Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-20
https://doi.org/10.22049/cco.2025.30326.2415

CCO
Commun. Comb. Optim.

Review Article

A survey of bipartite tournaments

Jay S. Bagga[1,*], **Lowell W. Beineke**[2]

[1]Ball State University, Muncie, Indiana, USA
*jbagga@bsu.edu

[2]Purdue University Fort Wayne, Fort Wayne, Indiana, USA
beineke@pfw.edu

Received: 1 February 2025; Accepted: 28 June 2025
Published Online: 17 July 2025

Abstract: The family of mathematical objects known as tournaments constitutes one of the main areas of directed graphs, being defined as having a set of vertices in which each pair is joined by exactly one arc. Here we are interested in a secondary family known as bipartite tournaments. These are defined as having two nonempty sets of vertices and one arc joining each pair that are in different sets. There is a substantial theory of this topic also, and this article presents many of its foundational areas. One key concept is that of the number of arcs going from a vertex, called its score, and the theory involves what lists of numbers can constitute the scores. The cycles in a bipartite tournament have a variety of interesting collections involving lengths. Other fundamental concepts that we discuss include bipartite tournaments with the property of being self-converse, the efficient removal of cycles by the reversal of arcs, development of a theory of upsets, and properties of automorphism groups of the structures. The concept of bipartite tournaments also generalizes naturally to multipartite tournaments with more parts. Many of the results we discuss in this paper lead to questions about analogous extensions to these structures. In the last section of the paper, we include a discussion of further research directions.

Keywords: directed graph, tournament, bipartite tournament.

AMS Subject classification: 05C20

1. Introduction

Consider a game in which two players compete with one another at most once. In one natural version, each person competes against each of the others, while in another natural version, there are two sets of players and each person competes against each

* *Corresponding Author*

one in the opposing set (and never against anyone in the same set). Both of these situations gives rise to natural mathematical models in the form of directed graphs. In order to distinguish between the two groups, we call the first "round-robin" and the second "bipartite". The first consists of a set V of vertices and a set A of arcs with exactly one arc joining a pair of vertices, while the vertices of the second consist of a pair of disjoint sets X and Y with an arc joining each pair of vertices in different sets (and none in the same set). Figure 1 gives an example of a *round-robin tournament* (or *tournament*, for brevity) with five players, while Figure 2 is a *bipartite tournament* with two players on one side (shown as open circles) and three players on the other (closed circles).

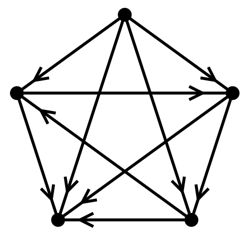

Figure 1. **A round-robin tournament with 5 vertices**

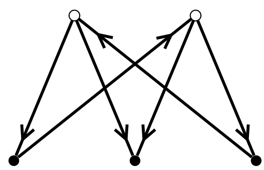

Figure 2. **A bipartite tournament with one set of 2 vertices and one of 3 vertices**

We call a bipartite tournament on two sets of vertices $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ an (r, s)-bipartite tournament (denoted $T_{r,s}$). We generally assume that $r \leq s$. The corresponding scores (out-degrees) are denoted as $A = [a_1, a_2, \ldots, a_r]$ and $B = [b_1, b_2, \ldots, b_s]$ (often assumed to be in nondecreasing order), while the corresponding co-scores (in-degrees) are $[\overline{a}_1, \overline{a}_2, \ldots, \overline{a}_r]$ and $[\overline{b}_1, \overline{b}_2, \ldots, \overline{b}_s]$.

In later sections, we will use matrices of zeros and ones to describe some results about bipartite tournaments. Just as graphs can be represented in terms of adjacency matrices, one can also give an equivalent matrix formulation of bipartite tournaments. The (r, s) bipartite tournament $T_{r,s}$ with partite sets $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ as above can be represented by an $r \times s$ matrix $M = [\alpha_{ij}]$ such that $\alpha_{ij} = 1$ if there is an arc from x_i to y_j and $\alpha_{ij} = 0$ otherwise. The matrix

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

is the 0-1 matrix associated with the bipartite tournament in Figure 2. It follows that the row sums of M are the scores of the vertices in X, while the column sums are the co-scores of vertices in Y.

