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Research Article

Distance spectra of neighbourhood corona of graphs

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Abstract: The neighbourhood corona $G \star H$ of two graphs G and H is obtained by taking one copy of G and $|V(G)|$ copies of H and making all the neighbours of the i^{th} vertex of G adjacent with all the vertices in the i^{th} copy of H. In this paper we describe the distance eigenvalues and corresponding eigenvectors of $G \star H$ in terms of the adjacency spectrum of G and H when G is a regular triangle-free graph with diameter 2 and H is regular. Several constructions are proposed using line graphs, iterated line graphs, double graphs, strong double graphs and complement graphs to obtain infinitely many distance non-cospectral pairs of distance equienergetic graphs and non-isomorphic pairs of distance cospectral graphs. Also we obtain the distance Laplacian spectrum of $G \star H$ in terms of the distance Laplacian spectrum of G and Laplacian spectrum of H when G is a transmission regular triangle-free graph with diameter 2. Further we find the distance signless Laplacian spectrum of $G \star H$ in terms of the distance signless Laplacian spectrum of G and signless Laplacian spectrum of H when G is a transmission regular triangle-free graph with diameter 2 and H is regular. We also construct infinitely many non-isomorphic pairs of distance Laplacian cospectral graphs and distance signless Laplacian cospectral graphs.

Keywords: distance spectrum, neighbourhood corona, distance Laplacian spectrum, distance signless Laplacian spectrum, distance equienergetic graphs.

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1. Introduction

Distance spectra and distance energy have been a keen area of research in spectral graph theory for many years. Studies on distance spectra gained increased fascination starting in 1971, when Graham and Pollack discovered a connection between the number of negative eigenvalues of the distance matrix and the addressing problem in data communication systems. Additionally, they proved that the determinant of the distance matrix of a tree is independent of its structure and solely relies on the number of vertices[6]. A lot of studies can be seen in literature based on the relationship between distance eigenvalues and graph structures and parameters. The studies on distance energy started in 2008. A wide variety of results on distance spectra and distance energy are available in the surveys [2, 14].

In this paper we only consider undirected graphs that are both simple and finite. Since we deal with different matrices associated with a graph we use the following notion for uniformity. Let $\mathcal{M}$ be a square matrix associated with G. The eigenvalues of $\mathcal{M}$ are the $\mathcal{M}$-eigenvalues and the collection of all $\mathcal{M}$-eigenvalues is the $\mathcal{M}$-spectrum of G, denoted by $Spec_{\mathcal{M}}(G)$. Two graphs G and G' are $\mathcal{M}$-cospectral if $Spec_{\mathcal{M}}(G) = Spec_{\mathcal{M}}(G')$. If $\lambda_1(G), \lambda_2(G), \ldots, \lambda_t(G)$ are distinct $\mathcal{M}$-eigenvalues of G with multiplicities $m_1, m_2, \ldots, m_t$, then we write $Spec_{\mathcal{M}}(G) = \{(\lambda_1(G))^{m_1}, (\lambda_2(G))^{m_2}, \ldots, (\lambda_t(G))^{m_t}\}$.

Let G be a graph on n vertices $u_1, \ldots, u_n$. Let $d(u_i)$ denote the degree of u_i. We write $u_i \sim u_j$ if u_i is adjacent to u_j, and $u_i \nsim u_j$ otherwise. The adjacency matrix $\mathcal{A}(G)$ of G is an $n \times n$ matrix whose (i, j)-th element is 1, if u_i and u_j are adjacent; and 0, otherwise. The $\mathcal{A}$-eigenvalues are denoted by $\mu_1(G) \geq \mu_2(G) \geq \ldots \geq \mu_n(G)$. For all connected graphs G, $\mu_1(G) > \mu_2(G)$. For an r-regular graph G, $\mu_1(G) = r$. We say that G is regular of degree r if G is r-regular. We denote the number of positive and negative $\mathcal{A}$-eigenvalues of G by $n^+(G)$ and $n^-(G)$ respectively.

The Laplacian matrix and signless Laplacian matrix of G are defined respectively by $\mathscr{L}(G) = \mathcal{D}_G - \mathcal{A}(G)$ and $\mathscr{Q}(G) = \mathcal{D}_G + \mathcal{A}(G)$, where $\mathcal{D}_G$ is an $n \times n$ diagonal matrix defined by $\mathcal{D}_G = Diag\big(d(u_i)\big)$. The $\mathscr{L}$-eigenvalues and $\mathscr{Q}$-eigenvalues of G are denoted respectively by $\eta_1(G) \geq \eta_2(G) \geq \ldots \geq \eta_n(G) = 0$ and $\delta_1(G) \geq \delta_2(G) \geq \ldots \geq \delta_n(G)$. For basic results on $\mathcal{A}$-spectrum, $\mathscr{L}$-spectrum and $\mathscr{Q}$-spectrum refer [4].

The distance between vertices u_i and u_j in a connected graph G, denoted by $d_G(u_i, u_j)$, is the length of a shortest path from u_i to u_j. If G is disconnected and u_i and u_j belong to different components of G, then $d_G(u_i, u_j)$ is defined to be ∞. The diameter $Diam(G)$, of G is the largest distance between any pair of distinct vertices in G. The distance matrix of a graph G, denoted by $\mathscr{D}(G)$, is an $n \times n$ matrix, whose (i, j)-th element is $d_G(u_i, u_j)$. $\mathscr{D}$-eigenvalues of G are denoted by $\rho_1(G) \geq \rho_2(G) \geq \ldots \geq \rho_n(G)$.

The transmission $Tr(u_i)$ of a vertex u_i in G is defined by $Tr(u_i) = \sum_{j=1}^{n} d_G(u_i, u_j)$. The transmission matrix is an $n \times n$ diagonal matrix defined by $Tr(G) = Diag\big(Tr(u_i)\big)$. G is s-transmission regular if $Tr(u_i) = s$ for all $i = 1, \ldots, n$. The distance Laplacian matrix and distance signless Laplacian matrix of G are defined by $\mathscr{D}^{\mathscr{L}}(G) = Tr(G) - \mathscr{D}(G)$ and $\mathscr{D}^{\mathscr{Q}}(G) = Tr(G) + \mathscr{D}(G)$ respectively[1]. $\mathscr{D}^{\mathscr{L}}$-

eigenvalues and $\mathscr{D}^{\mathscr{Q}}$-eigenvalues are denoted by $\rho_1^{\mathscr{L}}(G) \geq \rho_2^{\mathscr{L}}(G) \geq \ldots \geq \rho_n^{\mathscr{L}}(G) = 0$ and $\rho_1^{\mathscr{Q}}(G) \geq \rho_2^{\mathscr{Q}}(G) \geq \ldots \geq \rho_n^{\mathscr{Q}}(G)$ respectively.
The energy [7] $\mathcal{E}(G)$ of a graph G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\mu_i(G)|. \tag{1.1}$$

Two non-isomorphic graphs with the same energy are called equienergetic. More results on graph energy can be seen in [8, 13].
The distance energy [12] is defined analogous to the graph energy (1.1) as

$$\mathcal{DE}(G) = \sum_{i=1}^{n} |\rho_i(G)|. \tag{1.2}$$

Two non-isomorphic graphs G and G' are distance equienergetic ($\mathscr{D}$-equienergetic) if $\mathcal{DE}(G) = \mathcal{DE}(G')$. Clearly $\mathscr{D}$-cospectral graphs are $\mathscr{D}$-equienergetic. So it is interesting to find $\mathscr{D}$-equienergetic graphs that are $\mathscr{D}$-non-cospectral.
The line graph of G, denoted by $\mathrm{L}(G)$, is the graph whose vertices are the edges of G and two vertices of $\mathrm{L}(G)$ are adjacent if the corresponding edges in G have a common vertex. If G is an r-regular graph on n vertices then $\mathrm{L}(G)$ is a $(2r-2)$-regular graph on $nr/2$ vertices. The m^{th} iterated line graph $\mathrm{L}^m(G), m \geq 0$, is defined by $\mathrm{L}^m(G) = \mathrm{L}(\mathrm{L}^{m-1}(G))$, where $\mathrm{L}^1(G) = \mathrm{L}(G)$ and $\mathrm{L}^0(G) = G$ [3, 9]. If G is regular then $\mathrm{L}^m(G)$ is regular for all $m \geq 0$. Let $\mathrm{L}^m(G), m \geq 1$, be r_m-regular and of order n_m. Then $n_m = \dfrac{n_{m-1} \cdot r_{m-1}}{2}$ and $r_m = 2r_{m-1} - 2$, where $n_0 = n$ and $r_0 = r$. It can be deduced that

$$n_m = \frac{n}{2^m} \prod_{i=0}^{m-1} (2^i r - 2^{i+1} + 2) \tag{1.3}$$

and

$$r_m = 2^m r - 2^{m+1} + 2. \tag{1.4}$$

Graph operations have a pivotal role in graph theory since they are useful to produce graphs with specific structures. Corona[5] is a well-known operation which produces graphs with a specific pattern. Neighbourhood corona, a variant of corona, of two graphs G and H is denoted by $G \star H$ and is obtained by taking one copy of G and n copies of H and making all the neighbours of the i^{th} vertex of G adjacent with all the vertices in the i^{th} copy of H[11]. Various spectra of neighbourhood corona have been studied in [11, 15, 16].
There exists infinite number of regular graphs with diameter 2. Complete bipartite graphs $K_{n,n}$, $n \geq 2$, are well-known class of graphs with these properties. A graph is called triangle-free if it contains no induced subgraph isomorphic to C_3. In this paper we describe the $\mathscr{D}$- eigenvalues and corresponding eigenvectors of $G \star H$ in terms of the $\mathcal{A}$-spectrum of G and H when G is a regular triangle-free graph with

diameter 2 and H is regular. We construct infinitely many $\mathscr{D}$-non-cospectral pairs of $\mathscr{D}$-equienergetic graphs and non-isomorphic pairs of $\mathscr{D}$-cospectral graphs. We obtain the $\mathscr{D}^{\mathscr{L}}$-spectrum of $G \star H$ in terms of the $\mathscr{D}^{\mathscr{L}}$-spectrum of G and $\mathscr{L}$-spectrum of H when G is a transmission regular triangle-free graph with diameter 2 and H is arbitrary. We also find the $\mathscr{D}^{\mathscr{Q}}$-spectrum of $G \star H$ in terms of the $\mathscr{D}^{\mathscr{Q}}$-spectrum of G and $\mathscr{Q}$-spectrum of H when G is a transmission regular triangle-free graph with diameter 2 and H is regular. Finally we construct infinitely many non-isomorphic pairs of cospectral graphs with respect to $\mathscr{D}^{\mathscr{L}}$ and $\mathscr{D}^{\mathscr{Q}}$ matrices.
We denote the $n \times 1$ vector with all the entries as 1(respectively, 0) by $\mathbf{1}_n$(respectively, $\mathbf{0}_n$) and the $n \times 1$ vector in which the i-th entry is 1 and all other entries are 0 by $\mathbf{e}_i$. A square matrix of appropriate order in which all the entries are 1 is denoted by J.

2. Preliminaries

The following definitions and results will be used in the subsequent sections.

