' = (V'', E'')$, where the vertex set $V'' = V \times V'$ and the edge set $E'' = \Big\{\{(a, a'), (b, b')\} \Big| \{a, b\} \in E \text{ or } (a = b,\ \{a', b'\} \in E')\Big\}$.

The word-representability of lexicographical product of graphs is an open problem posed in [9, Chapter 7]. In connection to this problem, it was shown in [2, Section 6] that the class of word-representable graphs is not closed under the lexicographical product by constructing an explicit example. First, we observe the relation between lexicographical product and the operation of module replacing a vertex as per the following remark.

Remark 4. The lexicographical product $G \circ G'$ is nothing but replacing each vertex of G by the module G'. Hence, both G and G' are induced subgraphs of $G \circ G'$.

Accordingly, in the following theorem, we provide a necessary and sufficient condition for $G \circ G'$ to be a word-representable graph. Thus, we settle the above-mentioned open problem.

Theorem 10. *Let G and G' be two graphs. The lexicographical product $G \circ G'$ is word-representable if and only if G is word-representable and G' is a comparability graph. Moreover, if $\mathcal{R}(G) = k$ and $\mathcal{R}^p(G') = k'$, then $\mathcal{R}(G \circ G') = \max\{k, k'\}$.*

Proof. The proof follows from Remark 4, Theorem 6 and Theorem 7. □

We further state a characteristic property for $G \circ G'$ to be a comparability graph.

Theorem 11. *Let G and G' be two graphs. The lexicographical product $G \circ G'$ is a comparability graph if and only if G and G' are comparability graphs. Moreover, if $\mathcal{R}^p(G) = k$ and $\mathcal{R}^p(G') = k'$, then $\mathcal{R}^p(G \circ G') = \max\{k, k'\}$.*

We are now ready to present the main result of the paper on characterization of word-representable graphs with respect to the modular decomposition, through the following lemma.

Lemma 1. *Let G be a word-representable graph and A be any non-trivial module of G. Then the induced subgraph $G[A]$ is a comparability graph.*

Proof. Consider the graph G' which is obtained from G by replacing the module A by a new vertex, say a. Thus, G' is word-representable as it is isomorphic to an induced subgraph of G. Note that G can be reconstructed from G' by replacing the vertex a with the module $G[A]$. Hence, by Theorem 7, we have $G[A]$ is a comparability graph. □

Theorem 12. *Let G be a decomposable graph and $\mathcal{P} = \{A_1, A_2, \ldots, A_k\}$ be a modular partition of G, for some positive integer k. Then, we have the following:*

1. *G is word-representable if and only if for each $1 \leq i \leq k$, $G[A_i]$ is a comparability graph and ${}^{G}/_{\mathcal{P}}$ is a word-representable graph.*
2. *If G is word-representable, then*
$$\mathcal{R}(G) = \max\{\mathcal{R}({}^{G}/_{\mathcal{P}}), \mathcal{R}^p(G[A_1]), \ldots, \mathcal{R}^p(G[A_k])\}.$$
3. *If G is a comparability graph, then*
$$\mathcal{R}^p(G) = \max\{\mathcal{R}^p({}^{G}/_{\mathcal{P}}), \mathcal{R}^p(G[A_1]), \ldots, \mathcal{R}^p(G[A_k])\}.$$

Proof. 1. Suppose G is a word-representable graph. As ${}^{G}/_{\mathcal{P}}$ is an induced subgraph of G, it is a word-representable graph. Further, from Lemma 1, we have $G[A_i]$ is a comparability graph, for each $1 \leq i \leq k$. Note that the converse is evident from Remark 1 and Theorem 7.
2. If G is a word-representable graph, then from part (1) we have each $G[A_i]$ is a comparability graph and ${}^{G}/_{\mathcal{P}}$ is a word-representable graph. Hence, the result follows from Remark 1 and Theorem 6.
3. The result is evident from Remark 1, Theorem 4 and Theorem 8. □

4. Concluding Remarks

In this work, we characterized word-representable graphs with respect to the modular decomposition. Furthermore, we have established the word-representability of 3-leaf power graphs and lexicographical product of graphs and obtained their representation numbers and also their *prn*. It is interesting to find further special classes of word-representable graphs using modular decomposition.
While recognizing word-representability of graphs is NP-complete [7], it is interesting to study methods to prove the non-word-representability of a graph. To explore more in this direction, one may refer to [12]. In [13], it was shown that the recognition of comparability graphs as well as the computation of modular decomposition of a graph can be done in $O(n + m)$ time, where n and m are the number of vertices and the number of edges of the graph, respectively. Hence, in view of Theorem 12, we have a polynomial (in $m + n$) time test for non-word-representability of a graph G: Find the maximal modular partition of G in $O(m+n)$ time. Note that there are at most n modules in the partition. Then check the subgraph induced by each of these modules in $O(m + n)$ time whether they are comparability graphs. If a non-comparability module is found, G is not a word-representable graph. However, if all modules are comparability graphs, then the test gives no information. In that case, the problem is reduced to the problem of recognizing the word-representability of the associated quotient graph.

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