Starting with a variety of models including competing chickens as well as sports teams, round-robin tournaments have been widely studied, and two early surveys of results on tournaments were the book by John Moon, *Topics on Tournaments* [32], and the chapter by Reid and Beineke, "Tournaments", in the book by Beineke and Wilson *Selected Topics in Graph Theory* [35]. The article by Beineke [9] provides a comparative study of bipartite and round-robin tournaments.

Of course, tournaments and bipartite tournaments also have natural generalizations to multipartite tournaments, orientations of complete multipartite graphs, and these are well studied. See Volkmann [42] for an early survey on multipartite tournaments. Many of the results we describe below on bipartite tournaments lead to similar extensions or questions for multipartite tournaments. We will mention these as appropriate.

Figure 3 shows the three possible types of bipartite tournaments $T_{1,2}$ while Figure 4 shows the four possible arrangements of bipartite tournaments $T_{2,2}$ without regard to the sets.

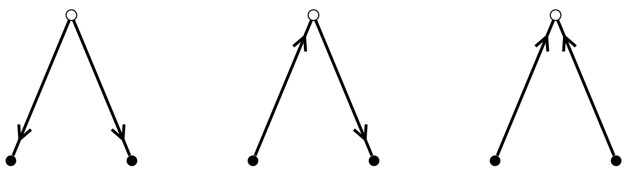

Figure 3. **The bipartite tournaments of type $T_{1,2}$**

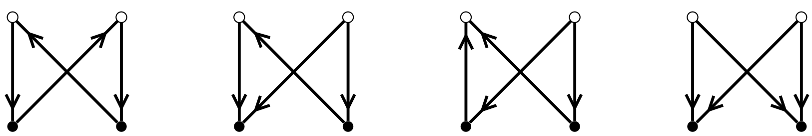

Figure 4. **The bipartite tournaments of type $T_{2,2}$**

Given two vertices v and w in a digraph (a directed graph), it is natural to ask whether there is a directed path from v to w. A digraph in which there is such a path from each vertex to each of the others is called *strongly connected*, generally shortened to *strong*.

2. Score Lists in Bipartite Tournaments

We describe several results related to score lists, and conditions on scores that lead to strong bipartite tournaments.

2.1. Realizable score lists

Our next issue is, given a pair of lists of integers, does there exist a bipartite tournament with these scores? The following repeated criteria for realizability are due to Gale [18].

Theorem 1. *Let $A = [a_1, a_2, \ldots, a_r]$ and $B = [b_1, b_2, \ldots, b_s]$ be two lists of integers, and let A' be the result of deleting a_r and let B' be obtained from B by deleting the largest a_r entries by 1 each. Then the pair (A, B) is realizable if and only if (A', B') is.*

Moon [30] provided this important computational result.

Theorem 2. *A pair of lists A and B of non-negative integers in nondecreasing order are the score lists of an (r, s) bipartite tournament if and only if for all $k = 1, 2, \ldots, r$, and $l = 1, 2, \ldots, s$,*

$$\sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j \geq kl,$$

with equality when $k = r$ and $l = s$.

Furthermore, the resulting bipartite tournament is strong if and only if all of the entries are non-zero and all of the given inequalities are strict except for the last.

Ryser [36] gave another formulation and proved it in terms of matrices of zeros and ones.

Theorem 3. *If A is a list in nonincreasing order, then the pair (A, B) is realizable if and only if for all $k = 1, 2, \ldots, r$,*

$$\sum_{i=1}^{k} a_i \leq \sum_{j=1}^{s} \min\{k, r - b_j\},$$

with equality when $k = r$.

Beineke and Eggleton [3, 9] independently observed that the only inequalities that need to be checked for realizability are where there are jumps. For convenience of list notation, given a list $L = [l_1, l_2, \ldots, l_h]$, we let $L_0 = 0$ and $L_i = l_1 + l_2 + \ldots + l_i$, with this applying to both lists A and B.