Theorem 1. *[4] Let G be an r-regular graph of order n with $Spec_{\mathcal{A}}(G) = \{r, \mu_2(G), \ldots, \mu_n(G)\}$. Then the complement $\overline{G}$ has the $\mathcal{A}$-eigenvalues $n-r-1, -(\mu_i(G)+1), i = 2, \ldots, n$.*

Theorem 2. *[4] Let G be an r-regular graph of order n with $Spec_{\mathcal{A}}(G) = \{r, \mu_2(G), \ldots, \mu_n(G)\}$. Then the $\mathcal{A}$-eigenvalues of $\mathrm{L}(G)$ are $2r-2, \mu_i(G)+r-2, i = 2, \ldots, n$ and -2 with multiplicity $\dfrac{n(r-2)}{2}$.*

Theorem 3. *[19] Let G be a regular graph of degree $r \geq 3$. Then for $m \geq 2$, all the negative eigenvalues of $\mathrm{L}^m(G)$ are equal to -2.*

Theorem 4. *[19] Let G be an r-regular graph of order n and degree $r \geq 3$. Then for $m \geq 2$, $\mathcal{E}\big(\mathrm{L}^m(G)\big) = 4n_m \frac{r_m - 2}{r_m + 2}$, where n_m and r_m are as defined in equations (1.3) and (1.4).*

Definition 1. [17] The double graph $D_2(G)$ of a graph G is the graph obtained by taking two copies of G say G_1 and G_2 and joining each vertex u_1 in G_1 to the neighbours of the corresponding vertex u_2 in G_2.

Definition 2. [18] The strong double graph $D_2^*(G)$ of a graph G is the graph obtained from $D_2(G)$ by joining each vertex u_1 in G_1 to the corresponding vertex u_2 in G_2.

If G is an r-regular graph then $D_2(G)$ and $D_2^*(G)$ are regular of degrees $2r$ and $2r+1$ respectively.

Theorem 5. *[17] Let G be a graph with $Spec_{\mathcal{A}}(G) = \{\mu_1(G), \ldots, \mu_n(G)\}$. Then the $\mathcal{A}$-eigenvalues of $D_2(G)$ are $2\mu_1(G), \ldots, 2\mu_n(G)$ and 0 with multiplicity n.*

Theorem 6. *[18] Let G be a graph with $Spec_{\mathcal{A}}(G) = \{\mu_1(G), \ldots, \mu_n(G)\}$. Then the $\mathcal{A}$-eigenvalues of $D_2^*(G)$ are $2\mu_1(G)+1, \ldots, 2\mu_n(G)+1$ and -1 with multiplicity n.*

3. Distance spectrum of $G \star H$

Definition 3. [11] The neighbourhood corona $G \star H$ of two graphs G and H is defined as a graph obtained by taking one copy of G and $|V(G)|$ copies of H and then joining all the neighbours of the i^{th} vertex of G to all the vertices in the i^{th} copy of H.

Let G be a connected (n, m) graph, $n \geq 2$, and H be a (p, q) graph. Then $G \star H$ is a graph of order $n(p+1)$ and size $m(2p+1)+nq$. Let $V(G) = \{u_1, \ldots, u_n\}$ and $V(H) = \{v_1, \ldots, v_p\}$ be the vertex sets of G and H respectively. For $i = 1, \ldots, n$, let $v_1^{(i)}, \ldots, v_p^{(i)}$ denote the vertices of the i^{th} copy of H in $G \star H$, such that $v_j^{(i)}$ is the copy of v_j in H for each $j = 1, \ldots, p$. Denote $V_j = \left\{v_j^{(1)}, \ldots, v_j^{(n)}\right\}$ for $j = 1, \ldots, p$. Then $V(G) \cup V_1 \cup \ldots \cup V_p$ is a partition of $V(G \star H)$.
Let $P = (p_{ij})$ and $Q = (q_{ij})$ be matrices of order $a \times b$ and $c \times d$ respectively. The Kronecker product[10] $P \otimes Q$ is of order $ac \times bd$ defined by $P \otimes Q = (p_{ij}Q)$. If k is any scalar, $(kP) \otimes Q = P \otimes (kQ) = k(P \otimes Q)$. For matrices P_1, P_2, Q_1 and Q_2, $(P_1 \otimes Q_1)(P_2 \otimes Q_2) = (P_1P_2) \otimes (Q_1Q_2)$, provided the products P_1P_2 and Q_1Q_2 exist. With respect to the above mentioned partition of $V(G \star H)$, the distance between vertices in $G \star H$ are described in four cases as follows.

Case (1): For $u_i, u_j \in V(G)$, $d_{G\star H}(u_i, u_j) = d_G(u_i, u_j)$.

Case (2): For $u_i \in V(G)$ and $v_a \in V(H)$,

$$\begin{aligned} d_{G\star H}\left(u_i, v_a^{(i)}\right) &= 2, && \text{for } a = 1, \ldots, p \text{ and } i = 1, \ldots, n \\ d_{G\star H}\left(u_i, v_a^{(k)}\right) &= d_G(u_i, u_k), && \text{for } i \neq k, \text{ and } a = 1, \ldots, p. \end{aligned}$$

Case (3): For $v_a, v_b \in V(H)$, where $a, b \in \{1, \ldots, p\}$ and for $i = 1, \ldots, n$,

$$d_{G\star H}\left(v_a^{(i)}, v_b^{(i)}\right) = \begin{cases} 1, & \text{if } v_a \sim v_b \text{ in } H \\ 2, & \text{if } v_a \nsim v_b \text{ in } H. \end{cases}$$

Case (4): For $v_a, v_b \in V(H)$, where $a, b \in \{1, \ldots, p\}$ and for $i, k \in \{1, \ldots, n\}$, where $i \neq k$,

$$d_{G\star H}\left(v_a^{(i)}, v_b^{(k)}\right) = \begin{cases} d_G(u_i, u_k) + 1, & \text{if } u_i \sim u_k \text{ and have a common neighbour in } G \\ d_G(u_i, u_k) + 2, & \text{if } u_i \sim u_k \text{ and have no common neighbour in } G \\ d_G(u_i, u_k), & \text{if } u_i \nsim u_k \text{ in } G. \end{cases}$$

From the distance relations described above, we can see that if $Diam(G)$ is d then $Diam(G \star H)$ can be d, $d+1$ or $d+2$. From the first case, it follows that the $\big(V(G), V(G)\big)$ block matrix in $\mathscr{D}(G\star H)$ is $\mathscr{D}(G)$. Case (2) implies that the $\big(V(G), V_j\big)$

block matrices, for $j = 1, \ldots, p$, in $\mathscr{D}(G \star H)$ is $\mathscr{D}(G) + 2I$. If G does not contain any triangle, then cases (3) and (4) imply that all the (V_i, V_j) block matrices in $\mathscr{D}(G \star H)$, where $i, j = 1, \ldots, p$, are in the form $\mathscr{D}(G) + 2\mathcal{A}(G) + tI_n$, where

$$t = \begin{cases} 0, & \text{if } i = j \\ 1, & \text{if } v_i \text{ is adjacent to } v_j \text{ in } H \\ 2, & \text{if } v_i \text{ is not adjacent to } v_j \text{ in } H. \end{cases}$$

Therefore, if G is a connected triangle-free graph then the distance matrix of $G \star H$ can be written as

$$\mathscr{D}(G \star H) = \begin{bmatrix} \mathscr{D}(G) & \mathbf{1}_p^T \otimes (\mathscr{D}(G) + 2I) \\ \mathbf{1}_p \otimes (\mathscr{D}(G) + 2I) & D' \end{bmatrix}, \tag{3.1}$$

where

$$D' = J_p \otimes \Big(\mathscr{D}(G) + 2\mathcal{A}(G)\Big) + \Big(2(J - I) - \mathcal{A}(H)\Big) \otimes I_n$$

and $\mathbf{1}_p^T$ denotes the transpose of $\mathbf{1}_p$.

Now, let G be a graph of diameter at most 2 such that any two adjacent vertices have a common neighbour. Then the distance matrix of $G \star H$ is as follows.

$$\mathscr{D}(G \star H) = \begin{bmatrix} \mathscr{D}(G) & \mathbf{1}_p^T \otimes (\mathscr{D}(G) + 2I) \\ \mathbf{1}_p \otimes (\mathscr{D}(G) + 2I) & J_p \otimes 2(J - I) + \Big(2(J - I) - \mathcal{A}(H)\Big) \otimes I_n \end{bmatrix}. \tag{3.2}$$

Remark 1. If G is any connected graph with at least two vertices then $G \vee \overline{K_{n_1}} \vee \overline{K_{n_2}} \vee \cdots \vee \overline{K_{n_p}}$, $p \geq 1$ and $n_i \geq 1$ for $i = 1, \ldots, p$, where $\vee$ denotes the join operation and is taken from left to right, is a family of graphs with diameter at most 2 such that any two adjacent vertices have a common neighbour. It can be seen that complete graph $K_n, n \geq 3$, complete split graphs except K_2, wheel graphs and fan graphs are subclasses of this family.

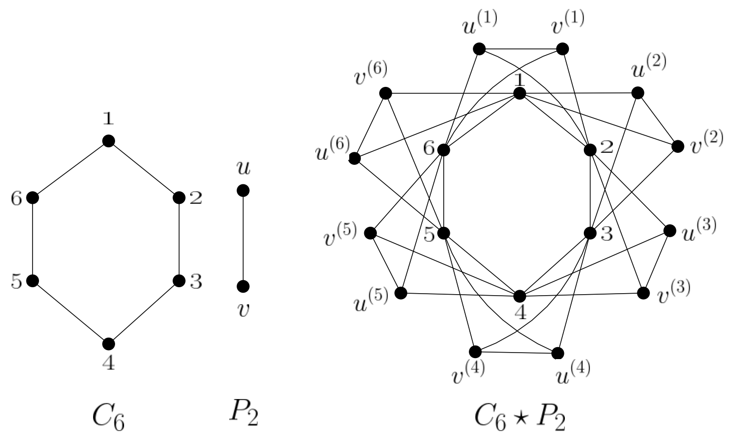

Figure 1. **Neighbourhood corona**

Remark 2. There exist graphs of diameter two with triangles that do not satisfy the condition required for equation (3.2). For example, $K_3 \square P_2$, where $\square$ denotes the Cartesian product operator, is a graph of diameter 2 containing triangles, but lacking the above property.

We illustrate the formation of the matrix $\mathscr{D}(G \star H)$ in the following examples.