Theorem 4. *Let $A = [a_1, a_2, \ldots, a_r]$ and $B = [b_1, b_2, \ldots, b_s]$ be nondecreasing integer lists with $0 \leq a_i \leq r$ and $0 \leq b_j \leq s$. If $A_r + B_s = rs$, and if $A_p + B_q \geq pq$ whenever $a_p < a_{p+1}$ and $b_q < b_{q+1}$, then (A, B) is a realizable bipartite tournament.*

Another view of bipartite tournaments is just the collection of degrees (out-degrees). Instead of the two lists of the degrees of all of the vertices, only those that are different are considered. For example, the tournament in Figure 5 has score lists $A = [0, 2, 2]$ and $B = [1, 2, 2]$ but the sets of degrees are the sets $C = \{0, 2\}$ and $D = \{1, 2\}$, and they are called the *score sets*. Wayland [43] investigated which pairs of sets of integers are the score sets of a bipartite tournament. Let $C = \{c_1, c_2, \ldots, c_m = c\}$ and $D = \{d_1, d_2, \ldots, d_n = d\}$ be nonempty sets of all of the nonnegative integers in strictly increasing order.

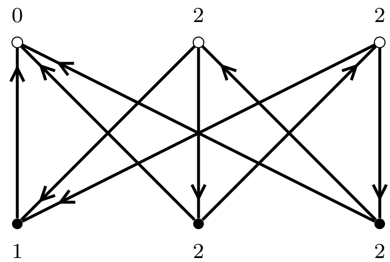

Figure 5. **A $(3, 3)$ bipartite tournament with degree sets $\{0, 2\}$ and $\{1, 2\}$**

Theorem 5. *There exists a bipartite tournament with bipartition (X, Y) whose score sets are C and D, such that $|X| > d$ if and only if the quantity*

$$\sum_{i=1}^{m} c_i + (d - m + 1)c + \sum_{j=1}^{n} d_j + 1 - n(d + 1)$$

is positive.

We refer the reader to Wayland [43] for more details.

2.2. Uniquely realizable score lists

We turn now to the types of bipartite tournaments for which all realizations are isomorphic, subject to interchanging the roles of the two sets of numbers. This result was due to the authors. We begin with the following three types, each of which has a unique representation.
Type 1: $A = [1, 1, \ldots, 1]$ and B arbitrary,
Type 2: $[1, 1, \ldots, 1, a]$ and $[b, b, \ldots, b]$,

Type 3: $[1, a, a, \ldots, a]$ and $[2, 2, \ldots, 2]$.

There are two variations of $T_{r,s}$ that are essentially equivalent and we show here the two possibilities for Type 1.
Complement: Instead of $[1, 1, \ldots, 1]$, take $A = [s-1, s-1, \ldots, s-1]$.
Interchange: If $|A| < |B|$, instead of $[1, 1, \ldots, 1]$, take $B = [r-1, r-1, \ldots, r-1]$.

The authors of this article proved the following theorem [3, 4]:

Theorem 6. *A strong pair (A, B) of score lists is uniquely realizable if and only if it is of Type 1, Type 2, or Type 3, or it is the complement or the interchange of one of these types.*

	A	B
Type 1	$[1, 1, \ldots, 1]$	arbitrary
Type 2	$[1, 1, \ldots, 1, a]$	$[b, b, \ldots, b]$
Type 3	$[1, a, a, \ldots, a]$	$[2, 2, \ldots, 2]$

3. Strong Connectedness and Cycles in Bipartite Tournaments

One of the early results on tournaments was the 1934 theorem by Rédei [34] that every round-robin tournament has a spanning path. An analogous strong result was proved by Camion [14]: Every strongly connected round-robin tournament has a spanning cycle. This was extended by Harary and Moser [22] and Moon [32] to cycles of all lengths from a 3-cycle on up to a largest cycle.

Theorem 7. *A tournament of order $n \geq 3$ is strong if and only if every vertex lies on a cycle of length k, for $k = 3, 4, \ldots, n$.*

As a consequence of this, it follows that a strong tournament has strong subtournaments of all orders from 3 up. In this section we will see that the corresponding results do not hold for bipartite tournaments.

We define a *bi-quadratic* bipartite tournament to be one in which each of the two partite sets X and Y is split into nonempty disjoint sets $X' \cup X''$ and $Y' \cup Y''$ with all arcs going between pairs of sets as shown in Figure 6.

3.1. Strong connectedness

Moon's result [30] in Section 2.1 provides conditions for pairs of lists (A, B) of strong bipartite tournaments. However, unlike tournaments, strong bipartite tournaments might not have spanning paths or cycles. Clearly, this is true when the partite sets X and Y have unequal orders, but even when $|X| = |Y|$, a bi-quadratic bipartite

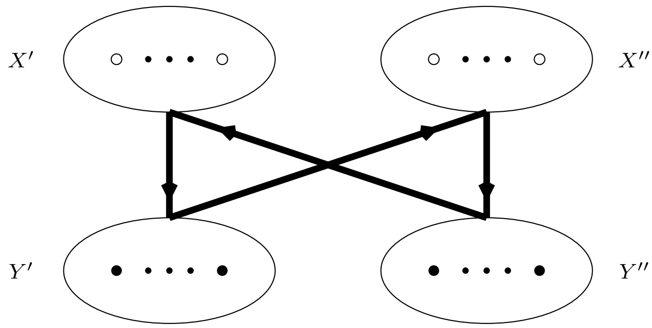

Figure 6. **A bi-quadratic bipartite tournament**

tournament with $|X'| \neq |Y'|$ provides an example of this.

As noted above, strong tournaments have strong subtournaments of all orders at least three. Even this weaker result does not hold for bipartite tournaments. For $r \geq 2$, consider the bipartite tournament $S_{r,r}$ of Figure 7, where only the X-to-Y arcs are shown. This bipartite tournament has score lists $[1, 1, \ldots, 1]$ and $[r-1, r-1, \ldots, r-1]$, and is strong (in fact, Hamiltonian). However, the removal of any vertex leaves a bipartite tournament which is not strong. It was shown by the authors [2, 3] that these examples are special, as the next result shows.

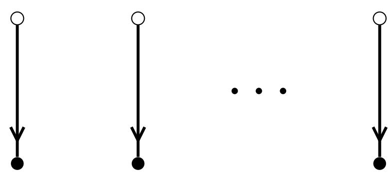

Figure 7. **The bipartite tournament $S_{r,r}$ with only the "downward" arcs shown**

Theorem 8. *Every strong bipartite tournament T contains a vertex v such that $T - v$ is strong unless $T = S_{r,r}$ for some r.*

Bagga and Beineke [2] found conditions on score lists of strong bipartite tournaments for which all vertex-deleted subtournaments are strong. A tournament or a bipartite tournament T is called *super-strong* if $T - v$ is strong for every vertex v. It is easily seen that every super-strong tournament and every super-strong bipartite tournament with more than two vertices is also strong, so it follows that this concept is the same as the 2-connectedness of graphs defined by Thomassen [38].

For the remainder of this section, we assume that $T_{r,s}$ is a strong bipartite tournament with score lists $A = [a_1, a_2, \ldots, a_r]$ and $B = [b_1, b_2, \ldots, b_s]$ in nondecreasing order, and $2 \le a_i \le s-2$ and $2 \le b_j \le r-2$ for all i and j. These conditions are needed since no score or co-score in a super-strong bipartite tournament can be 0 or 1. We also let

$$S_{k,l} = \sum_{i=1}^{k} a_i + \sum_{j=1}^{l} b_j$$

for $k = 1, 2, \ldots, r$ and $l = 1, 2, \ldots, s$.

The next two results were proved by Bagga and Beineke [2].

Theorem 9. *The pair of lists (A, B) belongs to a nonsuper-strong bipartite tournament if and only if for some k, l with $2 \le k \le s-2$ and $2 \le l \le r-2$,*
(a) $S_{k,l} \le kl + l$ and $S_{k+1,l} \le kl + s$, or
(b) $S_{k,l} \le kl + k$ and $S_{k,l+1} \le kl + r$.