Example 1. Consider the graph $G = C_6$ with $V(G) = \{1, 2, 3, 4, 5, 6\}$ and $H = P_2$ with $V(H) = \{u, v\}$, shown in Figure 1. $u^{(j)}$ and $v^{(j)}$ represents the vertices of P_2 in the j^{th} copy for $j = 1, 2, 3, 4, 5, 6$. Then the distance matrix of $C_6 \star P_2$ is

$\mathscr{D}(C_6 \star P_2) =$

	1	2	3	4	5	6	$u^{(1)}$	$u^{(2)}$	$u^{(3)}$	$u^{(4)}$	$u^{(5)}$	$u^{(6)}$	$v^{(1)}$	$v^{(2)}$	$v^{(3)}$	$v^{(4)}$	$v^{(5)}$	$v^{(6)}$
1	0	1	2	3	2	1	2	1	2	3	2	1	2	1	2	3	2	1
2	1	0	1	2	3	2	1	2	1	2	3	2	1	2	1	2	3	2
3	2	1	0	1	2	3	2	1	2	1	2	3	2	1	2	1	2	3
4	3	2	1	0	1	2	3	2	1	2	1	2	3	2	1	2	1	2
5	2	3	2	1	0	1	2	3	2	1	2	1	2	3	2	1	2	1
6	1	2	3	2	1	0	1	2	3	2	1	2	1	2	3	2	1	2
$u^{(1)}$	2	1	2	3	2	1	0	3	2	3	2	3	1	3	2	3	2	3
$u^{(2)}$	1	2	1	2	3	2	3	0	3	2	3	2	3	1	3	2	3	2
$u^{(3)}$	2	1	2	1	2	3	2	3	0	3	2	3	2	3	1	3	2	3
$u^{(4)}$	3	2	1	2	1	2	3	2	3	0	3	2	3	2	3	1	3	2
$u^{(5)}$	2	3	2	1	2	1	2	3	2	3	0	3	2	3	2	3	1	3
$u^{(6)}$	1	2	3	2	1	2	3	2	3	2	3	0	3	2	3	2	3	1
$v^{(1)}$	2	1	2	3	2	1	1	3	2	3	2	3	0	3	2	3	2	3
$v^{(2)}$	1	2	1	2	3	2	3	1	3	2	3	2	3	0	3	2	3	2
$v^{(3)}$	2	1	2	1	2	3	2	3	1	3	2	3	2	3	0	3	2	3
$v^{(4)}$	3	2	1	2	1	2	3	2	3	1	3	2	3	2	3	0	3	2
$v^{(5)}$	2	3	2	1	2	1	2	3	2	3	1	3	2	3	2	3	0	3
$v^{(6)}$	1	2	3	2	1	2	3	2	3	2	3	1	3	2	3	2	3	0

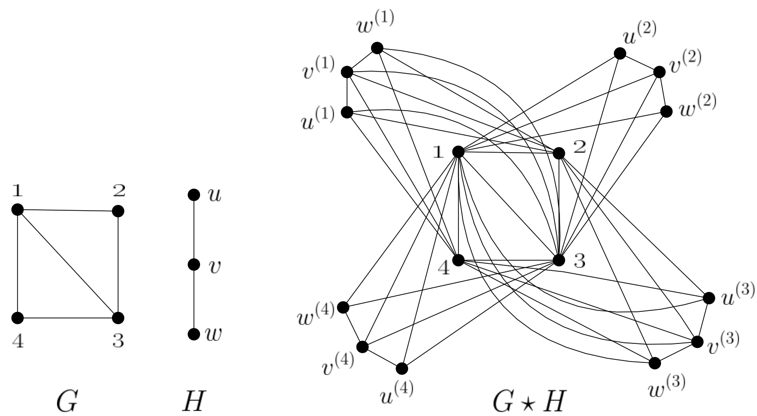

Figure 2. **Illustration of $G \star H$ when G is a graph with triangle**

Example 2. Figure 2 shows graphs G, H and $G \star H$, where G is a complete split graph. The distance matrix of $G \star H$ is

$$\mathscr{D}(G\star H) = \begin{array}{c|cccc|cccc|cccc|cccc|}
 & 1 & 2 & 3 & 4 & u^{(1)} & u^{(2)} & u^{(3)} & u^{(4)} & v^{(1)} & v^{(2)} & v^{(3)} & v^{(4)} & w^{(1)} & w^{(2)} & w^{(3)} & w^{(4)} \\
1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
2 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
3 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 2 & 1 \\
4 & 1 & 2 & 1 & 0 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
\hline
u^{(1)} & 2 & 1 & 1 & 1 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
u^{(2)} & 1 & 2 & 1 & 2 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
u^{(3)} & 1 & 1 & 2 & 1 & 2 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 & 2 \\
u^{(4)} & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 2 \\
\hline
v^{(1)} & 2 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 2 & 2 \\
v^{(2)} & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 1 & 2 & 2 \\
v^{(3)} & 1 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 1 & 2 \\
v^{(4)} & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 2 & 2 & 2 & 1 \\
\hline
w^{(1)} & 2 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 2 & 2 & 2 \\
w^{(2)} & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 2 & 2 \\
w^{(3)} & 1 & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0 & 2 \\
w^{(4)} & 1 & 2 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 2 & 2 & 2 & 0
\end{array}$$

The following result gives $n(p-1)$ $\mathscr{D}$-eigenvalues of $G \star H$, which depends only on H.

Theorem 7. *Let G be a connected graph of order n, which is either triangle-free or of diameter at most 2 such that any two adjacent vertices have a common neighbour in G. Let H be a k-regular graph of order p with $\mathcal{A}$-eigenvalues $k \geq \mu_2(H) \geq \ldots \geq \mu_p(H)$. Then $-\big(\mu_j(H)+2\big)$ is a $\mathscr{D}$-eigenvalue of $G \star H$ for $j = 2, \ldots, p$, each with multiplicity n.*

Proof. Let w_j be an eigenvector corresponding to $\mu_j(H), j = 2, \ldots, p$, that are orthogonal to $\mathbf{1}_p$. For $i = 1, \ldots, n$,

$$\mathscr{D}(G\star H)\begin{bmatrix}\mathbf{0}_n \\ w_j \otimes \mathbf{e}_i\end{bmatrix} = -\big(\mu_j(H)+2\big)\begin{bmatrix}\mathbf{0}_n \\ w_j \otimes \mathbf{e}_i\end{bmatrix}.$$

This implies that for each $j = 2, \ldots, p$, $\left\{\begin{bmatrix}\mathbf{0}_n \\ w_j \otimes \mathbf{e}_i\end{bmatrix}, i = 1, \ldots, n\right\}$ form a collection of n linearly independent eigenvectors of $\mathscr{D}(G \star H)$ corresponding to the $\mathscr{D}$-eigenvalue $-\big(\mu_j(H)+2\big)$. □

The next theorem describes completely the $\mathscr{D}$-spectrum of $G \star H$ in terms of $\mathcal{A}$-spectrum of G and H when both G and H are regular.

Theorem 8. *Let G be an r-regular triangle-free graph of order n with diameter 2 and H be a k-regular graph of order p. Let $r \geq \mu_2(G) \geq \ldots \geq \mu_n(G)$ and $k \geq \mu_2(H) \geq \ldots \geq \mu_p(H)$ be the $\mathcal{A}$-eigenvalues of G and H respectively. Then the $\mathscr{D}$-spectrum of $G \star H$ consists of the following.*

(a) $\frac{1}{2}\Big(p(2n+r)+2n-(r+k+4)\pm\sqrt{\big(p(2n+r)-2n+r-k\big)^2+4p\big(2n-r\big)^2}\Big)$

(b) $\frac{1}{2}\Big((p-1)\mu_i(G)-k-4\pm\sqrt{\big((p+1)\mu_i(G)-k\big)^2+4p\mu_i(G)^2}\Big)$, $i=2,\dots,n$, *provided* $\mu_i(G)\neq 0$ *for each* i.

(c) -2 *and* $-(k+2)$ *if* $0\in Spec_{\mathcal{A}}(G)$.

(d) $-\big(2+\mu_j(H)\big)$, $j=2,\dots,p$, *each with multiplicity* n.

Proof. Since $Diam(G)$ is 2, we have $\mathscr{D}(G)=2(J-I)-\mathcal{A}(G)$. Then equation (3.1) implies that

$$\mathscr{D}(G\star H)=\begin{bmatrix}2(J-I)-\mathcal{A}(G) & \mathbf{1}_p^T\otimes(2J-\mathcal{A}(G))\\ \mathbf{1}_p\otimes(2J-\mathcal{A}(G)) & D'\end{bmatrix}, \tag{3.3}$$

where

$$D'=J_p\otimes\Big(2(J-I)+\mathcal{A}(G)\Big)+\Big(2(J-I)-\mathcal{A}(H)\Big)\otimes I_n.$$

Let t_1 and α_1 be scalars such that

$$\mathscr{D}(G\star H)\begin{bmatrix}t_1\mathbf{1}_n\\ \mathbf{1}_p\otimes\mathbf{1}_n\end{bmatrix}=\alpha_1\begin{bmatrix}t_1\mathbf{1}_n\\ \mathbf{1}_p\otimes\mathbf{1}_n\end{bmatrix}. \tag{3.4}$$

Note that $\mathbf{1}_n$ is an eigenvector of G corresponding to r and $\mathbf{1}_p$ is an eigenvector of H corresponding to k. Then from equation (3.4) we have

$$t_1(2n-2-r)+p(2n-r)=\alpha_1 t_1 \tag{3.5}$$

and

$$t_1(2n-r)+p(2n+r)-2-k=\alpha_1. \tag{3.6}$$

Solving equations (3.5) and (3.6) we get

$$t_1=\frac{p(2n-r)}{\alpha_1-(2n-2-r)},\quad \text{provided}\ \ \alpha_1\neq 2n-2-r.$$

And in this case we obtain (a).
The case $\alpha_1=2n-2-r$ is never possible since this leads to $p(2n-r)=0$.
Now, let z_i be an eigenvector of G corresponding to $\mu_i(G), i=2,\dots,n$, that are orthogonal to $\mathbf{1}_n$. Also, let t_2 and α_2 be scalars such that

$$\mathscr{D}(G\star H)\begin{bmatrix}t_2 z_i\\ \mathbf{1}_p\otimes z_i\end{bmatrix}=\alpha_2\begin{bmatrix}t_2 z_i\\ \mathbf{1}_p\otimes z_i\end{bmatrix}. \tag{3.7}$$

This implies

$$t_2\big(2+\mu_i(G)\big)+p\mu_i(G)=-\alpha_2 t_2 \tag{3.8}$$

and

$$(p-t_2)\mu_i(G)-k-2=\alpha_2. \tag{3.9}$$

Equation (3.8) gives

$$t_2=\frac{-p\mu_i(G)}{\alpha_2+2+\mu_i(G)},\ \text{ provided }\ \alpha_2\neq-(2+\mu_i(G)).$$

Then from equation (3.9) we get
$\alpha_2=\frac{1}{2}\bigg((p-1)\mu_i(G)-k-4\pm\sqrt{\big((p+1)\mu_i(G)-k\big)^2+4p\mu_i(G)^2}\bigg).$
Now, $\alpha_2=-(2+\mu_i(G))\Rightarrow\mu_i(G)=0$. Hence we have (b).
If 0 is an $\mathcal{A}$-eigenvalue of G with multiplicity m then there exists m linearly independent eigenvectors $z_{i1},\ldots,z_{im}$ corresponding to 0. For each $j=1,\ldots,m$ consider $X_j=\begin{bmatrix}z_{ij}\\ \mathbf{0}_{pn}\end{bmatrix}$ and $Y_j=\begin{bmatrix}\mathbf{0}_n\\ \mathbf{1}_p\otimes z_{ij}\end{bmatrix}$. It can be seen that $\mathscr{D}(G\star H)X_j=-2X_j$ and $\mathscr{D}(G\star H)Y_j=-(k+2)Y_j$. Thus (c) follows. Finally, (d) follows from theorem 7. □

Remark 3. If the graph G in theorem 8 is s-transmission regular then the $\mathscr{D}$-eigenvalues (a) and (b) of $G\star H$ can be expressed in terms of $\mathscr{D}$-eigenvalues of G since $\mu_1(G)=2n-s-2$ and $\mu_i(G)=-(2+\rho_{n-i+2}),\ \ i=2,\ldots,n.$

3.1. Distance equienergetic graphs

In this section we construct infinitely many pairs of $\mathscr{D}$-non-cospectral and $\mathscr{D}$-equienergetic graphs.

Theorem 9. *Let G be an r-regular triangle-free graph on n vertices with diameter 2 and let H and H' be two $\mathcal{A}$-non-cospectral k-regular graphs on p vertices. Then for all $m\geq1$, $G\star\mathrm{L}^m(H)$ and $G\star\mathrm{L}^m(H')$ form a pair of $\mathscr{D}$-non-cospectral and $\mathscr{D}$-equienergetic graphs.*

Proof. The iterated line graph $\mathrm{L}^m(H)$, $m\geq1$, is of order

$$p_m=\frac{p}{2^m}\prod_{i=0}^{m-1}(2^ik-2^{i+1}+2).$$

Case 1: $m=1$
By theorems 2 and 8, the $\mathscr{D}$-eigenvalues of $G\star\mathrm{L}(H)$ are the following.

(a) $\frac{1}{2}\bigg(\frac{pk}{2}(2n+r)+2n-r-2k-2\pm\sqrt{\Big(\frac{pk}{2}(2n+r)-2n+r-2k+2\Big)^2+2pk(2n-r)^2}\bigg)$

(b) $\frac{1}{2}\left(\left(\frac{pk}{2}-1\right)\mu_i(G)-2k-2\pm\sqrt{\left(\left(\frac{pk}{2}+1\right)\mu_i(G)-2k+2\right)^2+2pk\mu_i(G)^2}\right)$, $i=2,\ldots,n$, provided $\mu_i(G)\neq 0$ for each i

(c) -2 and $-2k$ if $0\in Spec_{\mathcal{A}}(G)$

(d) $-(\mu_j(H)+k)$, $j=2,\ldots,p$, each with multiplicity n

(e) 0 with multiplicity $\frac{np(k-2)}{2}$.

Similarly we obtain the $\mathscr{D}$-eigenvalues of $G\star\mathrm{L}(H')$. Note that the $\mathscr{D}$-spectra differ only in the $\mathscr{D}$-eigenvalues given by (d).
Consider

$$\begin{aligned} n\sum_{j=2}^{p}\left|-\left(\mu_j(H)+k\right)\right| &= n\sum_{j=2}^{p}(\mu_j(H)+k) \text{ since } \mu_j(H)\geq -k \text{ for } j=2,\ldots,p \\ &= nk(p-2). \end{aligned}$$

The same argument can be applied for the $\mathscr{D}$-eigenvalues $-(\mu_j(H')+k), j=2,\ldots,p$ of $G\star\mathrm{L}(H')$. Then by equation (1.2), it follows that $\mathcal{DE}(G\star\mathrm{L}(H))=\mathcal{DE}(G\star\mathrm{L}(H'))$.
Case 2: $m\geq 2$
The m-fold application of theorem 2 gives the $\mathcal{A}$-eigenvalues of $L^m(H)$, $m\geq 2$, as listed below.

$$\left.\begin{array}{l} \mu_j(H)+(2^m-1)(k-2), \quad j=1,\ldots,p, \\ (2^m-2)k-2(2^m-1), \ \dfrac{p(k-2)}{2} \ \text{ times }, \\ (2^m-2^i)k-2(2^m-2^i+1), \ R_i \ \text{ times }, \ i=2,\ldots,m, \\ \text{where } \quad R_i=\dfrac{p(k-2)}{2}\displaystyle\prod_{j=0}^{i-2}(2^jk-2^{j+1}+2). \end{array}\right\} \qquad (3.10)$$

By theorem 8 and using equations (1.3) and (1.4), $Spec_{\mathscr{D}}(G\star\mathrm{L}^m(H))$ consists of the following.

(a) $\frac{1}{2}\Big((2n+r)p_m+2n-r-6-2^m(k-2)\pm$
$\sqrt{\left((2n+r)p_m-2n+r-2-2^m(k-2)\right)^2+4p_m\left(2n-2\right)^2}\Big)$

(b) $\frac{1}{2}\Big(\left(p_m-1\right)\mu_i(G)-2^m(k-2)-6\pm$
$\sqrt{\left(\left(p_m+1\right)\mu_i(G)-(2^mk-2^{m+1}+2)\right)^2+4p_m\mu_i(G)^2}\Big)$,
$i=2,\ldots,n$, provided $\mu_i(G)\neq 0$ for each i

(c) -2 and $-\big(2^m(k-2)+4\big)$, if $0 \in Spec_{\mathcal{A}}(G)$

(d) $-\big(\mu_j(H)+k+(2^m-2)(k-2)\big)$, $j=2,\dots,p$, each repeated n times

(e) $-(2^m-2)(k-2)$, $\frac{np(k-2)}{2}$ times

(f) $-(2^m-2^i)(k-2)$, nR_i times, $i=2,\dots,m$.

In the same way we obtain $Spec_{\mathscr{D}}(G \star \mathrm{L}^m(H'))$. The two $\mathscr{D}$-spectra differ only in the set of $\mathscr{D}$-eigenvalues given in (d). Since there exists no graphs H and H' as in the assertion of the theorem for $k=0$ and 1 we let $k \geq 2$. Therefore,

$$\begin{aligned} n\sum_{j=2}^{p}\left|-\big(\mu_j(H)+k+(2^m-2)(k-2)\big)\right| =& n\sum_{j=2}^{p}\big(\mu_j(H)+k+(2^m-2)(k-2)\big) \\ =& n\big((p-1)(2^m k-2^{m+1}+4)-pk\big). \end{aligned}$$

We can apply the same arguments to the $\mathscr{D}$-eigenvalues $-\big(\mu_j(H')+k+(2^m-2)(k-2)\big)$, $j=2,\dots,p$ of $G\star \mathrm{L}^m(H')$. Then the result follows by equation (1.2). □

Theorem 10. *Let G be an r-regular triangle-free graph of order n with diameter 2 and H and H' be two $\mathcal{A}$-non-cospectral k-regular graphs of order p. Then $G\star\overline{\mathrm{L}^m(H)}$ and $G\star\overline{\mathrm{L}^m(H')}$, $m \geq 1$, form a pair of $\mathscr{D}$-non-cospectral and $\mathscr{D}$-equienergetic graphs if $\mu_p(H), \mu_p(H') \geq 2^{m+1}-1-(2^m-1)k$.*

Proof. **Case 1:** $m=1$
From theorems 1, 2 and 8, we obtain the $\mathscr{D}$-eigenvalues of $G\star\overline{\mathrm{L}(H)}$ as follows.

(a) $\frac{1}{2}\Bigg(\frac{pk}{2}(2n+r-1)+2n-r+2k-5 \pm \sqrt{\big(\frac{pk}{2}(2n+r-1)-2n+r+2k-1\big)^2+2pk(2n-r)^2}\Bigg)$

(b) $\frac{1}{2}\Bigg(\Big(\frac{pk}{2}-1\Big)\mu_i(G)-\frac{pk}{2}+2k-5\pm\sqrt{\Big(\big(\frac{pk}{2}+1\big)\mu_i(G)-\frac{pk}{2}+2k-1\Big)^2+2pk\mu_i(G)^2}\Bigg)$, $i=2,\dots,n$, provided $\mu_i(G)\neq 0$ for each i

(c) -2 and $-\frac{pk}{2}+2k-3$ if $0\in Spec_{\mathcal{A}}(G)$

(d) $\mu_j(H)+k-3$, $j=2,\dots,p$, each with multiplicity n

(e) -3 with multiplicity $\frac{np(k-2)}{2}$.

Similarly we get the $\mathscr{D}$-eigenvalues of $G\star\overline{\mathrm{L}(H')}$. Note that the two $\mathscr{D}$-spectra differ only in the set of $\mathscr{D}$-eigenvalues given by (d).

If $\mu_p(H) \geq 3-k$, then $\mu_j(H)+k-3 \geq 0$ for $j=2,\ldots,p$.

$$\begin{aligned}\therefore n\sum_{j=2}^{p}\big|\mu_j(H)+k-3\big| &= n\big(k(p-2)-3p+3\big)\\ &= n\sum_{j=2}^{p}\big|\mu_j(H')+k-3\big|.\end{aligned}$$

Then, by equation (1.2), it follows that $\mathcal{DE}\big(G\star\overline{\mathrm{L}(H)}\big)=\mathcal{DE}\big(G\star\overline{\mathrm{L}(H')}\big)$.
Case 2: $m \geq 2$
From the $\mathcal{A}$-eigenvalues of $\mathrm{L}^m(H)$, $m\geq 2$, given by (3.10) and using theorems 1 and 8, the $\mathscr{D}$-eigenvalues of $G\star\overline{\mathrm{L}^m(H)}$ are as follows.

(a) $\frac{1}{2}\Bigg((2n+r)p_m+2n-r-p_m+2^m(k-2)-1\pm \sqrt{\big((2n+r)p_m-2n+r-p_m+2^m(k-2)+3\big)^2+4p_m\big(2n-r\big)^2}\Bigg)$

(b) $\frac{1}{2}\Bigg(\left(p_m-1\right)\mu_i(G)-p_m+2^m(k-2)-1\pm \sqrt{\big(\left(p_m+1\right)\mu_i(G)-p_m+2^m(k-2)+3\big)^2+4p_m\mu_i(G)^2}\Bigg)$,
$i=2,\ldots,n$, provided $\mu_i(G)\neq 0$ for each i

(c) -2 and $-\big(p_m-2^m(k-2)-1\big)$, if $0\in Spec_{\mathcal{A}}(G)$

(d) $\mu_j(H)+(2^m-1)k-2^{m+1}+1$, $j=2,\ldots,p$, each repeated n times

(e) $(2^m-2)k-2^{m+1}+1$, $\frac{np(k-2)}{2}$ times

(f) $(2^m-2^i)(k-2)-3$, nR_i times, $\quad i=2,\ldots,m$.

Similarly we obtain the $Spec_{\mathscr{D}}(G\star\overline{\mathrm{L}^m(H')})$. Note that the two $\mathscr{D}$-spectra differ only in the set of $\mathscr{D}$-eigenvalues given by (d). Now, if $\mu_p(H)\geq 2^{m+1}-1-(2^m-1)k$ then $\mu_j(H)+(2^m-1)k-2^{m+1}+1\geq 0$ for $j=2,\ldots,p$. Therefore,

$$\begin{aligned}n\sum_{j=2}^{p}\big|\mu_j(H)+(2^m-1)k-2^{m+1}+1\big| =& n\sum_{j=2}^{p}\big(\mu_j(H)+(2^m-1)k-2^{m+1}+1\big)\\ =& n\Big(p\big((2^m-1)k-2^{m+1}+1\big)-2^m(k-2)-1\Big)\\ =& n\sum_{j=2}^{p}\big|\mu_j(H')+(2^m-1)k-2^{m+1}+1\big|.\end{aligned}$$

Then, by equation (1.2), it follows that $\mathcal{DE}\big(G\star\overline{\mathrm{L}^m(H)}\big)=\mathcal{DE}\big(G\star\overline{\mathrm{L}^m(H')}\big)$. □

Theorem 11. *Let G be an r-regular triangle-free graph on n vertices with diameter 2 and H and H' be k-regular graphs on p vertices that are $\mathcal{A}$-non-cospectral and equienergetic. Then $G\star\overline{D_2^*(H)}$ and $G\star\overline{D_2^*(H')}$ form a pair of $\mathscr{D}$-non-cospectral and $\mathscr{D}$-equienergetic graphs.*

Proof. By theorems 1, 6 and 8, the $\mathscr{D}$-eigenvalues of $G\star\overline{D_2^*(H)}$ are as follows.