Theorem 10. *The pair of lists (A, B) belongs to a super-strong bipartite tournament if and only if for every k and l with $1 \le k \le r$ and $1 \le l \le s$, $S_{k,l} \ge kl + 2$, except that $S_{r,s} = rs$.*

3.2. Cycles

Obviously, there are no cycles of odd length in a bipartite tournament. However, as shown in Figure 8, a bipartite tournament of type $T_{3,3}$ might not have a 6-cycle.

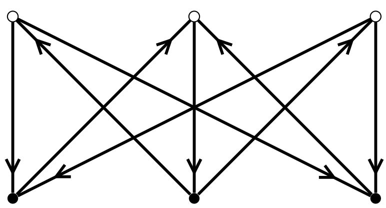

Figure 8. **A $(3, 3)$ bipartite tournament with no 6-cycle**

The $(4, 4)$ bipartite tournament in Figure 9 is a bi-quadratic tournament in which each cycle has length either 4 or 8. Clearly, if a bipartite tournament of type $T_{r,s}$ (with $r \le s$) has a cycle C_p, then $r > 1$ and $p \le 2r$. Beineke and Little [26] proved the following theorem on cycle lengths.

Theorem 11. *Let T be a bipartite tournament with a cycle of length $2k$.*
(a) If T is bi-quadratic, then k is even and T has cycles of all lengths $4, 8, \ldots, 2k$.
(b) Otherwise, T has cycles of all lengths $4, 6, \ldots, 2k$.

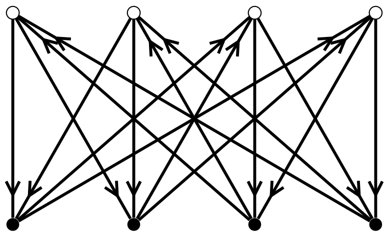

Figure 9. **A bi-quadratic tournament**

Bill Jackson [27] established the existence of a cycle of a certain length depending on the scores and co-scores of the vertices. His result includes the following consequence for bipartite tournaments.

Theorem 12. *A strong bipartite tournament with minimum score p and minimum co-score q has a cycle of length at least $2(p+q)$.*

This result is sharp in that for $p+q=r \leq s$, there is a strong bipartite tournament $T_{r,s}$ with minimum score p and minimum co-score q and with a cycle of length $2(p+q)$. The theorem also has a nice corollary.

Corollary 1. *Every regular bipartite tournament is Hamiltonian.*

4. Numbers of cycles in strong bipartite tournaments

The scores s_i in a round-robin tournament determine the number of 3-cycles by the elementary formula: $\binom{n}{3} - \sum \binom{s_i}{2}$. However, the scores do not determine the exact number of 4-cycles even in some tournaments of order 5. For example, the two 5-tournaments in Figure 10 have the same score list, but the first has four 4-cycles while the second has just three (see Reid and Beineke [35]).

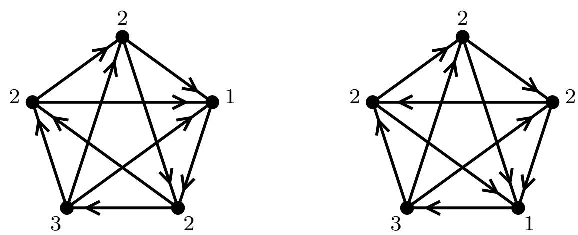

Figure 10. Two round-robin tournaments with same scores but different number of 4-cycles

Similarly, Figure 11 shows two $(3,3)$ bipartite tournaments with the same sets of scores, $\{1,1,2\}$ and $\{1,2,2\}$, but different numbers of 4-cycles.

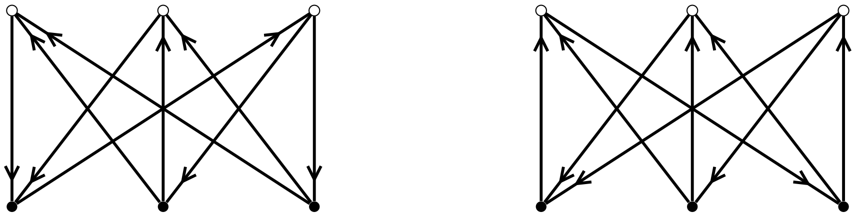

Figure 11. Two $(3,3)$ bipartite tournaments same score lists, but different numbers of 4-cycles

Not surprisingly, the maximum number of 4-cycles in a bipartite tournament occurs when the scores in each partite set are as nearly equal as possible, as was first established by Moon and Moser [33], among other distributions.

Theorem 13. *The maximum number of 4-cycles in an (r,s) bipartite tournament is* $\lfloor \frac{r}{2} \rfloor \lceil \frac{r}{2} \rceil \lfloor \frac{s}{2} \rfloor \lceil \frac{s}{2} \rceil$.

Among many results on random bipartite tournaments, Bollobás, Frank, and Karoński [12] showed that the expected number of 4-cycles in a random (r,s) bipartite tournament, where each arc is chosen with probability p, is

$$2\binom{r}{2}\binom{s}{2}p^2(1-p)^2.$$

Bagga and Beineke [5] obtained several results on the numbers of some subtournaments of a bipartite tournament.

5. Disjoint Cycles in Bipartite Tournaments

Some questions and some results apply to digraphs other than bipartite tournaments, but for much of this audience, the focus on this tournament family gives a nice collection of material.

There are several interesting results on disjoint cycles in bipartite tournaments. The first was proved by Bai, Li, and Li [6] that, in loose terms, there will be a set of about half as many disjoint cycles as the minimum score in the tournament. The problem originated in a conjecture by Bermond and Thomassen [11] for digraphs in general. Their basic version is the following.

Theorem 14. *If $k \geq 2$ and T is a bipartite tournament with minimum score (or minimum co-score) at least $2k-1$, then T has a set of at least k disjoint cycles.*

The following is a more general result of theirs (among others).

Theorem 15. *Every bipartite tournament T with minimum score (or minimum co-score) at least $kl-1$ has a set of k disjoint cycles*
(a) each of length $2l$ if l is even, and
(b) each of length $2l$ or $2l+2$ if l is odd.

These results were extended to stronger results by Balbuena, González, and Olsen [7], some of which are elegant with lower bounds for both scores and co-scores.

Theorem 16. *Every bipartite tournament T with each score and co-score at least 2 has a pair of disjoint 4-cycles.*

Theorem 17. *Every bipartite tournament T with each partite set of order at least $2k$ with $k \geq 3$ and minimum score at least $2k-2$ and minimum co-score at least 1 has a set of k disjoint cycles.*

One interesting fact is that the minimum value of k is 3. The bipartite tournament in Figure 12 has $k = 2$ but no disjoint 4-cycles.

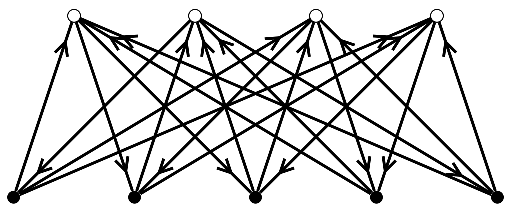

Figure 12. **A strong bipartite tournament $T_{4,5}$ without any disjoint 4-cycles**

Theorem 18. *Every bipartite tournament T with each score and co-score at least $(3k-1)/2$ and with $k \geq 2$ has a collection of k disjoint cycles.*

The following result of Lichiardopol [29] is a special case of a conjecture of Henning and Yeo [24].

Theorem 19. *Every bipartite tournament T with each score at least 4 has a pair of disjoint cycles of different lengths.*

We now turn to the opposite family of cycles. Brualdi and Shen [13] conjectured that the arcs of every Eulerian bipartite tournament can be partitioned into cycles of length 4, a question that remains open. However, one special tournament family was established by Del Valle Vega [40], that being where the partite sets are the same size.