(a) $\frac{1}{2}\Big(2p(2n+r)+2n-r-2p+2k-2\pm \sqrt{\big(2p(2n+r)-2n+r-2p+2k+2\big)^2+8p(2n-r)^2}\Big)$

(b) $\frac{1}{2}\Bigg(\big(2p-1\big)\mu_i(G)-2p+2k-2\pm\sqrt{\Big(\big(2p+1\big)\mu_i(G)-2p+2k+2\Big)^2+8p\mu_i(G)^2}\Bigg)$, $i=2,\dots,n$, provided $\mu_i(G)\neq 0$ for each i

(c) -2 and $-2(p-k)$ if $0\in Spec_{\mathcal{A}}(G)$

(d) $2\mu_j(H)$, $j=2,\dots,p$, each with multiplicity n

(e) -2 with multiplicity np.

The above $\mathscr{D}$-spectrum differs from $Spec_{\mathscr{D}}\big(G\star\overline{D_2^*(H')}\big)$ only in the set of $\mathscr{D}$-eigenvalues given by (d). Now,

$$n\sum_{j=2}^{p}|2\mu_j(H)|=2n\sum_{j=2}^{p}|\mu_j(H)|=2n\big(\mathcal{E}(H)-k\big).$$

Then the result follows since $\mathcal{E}(H)=\mathcal{E}(H')$. □

Theorem 12. *Let G be an r-regular triangle-free graph on n vertices with diameter 2. Let H and H' be k-regular graphs on p vertices that are $\mathcal{A}$-non-cospectral and equienergetic such that $n^+(H)-n^-(H)=d=n^+(H')-n^-(H')$. Then $G\star\overline{H}$ and $G\star\overline{H'}$ form a pair of $\mathscr{D}$-non-cospectral and $\mathscr{D}$-equienergetic graphs provided $|\mu_j(H)|,|\mu_j(H')|\geq 1$ for $j=2,\dots,p$.*

Proof. By theorems 1 and 8 the $Spec_{\mathscr{D}}(G\star\overline{H})$ consists of the following.

(a) $\frac{1}{2}\Big(p(2n+r)+2n-r-p+k-3\pm \sqrt{\big(p(2n+r)-2n+r-p+k+1\big)^2+4p(2n-r)^2}\Big)$

(b) $\frac{1}{2}\Big(\big(p-1\big)\mu_i(G)-p+k-3\pm\sqrt{\big(\big(p+1\big)\mu_i(G)-p+k+1\big)^2+4p\mu_i(G)^2}\Big)$, $i=2,\dots,n$, provided $\mu_i(G)\neq 0$ for each i

(c) -2 and $-(p-k+1)$ if $0\in Spec_{\mathcal{A}}(G)$

(d) $\mu_j(H)-1$, $j=2,\dots,p$, each with multiplicity n.

For $j = 2, \ldots, p$,

$$|\mu_j(H) - 1| = \begin{cases} |\mu_j(H)| - 1, & \text{if } \mu_j(H) > 0 \\ |\mu_j(H)| + 1, & \text{if } \mu_j(H) < 0 \end{cases}$$

$$\begin{aligned} \therefore n\sum_{j=2}^{p} |\mu_j(H) - 1| &= n\big(\sum_{j=1}^{p} |\mu_j(H) - 1| - (k-1)\big) \\ &= n\Bigg(\sum_{\mu_j(H)>0} \big(|\mu_j(H)| - 1\big) + \sum_{\mu_j(H)<0} \big(|\mu_j(H)| + 1\big) - k + 1\Bigg) \\ &= n\big(\mathcal{E}(H) - d - k + 1\big). \end{aligned}$$

The same arguments can be applied for the $\mathscr{D}$-eigenvalues of $G \star \overline{H'}$. Then the result follows from equation (1.2). □

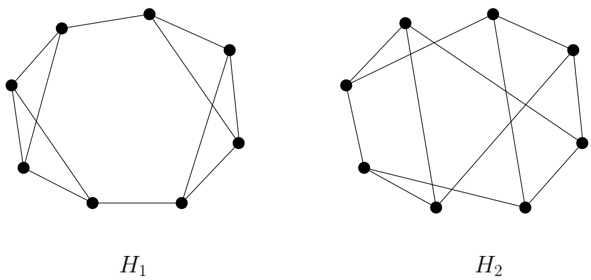

Figure 3. **Two non-isomorphic 3-regular graphs of order 8**

Example 3. Let $G = K_{3,3}$ and H_1 and H_2 be the graphs as shown in figure 3. Then $Spec_{\mathcal{A}}(G) = \{3, 0^4, -3\}$, $Spec_{\mathcal{A}}(H_1) = \{3, \sqrt{5}, 1, (-1)^4, -\sqrt{5}\}$ and $Spec_{\mathcal{A}}(H_2) = \{3, 1^2, (\sqrt{2}-1)^2, -1, (-1-\sqrt{2})^2\}$. Let $H = \mathrm{L}^3(H_1)$ and $H' = \mathrm{L}^3(H_2)$. Then $Spec_{\mathcal{A}}(H) = \{10, 7+\sqrt{5}, 8, 6^4, 7-\sqrt{5}, 4^4, 2^{12}, (-2)^{48}\}$ and $Spec_{\mathcal{A}}(H') = \{10, 8^2, (6+\sqrt{2})^2, 6, (6-\sqrt{2})^2, 4^4, 2^{12}, (-2)^{48}\}$. It can be seen that $\mathcal{E}(H) = 192 = \mathcal{E}(H')$ and $n^+(H) - n^-(H) = -24 = n^+(H') - n^-(H')$. Also $|\mu_j(H)|, |\mu_j(H')| > 1$ for $j = 2, \ldots, 72$. We have $Spec_{\mathcal{D}}(G \star \overline{H}) = \{512 + \sqrt{260857}, 512 - \sqrt{260857}, -139 + 2\sqrt{5062}, -139 - 2\sqrt{5062}, (-2)^4, (-63)^4, (6-\sqrt{5})^6, 5^{24}, 7^6, (6+\sqrt{5})^6, 3^{24}, 1^{72}, (-3)^{288}\}$ and $Spec_{\mathcal{D}}(G \star \overline{H'}) = \{512 + \sqrt{260857}, 512 - \sqrt{260857}, -139 + 2\sqrt{5062}, -139 - 2\sqrt{5062}, (-2)^4, (-63)^4, (5-\sqrt{2})^{12}, 5^6, (5+\sqrt{2})^{12}, 7^{12}, 3^{24}, 1^{72}, (-3)^{288}\}$. It is obtained that $\mathcal{DE}(G \star \overline{H}) = 2526 + 4\sqrt{5062} = \mathcal{DE}(G \star \overline{H'})$.

Theorem 13. *Let G be an r-regular triangle-free graph on n vertices with diameter 2. Let H and H' be k-regular graphs on p vertices that are $\mathcal{A}$-non-cospectral and equienergetic* *such that $n^+(H) - n^-(H) = d = n^+(H') - n^-(H')$. Then $G \star D_2^*(H)$ and $G \star D_2^*(H')$ form a pair of $\mathscr{D}$-non-cospectral and $\mathscr{D}$-equienergetic graphs provided $|\mu_j(H)|, |\mu_j(H')| \geq \frac{3}{2}$ for $j = 2, \ldots, p$.*

Proof. By theorems 6 and 8 $Spec_{\mathscr{D}}(G \star D_2^*(H))$ consists of the following.

(a) $\frac{1}{2}\Big(2p(2n+r)+2n-r-2k-5 \pm \sqrt{\big(2p(2n+r)-2n+r-2k-1\big)^2 + 8p(2n-r)^2}\Big)$

(b) $\frac{1}{2}\Bigg(\big(2p-1\big)\mu_i(G) - 2k - 5 \pm \sqrt{\Big(\big(2p+1\big)\mu_i(G) - 2k - 1\Big)^2 + 8p\mu_i(G)^2}\Bigg)$,
$i = 2, \ldots, n$, provided $\mu_i(G) \neq 0$ for each i

(c) -2 and $-(2k+3)$ if $0 \in Spec_{\mathcal{A}}(G)$

(d) $-(2\mu_j(H)+3)$, $j = 2, \ldots, p$, each with multiplicity n

(e) -1 with multiplicity np.

Similarly we obtain $Spec_{\mathscr{D}}(G \star D_2^*(H'))$.
For $j = 2, \ldots,$ p,

$$\big| -(2\mu_j(H)+3)\big| = 2\left|\mu_j(H) + \frac{3}{2}\right|$$
$$= \begin{cases} 2\big(|\mu_j(H)| + \frac{3}{2}\big), & \text{if } \mu_j(H) > 0 \\ 2\big(|\mu_j(H)| - \frac{3}{2}\big), & \text{if } \mu_j(H) < 0 \end{cases}$$
$$\therefore n\sum_{j=2}^{p} \big| -(2\mu_j(H)+3)\big| = n\big(2\mathcal{E}(H) + 3d - (2k+3)\big).$$

Then the result follows from equation (1.2). ☐

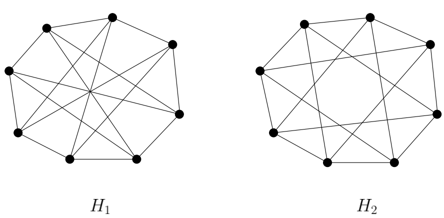

Figure 4. **Two non-isomorphic 4-regular graphs of order 8**

Example 4. Let $G = K_{3,3}$ and H_1 and H_2 be the graphs as shown in figure 4. Then $Spec_{\mathcal{A}}(H_1) = \{4, 2, 0^3, (-2)^3\}$ and $Spec_{\mathcal{A}}(H_2) = \{4, 0^6, -4\}$. Let $H = \mathrm{L}^2(H_1)$ and $H' = \mathrm{L}^2(H_2)$. Then $Spec_{\mathcal{A}}(H) = \{10, 8, 6^3, 4^3, 2^8, (-2)^{32}\}$ and

$Spec_{\mathcal{A}}(H') = \{10, 6^6, 2^9, (-2)^{32}\}$. It can be seen that $\mathcal{E}(H) = 128 = \mathcal{E}(H')$ and $n^+(H) - n^-(H) = -16 = n^+(H') - n^-(H')$. Also $|\mu_j(H)|, |\mu_j(H')| > \frac{3}{2}$ for $j = 2, \ldots, 48$. Then $Spec_{\mathcal{D}}(G \star D_2^*(H)) = \{712 + 3\sqrt{56089}, 712 - 3\sqrt{56089}, -155 + 60\sqrt{7}, -155 - 60\sqrt{7}, (-2)^4, (-23)^4, (-19)^6, (-15)^{18}, (-11)^{18}, (-7)^{48}, 1^{192}, (-1)^{288}\}$ and $Spec_{\mathcal{D}}(G \star D_2^*(H')) = \{712 + 3\sqrt{56089}, 712 - 3\sqrt{56089}, -155 + 60\sqrt{7}, -155 - 60\sqrt{7}, (-2)^4, (-23)^4, (-15)^{36}, (-7)^{54}, 1^{192}, (-1)^{288}\}$. We get $\mathcal{DE}(G \star D_2^*(H)) = 2922 + 120\sqrt{7} = \mathcal{DE}(G \star D_2^*(H'))$.

Theorem 14. *Let G be an r-regular triangle-free graph on n vertices with diameter 2 and H and H' be k-regular graphs on p vertices that are $\mathcal{A}$-non-cospectral and equienergetic such that $n^+(H) - n^-(H) = d = n^+(H') - n^-(H')$. Then $G \star D_2(H)$ and $G \star D_2(H')$ form a pair of $\mathscr{D}$-non-cospectral and $\mathscr{D}$-equienergetic graphs provided $|\mu_j(H)|, |\mu_j(H')| \geq 1$ for $j = 2, \ldots, p$.*

Proof. By theorems 5 and 8 the $Spec_{\mathscr{D}}(G \star D_2(H))$ consists of the following.

(a) $\frac{1}{2}\left(2p(2n+r) + 2n - r - 2k - 4 \pm \sqrt{\left(2p(2n+r) - 2n + r - 2k\right)^2 + 8p(2n-r)^2}\right)$

(b) $\frac{1}{2}\left((2p-1)\mu_i(G) - 2k - 4 \pm \sqrt{\left((2p+1)\mu_i(G) - 2k\right)^2 + 8p\mu_i(G)^2}\right)$,
$i = 2, \ldots, n$, provided $\mu_i(G) \neq 0$ for each i

(c) -2 and $-2(k+1)$ if $0 \in Spec_{\mathcal{A}}(G)$

(d) $-2(\mu_j(H) + 1)$, $j = 2, \ldots, p$, each with multiplicity n

(e) -2 with multiplicity np.

Similarly we obtain the $Spec_{\mathscr{D}}(G \star D_2(H'))$. For $j = 2, \ldots,$ p,

$$\left|\mu_j(H) + 1\right| = \begin{cases} |\mu_j(H)| + 1, & \text{if } \mu_j(H) > 0 \\ |\mu_j(H)| - 1, & \text{if } \mu_j(H) < 0 \end{cases}$$

$$\therefore n \sum_{j=2}^{p} \left| -2\left(\mu_j(H) + 1\right)\right| = 2n\left(\mathcal{E}(H) + d - (k+1)\right).$$

Then the result follows from equation (1.2). □

Example 5. Let $G = K_{3,3}$ and H_1 and H_2 be the graphs as shown in Figure 5. Then $Spec_{\mathcal{A}}(H_1) = \{4, 1^4, (-1)^4, -4\}$ and $Spec_{\mathcal{A}}(H_2) = \left\{4, -1, 0^2, \frac{\sqrt{5}-1}{2}, \frac{-\sqrt{5}-1}{2}, \frac{\sqrt{19+2\sqrt{17}}-1}{2}, \frac{-\sqrt{19+2\sqrt{17}}-1}{2}, \frac{\sqrt{19-2\sqrt{17}}-1}{2}, \frac{-\sqrt{19-2\sqrt{17}}-1}{2}\right\}$.
Let $H = \mathrm{L}^2(H_1)$ and $H' = \mathrm{L}^2(H_2)$. Then $Spec_{\mathcal{A}}(H) = \{10, 7^4, 5^4, 2^{11}, (-2)^{40}\}$ and $Spec_{\mathcal{A}}(H') = \left\{10, 6^2, \frac{\sqrt{19+2\sqrt{17}}+11}{2}, \frac{\sqrt{19-2\sqrt{17}}+11}{2}, 5, \frac{11+\sqrt{5}}{2}, \frac{11-\sqrt{5}}{2}, \frac{-\sqrt{19+2\sqrt{17}}+11}{2}, \frac{-\sqrt{19-2\sqrt{17}}+11}{2}, 2^{10}, (-2)^{40}\right\}$. It can be seen that $\mathcal{E}(H) = 160 = \mathcal{E}(H')$ and

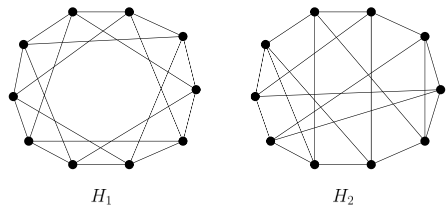

Figure 5. **Two non-isomorphic 4-regular graphs of order 10**

$n^+(H) - n^-(H) = -20 = n^+(H') - n^-(H')$. Also $|\mu_j(H)|, |\mu_j(H')| > 1$ for $j = 2, \ldots, 60$. Then $Spec_{\mathcal{D}}(G \star D_2(H)) = \Big\{ \frac{1785}{2} + \frac{1}{2}\sqrt{3175321}, \frac{1785}{2} - \frac{1}{2}\sqrt{3175321}, \frac{-381}{2} + \frac{1}{2}\sqrt{151009}, \frac{-381}{2} - \frac{1}{2}\sqrt{151009}, (-22)^4, (-2)^{364}, (-16)^{24}, (-12)^{24}, (-6)^{66}, 2^{240} \Big\}$ and $Spec_{\mathcal{D}}(G \star D_2(H')) = \Big\{ \frac{1785}{2} + \frac{1}{2}\sqrt{3175321}, \frac{1785}{2} - \frac{1}{2}\sqrt{3175321}, \frac{-381}{2} + \frac{1}{2}\sqrt{151009}, \frac{-381}{2} - \frac{1}{2}\sqrt{151009}, (-22)^4, (-2)^{364}, \big(-13-\sqrt{19+2\sqrt{17}}\big)^6, \big(-13-\sqrt{19-2\sqrt{17}}\big)^6, (-14)^{12}, (-12)^6, (-13+\sqrt{5})^6, (-13-\sqrt{5})^6, \big(-13+\sqrt{19+2\sqrt{17}}\big)^6, \big(-13+\sqrt{19-2\sqrt{17}}\big)^6, (-6)^{60}, 2^{240} \Big\}$. We get $\mathcal{DE}(G\star D_2(H)) = 4149+\sqrt{151009} = \mathcal{DE}(G\star D_2(H'))$.

Theorem 15. *Let G be an r-regular triangle-free graph on n vertices with diameter 2. Let H and H' be k-regular graphs on p vertices that are $\mathcal{A}$-non-cospectral and equienergetic such that $n^+(H) - n^-(H) = d = n^+(H') - n^-(H')$. Then $G \star \overline{D_2(H)}$ and $G \star \overline{D_2(H')}$ form a pair of $\mathscr{D}$-non-cospectral and $\mathscr{D}$-equienergetic graphs provided $|\mu_j(H)|, |\mu_j(H')| \geq \frac{1}{2}$ for $j = 2, \ldots, p$.*

Proof. By theorems 1, 5 and 8 $Spec_{\mathscr{D}}(G \star \overline{D_2(H)})$ consists of the following.

(a) $\frac{1}{2}\Bigg(2p(2n+r) + 2n - r - 2p + 2k - 3 \pm \sqrt{\big(2p(2n+r) - 2n + r - 2p + 2k + 1\big)^2 + 8p(2n-r)^2}\Bigg)$

(b) $\frac{1}{2}\Bigg(\big(2p-1\big)\mu_i(G) - 2p + 2k - 3 \pm \sqrt{\big(\big(2p+1\big)\mu_i(G) - 2p + 2k + 1\big)^2 + 8p\mu_i(G)^2}\Bigg)$, $i = 2, \ldots, n$, provided $\mu_i(G) \neq 0$ for each i

(c) -2 and $-(2p - 2k + 1)$ if $0 \in Spec_{\mathcal{A}}(G)$

(d) $2\mu_j(H) - 1$, $j = 2, \ldots, p$, each with multiplicity n

(e) -1 with multiplicity np.

In the same way we obtain $Spec_{\mathscr{D}}(G \star \overline{D_2(H')})$. Now, for $j = 2, \ldots, p$,

$$|2\mu_j(H) - 1| = \begin{cases} 2\big(\,|\mu_j(H)| - \frac{1}{2}\big), & \text{if } \mu_j(H) > 0 \\ 2\big(\,|\mu_j(H)| + \frac{1}{2}\big), & \text{if } \mu_j(H) < 0 \end{cases}$$

$$\therefore n \sum_{j=2}^{p} \big|2\mu_j(H) - 1\big| = n\big(2\mathcal{E}(H) - d - 2k + 1\big).$$

The result then follows from equation (1.2). □

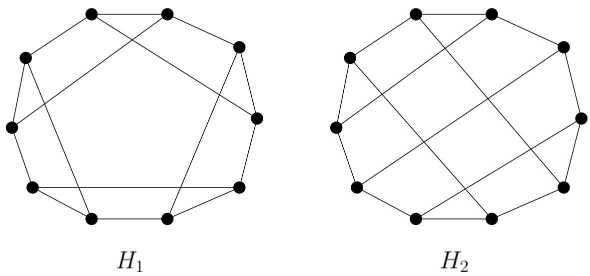

Figure 6. **Two non-isomorphic 3-regular graphs of order 10**

Example 6. Let $G = K_{3,3}$ and H_1 and H_2 be the graphs as shown in figure 6. Then $Spec_{\mathcal{A}}(H_1) = \left\{3, \left(\sqrt{\frac{3+\sqrt{5}}{2}}\right)^2, \left(\sqrt{\frac{3-\sqrt{5}}{2}}\right)^2, \left(-\sqrt{\frac{3-\sqrt{5}}{2}}\right)^2, \left(-\sqrt{\frac{3+\sqrt{5}}{2}}\right)^2, -3\right\}$ and $Spec_{\mathcal{A}}(H_2) = \left\{3, \left(\frac{1+\sqrt{5}}{2}\right)^2, 1, \left(\frac{\sqrt{5}-3}{2}\right)^2, \left(\frac{1-\sqrt{5}}{2}\right)^2, \left(\frac{-3-\sqrt{5}}{2}\right)^2\right\}$. Let $H = \mathrm{L}^3(H_1)$ and $H' = \mathrm{L}^3(H_2)$. Then $Spec_{\mathcal{A}}(H) = \left\{10, \left(7 + \sqrt{\frac{3+\sqrt{5}}{2}}\right)^2, \left(7 + \sqrt{\frac{3-\sqrt{5}}{2}}\right)^2, \left(7 - \sqrt{\frac{3-\sqrt{5}}{2}}\right)^2, \left(7 - \sqrt{\frac{3+\sqrt{5}}{2}}\right)^2, 4^6, 2^{15}, (-2)^{60}\right\}$ and $Spec_{\mathcal{A}}(H') = \left\{10, \left(\frac{15+\sqrt{5}}{2}\right)^2, 8, \left(\frac{11+\sqrt{5}}{2}\right)^2, \left(\frac{15-\sqrt{5}}{2}\right)^2, \left(\frac{11-\sqrt{5}}{2}\right)^2, 4^5, 2^{15}, (-2)^{60}\right\}$. It can be seen that $\mathcal{E}(H) = 240 = \mathcal{E}(H')$ and $n^+(H) - n^-(H) = -30 = n^+(H') - n^-(H')$. Also $|\mu_j(H)|, |\mu_j(H')| > 1$ for $j = 2, \ldots, 90$. Then $Spec_{\mathcal{D}}\big(G \star \overline{D_2(H)}\big) = \Big\{1273 + 6\sqrt{44926}, 1273 - 6\sqrt{44926}, -350 + 9\sqrt{1541}, -350 - 9\sqrt{1541}, (-2)^4, (-161)^4, \left(13 + 2\sqrt{\frac{3+\sqrt{5}}{2}}\right)^{12}, \left(13 + 2\sqrt{\frac{3-\sqrt{5}}{2}}\right)^{12}, \left(13 - 2\sqrt{\frac{3-\sqrt{5}}{2}}\right)^{12}, \left(13 - 2\sqrt{\frac{3+\sqrt{5}}{2}}\right)^{12}, 7^{36}, (3)^{90}, (-5)^{360}, (-1)^{540}\Big\}$ and $Spec_{\mathcal{D}}\big(G \star \overline{D_2(H')}\big) = \Big\{1273 + 6\sqrt{44926}, 1273 - 6\sqrt{44926}, -350 + 9\sqrt{1541}, -350 - 9\sqrt{1541}, (-2)^4, (-161)^4, \big(14 + \sqrt{5}\big)^{12},$

$(15)^6, \left(10+\sqrt{5}\right)^{12}, \left(14-\sqrt{5}\right)^{12}, \left(10-\sqrt{5}\right)^{12}, (7)^{30}, (3)^{90}, (-5)^{360}, (-1)^{540}\Big\}$. We get $\mathcal{DE}\big(G\star\overline{D_2(H)}\big) = 6684 + 18\sqrt{1541} = \mathcal{DE}\big(G\star\overline{D_2(H')}\big)$.

Remark 4. There are well-known families of graphs satisfying the eigenvalue constraints in theorems 12 to 15. For, if H_1 and H_2 are any two regular, non-cospectral graphs of order p and degree $k \geq 3$, then it follows from theorems 3 and 4 that the iterated line graphs $H = \mathrm{L}^m(H_1)$ and $H' = \mathrm{L}^m(H_2)$, for $m \geq 3$, form a pair of graphs satisfying the eigenvalue constraints in theorems 12 to 15.

Remark 5. The eigenvalue constraints in theorems 12 to 15 can be relaxed. For example, in theorem 12, $K_{3,3}$ and $C_3\Box P_2$ do not satisfy the eigenvalue constraints. But for any regular triangle-free graph G of diameter 2, $G\star\overline{K_{3,3}}$ and $G\star\overline{C_3\Box P_2}$ are $\mathscr{D}$-non-cospectral and $\mathscr{D}$-equienergetic graphs.

3.2. Distance cospectral graphs

In this section we construct infinitely many non-isomorphic pairs of $\mathscr{D}$-cospectral graphs using theorem 8.

Theorem 16. *Let G be a regular triangle-free graph with diameter 2 and let H and H' be two non-isomorphic k-regular $\mathcal{A}$-cospectral graphs of order p. Then $G\star H$ and $G\star H'$ form a pair of non-isomorphic $\mathscr{D}$-cospectral graphs.*

Corollary 1. *Let G be a regular triangle-free graph with diameter 2 and let H and H' be two non-isomorphic k-regular $\mathcal{A}$-cospectral graphs of order p. Then*

(a) $G\star\mathrm{L}^m(H)$ and $G\star\mathrm{L}^m(H')$, $m\geq 1$, form a pair of non-isomorphic $\mathscr{D}$-cospectral graphs.

(b) $G\star D_2(H)$ and $G\star D_2(H')$ form a pair of non-isomorphic $\mathscr{D}$-cospectral graphs.

(c) $G\star D_2^*(H)$ and $G\star D_2^*(H')$ form a pair of non-isomorphic $\mathscr{D}$-cospectral graphs.

(d) $G\star\overline{H}$ and $G\star\overline{H'}$ form a pair of non-isomorphic $\mathscr{D}$-cospectral graphs.

4. Distance Laplacian spectrum of $G\star H$

With respect to the labeling of the vertex set of $G\star H$, described in section 3, the distance Laplacian matrix of $G\star H$, when G is a connected triangle-free graph, can be written as

$$\mathscr{D}^{\mathscr{L}}(G\star H) = \begin{bmatrix} \mathscr{D}^{\mathscr{L}}(G) + pTr(G) + 2pI & \mathbf{1}_p^T \otimes -(\mathscr{D}(G)+2I) \\ \mathbf{1}_p \otimes -(\mathscr{D}(G)+2I) & D^{L'} \end{bmatrix}, \tag{4.1}$$

where

$$\begin{aligned} D^{L'} =& J_p \otimes -\big(\mathscr{D}(G) + 2\mathcal{A}(G)\big) + I_p \otimes \big((p+1)Tr(G) + 2I + 2p\mathcal{D}_G\big) \\ &+ \big(2pI - 2J - \mathscr{L}(H)\big) \otimes I_n. \end{aligned}$$

The following result gives $n(p-1)$ $\mathscr{D}^{\mathscr{L}}$-eigenvalues of $G \star H$ when G is a connected triangle-free graph and H is arbitrary.

Theorem 17. *Let G be a connected triangle-free graph with $V(G) = \{u_1, \ldots, u_n\}$ and H be a graph of order p with $\mathscr{L}$-eigenvalues $\eta_1(H) \geq \eta_2(H) \geq \ldots \geq \eta_p(H) = 0$. Then $(p+1)Tr(u_i) + 2p\big(d(u_i)+1\big) + 2 - \eta_j(H)$ are $\mathscr{D}^{\mathscr{L}}$-eigenvalues of $G \star H$, for $i = 1, \ldots, n$ and $j = 1, \ldots, p-1$.*

Proof. Clearly $\mathbf{1}_p$ is an eigenvector of $\mathscr{L}(H)$ corresponding to $\eta_p(H) = 0$. Let $f_1, \ldots, f_{p-1}$ be eigenvectors of $\mathscr{L}(H)$ corresponding to $\eta_1(H), \ldots, \eta_{p-1}(H)$ respectively, that are orthogonal to $\mathbf{1}_p$. Then for $i = 1, \ldots, n$ and $j = 1, \ldots, p-1$,

$$\mathscr{D}^{\mathscr{L}}(G \star H) \begin{bmatrix} \mathbf{0}_n \\ f_j \otimes \mathbf{e}_i \end{bmatrix} = \Big((p+1)Tr(u_i) + 2p\big(d(u_i)+1\big) + 2 - \eta_j(H)\Big) \begin{bmatrix} \mathbf{0}_n \\ f_j \otimes \mathbf{e}_i \end{bmatrix}.$$

□

In the next theorem we obtain the $Spec_{\mathscr{D}^{\mathscr{L}}}(G \star H)$ in terms of $\mathscr{D}^{\mathscr{L}}$-eigenvalues of G and $\mathscr{L}$-eigenvalues of H when G is a transmission regular triangle-free graph and H is arbitrary.

Theorem 18. *Let G be a s-transmission regular triangle-free graph of diameter 2 with $\mathscr{D}^{\mathscr{L}}$-eigenvalues $\rho_1^{\mathscr{L}}(G) \geq \rho_2^{\mathscr{L}}(G) \geq \ldots \geq \rho_n^{\mathscr{L}}(G) = 0$. Let H be a graph of order p with $\mathscr{L}$-eigenvalues $\eta_1(H) \geq \eta_2(H) \geq \ldots \geq \eta_p(H) = 0$. Then the distance Laplacian eigenvalues of $G \star H$ are the following.*

(a) $0, \ (p+1)(s+2)$

(b) $\frac{1}{2}\Big(p(4n+s+2) + s + 2 + (1-p)\rho_i^L(G) \pm$
$\sqrt{\big(p(s+2-4n) - s - 2 + (1+p)\rho_i^L(G)\big)^2 + 4p\big(s+2-\rho_i^L(G)\big)^2}\Big)$, $i = 1, \ldots, n-1$,
provided $\rho_i^L(G) \neq s+2$ for any i

(c) $(p+1)(s+2)$ *and* $p(4n-s-2) + s + 2$ *if* $s+2 \in Spec_{\mathscr{D}^{\mathscr{L}}}(G)$

(d) $p(4n-s-2) + s + 2 - \eta_j(H)$, $j = 1, \ldots, p-1$, *each with multiplicity n.*

Proof. Clearly G is $2n-s-2$ regular. Let t_1 and β_1 be scalars such that

$$\mathscr{D}^{\mathscr{L}}(G \star H) \begin{bmatrix} t_1\mathbf{1}_n \\ \mathbf{1}_p \otimes \mathbf{1}_n \end{bmatrix} = \beta_1 \begin{bmatrix} t_1\mathbf{1}_n \\ \mathbf{1}_p \otimes \mathbf{1}_n \end{bmatrix}. \tag{4.2}$$

Note that $\mathbf{1}_n$ is an eigenvector corresponding to the $\mathscr{D}^{\mathscr{L}}$-eigenvalue 0 of G, the $\mathscr{D}$-eigenvalue s of G and the $\mathcal{A}$-eigenvalue $2n-2-s$ of G. Therefore, equation (4.2) implies

$$(t_1 - 1)p(s+2) = t_1\beta_1 \tag{4.3}$$

and

$$(1 - t_1)(s+2) = \beta_1. \tag{4.4}$$

Solving equations (4.3) and (4.4) we get

$$t_1 = \frac{p(s+2)}{p(s+2) - \beta_1}, \text{ provided } \beta_1 \neq p(s+2),$$

and, in this case we obtain $\beta_1 = 0$ and $(p+1)(s+2)$.
The case $\beta_1 = p(s+2)$ can be omitted since it implies $p = 0$ or $s = -2$, both are invalid. Hence we have (a).
Let x_i be an eigenvector corresponding to $\rho_i^{\mathscr{L}}(G)$, $i = 1, \ldots, n-1$, that are orthogonal to $\mathbf{1}_n$. Let t_2 and β_2 be scalars such that

$$\mathscr{D}^{\mathscr{L}}(G \star H) \begin{bmatrix} t_2 x_i \\ \mathbf{1}_p \otimes x_i \end{bmatrix} = \beta_2 \begin{bmatrix} t_2 x_i \\ \mathbf{1}_p \otimes x_i \end{bmatrix}. \tag{4.5}$$

Since $\mathscr{D}(G) = Tr(G) - \mathscr{D}^{\mathscr{L}}(G)$ and $\mathcal{A}(G) = 2(J - I) - \mathscr{D}(G)$, from equation (4.5) we have

$$t_2\left(\rho_i^{\mathscr{L}}(G) + ps + 2p\right) + p\left(\rho_i^{\mathscr{L}}(G) - s - 2\right) = t_2\beta_2 \tag{4.6}$$

and

$$-t_2\left(s + 2 - \rho_i^{\mathscr{L}}(G)\right) + p\left(4n - \rho_i^{\mathscr{L}}(G)\right) + s + 2 = \beta_2. \tag{4.7}$$

Solving equations (4.6) and (4.7),

$$t_2 = \frac{p(s + 2 - \rho_i^{\mathscr{L}}(G))}{\rho_i^{\mathscr{L}}(G) + ps + 2p - \beta_2}, \text{ provided } \beta_2 \neq \rho_i^{\mathscr{L}}(G) + ps + 2p,$$

and, in this case we get

$$\begin{aligned}\beta_2 =& \frac{1}{2}\Bigg(p(4n + s + 2) + s + 2 + (1 - p)\rho_i^{\mathscr{L}}(G) \pm \\ & \sqrt{\left(p(s + 2 - 4n) - s - 2 + (1 + p)\rho_i^{\mathscr{L}}(G)\right)^2 + 4p\left(s + 2 - \rho_i^{\mathscr{L}}(G)\right)^2}\Bigg).\end{aligned}$$

Now, if $\beta_2 = \rho_i^{\mathscr{L}}(G) + ps + 2p$ then we get $\rho_i^{\mathscr{L}}(G) = s + 2$. Hence (b) follows.
If $\rho_i^{\mathscr{L}}(G) = s + 2$ for any i then

$$\mathscr{D}^{\mathscr{L}}(G \star H) \begin{bmatrix} x_i \\ \mathbf{0}_{pn} \end{bmatrix} = (p+1)(s+2) \begin{bmatrix} x_i \\ \mathbf{0}_{pn} \end{bmatrix} \tag{4.8}$$

and

$$\mathscr{D}^{\mathscr{L}}(G\star H)\begin{bmatrix}\mathbf{0}_n\\ \mathbf{1}_p\otimes x_i\end{bmatrix}=\Big(p(4n-s-2)+s+2\Big)\begin{bmatrix}\mathbf{0}_n\\ \mathbf{1}_p\otimes x_i\end{bmatrix}. \tag{4.9}$$

From equations (4.8) and (4.9), we obtain (c).
Finally, (d) follows from theorem 17 since $Tr(u_i) = s$ and $d(u_i) = 2n-2-s$ for $i = 1,\ldots,n$. □

Remark 6. In the above theorem the $\mathscr{D}^{\mathscr{L}}$-eigenvalues (b) of $G\star H$ can be expressed by the $\mathscr{D}$-eigenvalues of G since $\rho_i^{\mathscr{L}}(G) = s-\rho_{n-i+1}(G), i = 1,\ldots,n$.

As applications of theorem 18 we construct distance Laplacian cospectral graphs in the following two corollaries.

Corollary 2. *Let G be a transmission regular triangle-free graph of diameter 2. Let H and H' be two non-isomorphic $\mathscr{L}$-cospectral graphs of the same order. Then $G\star H$ and $G\star H'$ form a pair of non-isomorphic $\mathscr{D}^{\mathscr{L}}$-cospectral graphs.*

5. Distance signless Laplacian spectrum of $G\star H$

The distance signless Laplacian matrix of $G\star H$, when G is a connected triangle-free graph, can be written as

$$\mathscr{D}^{\mathscr{Q}}(G\star H)=\begin{bmatrix}\mathscr{D}^{\mathscr{Q}}(G)+pTr(G)+2pI & \mathbf{1}_p^T\otimes(\mathscr{D}(G)+2I)\\ \mathbf{1}_p\otimes(\mathscr{D}(G)+2I) & D^{Q'}\end{bmatrix}, \tag{5.1}$$

where

$$\begin{aligned}D^{Q'} =& J_p\otimes\Big(\mathscr{D}(G)+2\mathcal{A}(G)\Big)+I_p\otimes\Big((p+1)Tr(G)+2I+2p\mathcal{D}_G\Big)+\\ &\Big((2p-4)I+2J-\mathscr{Q}(H)\Big)\otimes I_n.\end{aligned}$$

The following theorem gives $n(p-1)$ $\mathscr{D}^{\mathscr{Q}}$-eigenvalues of $G\star H$ when G is connected triangle-free and H is regular.

Theorem 19. *Let G be a connected triangle-free graph with $V(G)=\{u_1,\ldots,u_n\}$ and H be a k-regular graph of order p with $\mathscr{Q}$-eigenvalues $2k=\delta_1(H)\geq\delta_2(H)\geq\ldots\geq\delta_p(H)$. Then $(p+1)Tr(u_i)+2p\big(d(u_i)+1\big)-2-\delta_j(H)$, for $i=1,\ldots,n$ and $j=2,\ldots,p$ are $\mathscr{D}^{\mathscr{Q}}$-eigenvalues of $G\star H$.*

Proof. Let $g_2,\ldots,g_p$ be eigenvectors of $\mathscr{Q}(H)$ corresponding to $\delta_2(H),\ldots,\delta_p(H)$ respectively, that are orthogonal to $\mathbf{1}_p$. Then, for $i=1,\ldots,n$ and $j=2,\ldots,p$,

$$\mathscr{D}^{\mathscr{Q}}(G\star H)\begin{bmatrix}\mathbf{0}_n\\ g_j\otimes\mathbf{e}_i\end{bmatrix}=\Big((p+1)Tr(u_i)+2p\big(d(u_i)+1\big)-2-\delta_j(H)\Big)\begin{bmatrix}\mathbf{0}_n\\ g_j\otimes\mathbf{e}_i\end{bmatrix}.$$

□

The next result gives all the $\mathscr{D}^{\mathscr{Q}}$-eigenvalues of $G\star H$ in terms of $\mathscr{D}^{\mathscr{Q}}$-eigenvalues of G and $\mathscr{Q}$-eigenvalues of H when G is connected, triangle-free and transmission regular and H is regular.

Theorem 20. *Let G be a s-transmission regular triangle-free graph of diameter 2 with $\mathscr{D}^{\mathscr{Q}}$-eigenvalues $2s = \rho_1^{\mathscr{Q}}(G) \geq \rho_2^{\mathscr{Q}}(G) \geq \ldots \geq \rho_n^{\mathscr{Q}}(G)$. Let H be a k-regular graph of order p with $\mathscr{Q}$-eigenvalues $2k = \delta_1(H) \geq \delta_2(H) \geq \ldots \geq \delta_p(H)$. Then the distance signless Laplacian eigenvalues of $G\star H$ are the following.*

(a) $\frac{1}{2}\left(2(4pn-k) - p(s+2) + 3s - 2 \pm \sqrt{\left(3p(s+2) - 2(4pn-k) + s + 2\right)^2 + 4p(s+2)^2}\right)$

(b) $\frac{1}{2}\left((1-p)\rho_i^{\mathscr{Q}}(G) + p(4n+s-2) + s + 2 - 2(k+2) \pm \right.$
$\left.\sqrt{\left((1+p)\rho_i^{\mathscr{Q}}(G) + p(s+6-4n) - s + 2(k+1)\right)^2 + 4p\left(\rho_i^{\mathscr{Q}}(G) - s + 2\right)^2}\right),$
$i = 2, \ldots, n,$ *provided* $\rho_i^{\mathscr{Q}}(G) \neq s-2$ *for any* i

(c) $p(s+2) + s - 2$ *and* $4pn - p(s+2) + s - 2 - 2k$ *if* $s - 2 \in Spec_{\mathscr{D}^{\mathscr{Q}}}(G)$

(d) $4pn - p(s+2) + s - 2 - \delta_j(H),\ \ j = 2, \ldots, p,$ *each with multiplicity* n.

Proof. Let t_1 and γ_1 be scalars such that

$$\mathscr{D}^{\mathscr{Q}}(G\star H)\begin{bmatrix} t_1\mathbf{1}_n \\ \mathbf{1}_p \otimes \mathbf{1}_n\end{bmatrix} = \gamma_1 \begin{bmatrix} t_1\mathbf{1}_n \\ \mathbf{1}_p \otimes \mathbf{1}_n\end{bmatrix}. \tag{5.2}$$

Note that $\mathbf{1}_n$ is an eigenvector corresponding to $\rho_1^{\mathscr{Q}}(G)$ as well as s, the largest $\mathscr{D}$-eigenvalue of G. Then equation (5.2) implies

$$t_1\big(2s + p(s+2)\big) + p(s+2) = t_1\gamma_1 \tag{5.3}$$

and

$$t_1(s+2) + 2p(4n-s-2) - 2k + s - 2 = \gamma_1. \tag{5.4}$$

Solving equations (5.3) and (5.4) for t_1 and γ_1 we obtain

$$t_1 = \frac{-p(s+2)}{2s+p(s+2)-\gamma_1}, \ \text{ provided } \ \gamma_1 \neq 2s + p(s+2),$$

and in this case we obtain (a).
As in the proof of theorem 18, the case $\gamma_1 = 2s + p(s+2)$ can be rejected.

Let y_i be an eigenvector corresponding to $\rho_i^{\mathscr{Q}}(G), i = 2, \ldots, n$, and is orthogonal to $\mathbf{1}_n$. Let t_2 and γ_2 be scalars such that

$$\mathscr{D}^{\mathscr{Q}}(G\star H)\begin{bmatrix} t_2 y_i \\ \mathbf{1}_p \otimes y_i\end{bmatrix} = \gamma_2 \begin{bmatrix} t_2 y_i \\ \mathbf{1}_p \otimes y_i\end{bmatrix}. \tag{5.5}$$

Since $\mathscr{D}(G) = \mathscr{D}^{\mathscr{Q}}(G) - Tr(G)$, from equation (5.5) we have

$$t_2\Big(\rho_i^{\mathscr{Q}}(G) + p(s+2)\Big) + p\Big(\rho_i^{\mathscr{Q}}(G) - s + 2\Big) = t_2\gamma_2 \tag{5.6}$$

and

$$t_2\Big(\rho_i^{\mathscr{Q}}(G) - s + 2\Big) + p\Big(4n - 4 - \rho_i^{\mathscr{Q}}(G)\Big) - 2(k+2) + s + 2 = \gamma_2. \tag{5.7}$$

Solving for t_2 and γ_2 from equations (5.6) and (5.7) we get

$$t_2 = \frac{-p\Big(\rho_i^{\mathscr{Q}}(G) - s + 2\Big)}{\rho_i^{\mathscr{Q}}(G) + p(s+2) - \gamma_2}, \quad \text{provided} \quad \gamma_2 \neq \rho_i^{\mathscr{Q}}(G) + p(s+2).$$

In this case,

$$\gamma_2 = \frac{1}{2}\Bigg((1-p)\rho_i^{\mathscr{Q}}(G) + p(4n+s-2) + s + 2 - 2(k+2) \pm \sqrt{\Big((1+p)\rho_i^{\mathscr{Q}}(G) + p(s+6-4n) - s + 2(k+1)\Big)^2 + 4p\Big(\rho_i^{\mathscr{Q}}(G) - s + 2\Big)^2}\Bigg).$$

$\gamma_2 = \rho_i^{\mathscr{Q}}(G) + p(s+2) \Rightarrow \rho_i^{\mathscr{Q}}(G) = s - 2$. Hence (b) follows.
Now, if $\rho_i^{\mathscr{Q}}(G) = s - 2$ for any i then

$$\mathscr{D}^{\mathscr{Q}}(G \star H)\begin{bmatrix} y_i \\ \mathbf{0}_{pn} \end{bmatrix} = \Big(p(s+2) + s - 2\Big)\begin{bmatrix} y_i \\ \mathbf{0}_{pn} \end{bmatrix}$$

and

$$\mathscr{D}^{\mathscr{Q}}(G \star H)\begin{bmatrix} \mathbf{0}_n \\ \mathbf{1}_p \otimes y_i \end{bmatrix} = \Big(4pn - p(s+2) + s - 2 - 2k\Big)\begin{bmatrix} \mathbf{0}_n \\ \mathbf{1}_p \otimes y_i \end{bmatrix}.$$

Hence (c) follows. Since $Tr(u_i) = s$ and $d(u_i) = 2n - s - 2$ for $i = 1, \dots, n$, we obtain (d) from Theorem 19. □

Remark 7. The $\mathscr{D}^{\mathscr{Q}}$-eigenvalues of $G \star H$ given in (b) in the above theorem can be expressed in terms of the $\mathscr{D}$-eigenvalues of G since $\rho_i^{\mathscr{Q}}(G) = s + \rho_i(G), i = 1, \dots, n$.

Corollary 3. *Let G be a transmission regular triangle-free graph of diameter 2. Let H and H' be two non-isomorphic k-regular $\mathscr{Q}$-cospectral graphs of the same order. Then $G \star H$ and $G \star H'$ form a pair of non-isomorphic $\mathscr{D}^{\mathscr{Q}}$-cospectral graphs.*

6. Conclusion

The neighbourhood corona of graphs is a variation of the classical corona graph operation and has been widely studied in areas such as spectral graph theory, domination, metric dimension, coloring, and more. However, its distance-based spectra have remained largely unexplored. In this work, we have obtained the distance (respectively, distance Laplacian, distance signless Laplacian) spectra of $G \star H$, when G is a regular, triangle-free graph of diameter 2 and H is regular (respectively, arbitrary, regular). Additionally, we have constructed several distance equienergetic graphs and graphs that are cospectral with respect to distance, distance Laplacian, and distance signless Laplacian matrices.
In future work, we intend to investigate the distance-based spectra and related properties of $G \star H$, when G belongs to the class of graphs associated with equation (3.2) and remark 1. Determining the distance spectra of $G \star H$ when G is any connected triangle-free graph also remains as an open area of research.

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