Theorem 20. *Every Eulerian bipartite tournament $T_{2r,2r}$ can be partitioned into 4-cycles.*

6. Decycling Bipartite Tournaments

One of the key features of graphs (or digraphs) is for a subgraph to have all cycles removed, an action we call *decycling*. In the case of graphs this is usually simply deleting vertices or edges, and the goal then is to retain as large a subgraph as possible. However, in the case of digraphs without any symmetric pairs of arcs (sometimes called oriented graphs), things are more varied and the action taken is the reversal of some of the arcs.

Formally, the problem considered is to find the smallest number of arcs in a bipartite tournament whose reversal yields a tournament that is acyclic. The decycling number of a tournament is denoted $\nabla(T)$ and is called the *decycling index*. For example, in Figure 13, the bipartite tournament $T_{2,3}$ has two cycles, $vywxv$ and $vywzv$, but if the arc wy is reversed, then the result $T'_{2,3}$ is acyclic. Thus, in this case, $\nabla(T) = 1$.

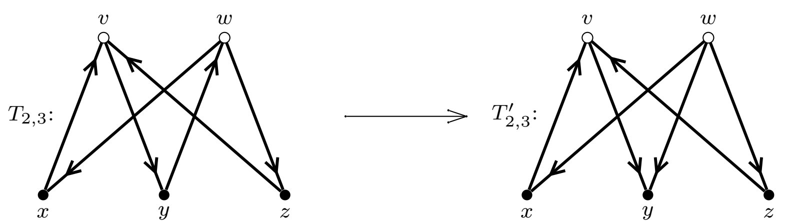

Figure 13. **The decycling index is 1 for the bipartite tournament T**

As usual, for a given bipartite tournament $T_{r,s}$ we assume that $r \leq s$ unless otherwise specified. Basic results are known only for small values of r. Vandell [39] began with the strong bipartite tournament $T_{2,s}$.

Theorem 21. *The decycling index of a strong bipartite tournament $T_{2,s}$ is the minimum score of a vertex in the partite set of order 2.*

Consider now a strong bipartite tournament $T_{3,s}$ with bipartition (X, Y). Each vertex in Y has only one arc, α^+ or α^- in one direction (and of course two in the other). Vandell constructed an elementary algorithm for computing the maximum value β of a combination of a pair of αs.

Theorem 22. *The decycling index of a $(3, s)$ strong bipartite tournament $\nabla(T_{3,s})$ is $s - \beta$, where β is as described above.*

A common problem in math is to find the maximum value of a graph parameter. In this case, an interesting question asks for the maximum decycling index of a family of bipartite tournaments. For given r, the maximum decycling index of the bipartite tournament $T_{r,s}$ is denoted $\nabla_r(s)$. Vandell proved the following three results.

Theorem 23. *(a)* $\nabla_2(s) = \lfloor \frac{s}{2} \rfloor$.

(b) $\nabla_3(s) = \lfloor \frac{2s}{3} \rfloor$.

(c) $$\nabla_4(s) = \begin{cases} \lfloor \frac{7s}{6} \rfloor - 1 & \text{if } s \equiv 1,3 \pmod 6, \\ \lfloor \frac{7s}{6} \rfloor & \text{if } s \equiv 0,2,4,5 \pmod 6. \end{cases}$$

Larger values of r and s turn out to be significantly harder problems. The maximum decycling index of $T_{5,5}$ was determined by Vandell to be 6 and he conjectured that of $T_{6,6}$ would be 10. In fact, there have not been many more values found, with Andreas Holtkamp and Lutz Volkmann [25] determining only the values for the remaining sets of up to 12 vertices. Note that these last results are determined through the exponential complexity of computing. This theorem gives the combined results of Vandell and of Holtkamp and Volkmann – the remaining cases are apparently still open.

Theorem 24. $\nabla_5(5) = 6$, $\nabla_5(6) = 7$, $\nabla_5(7) = 9$, *and* $\nabla_6(6) = 10$.

7. Self-Converse Bipartite Tournaments

A graph G is *self-complementary* if $\overline{G}$, the complement of G, is isomorphic to G. Self-complementary graphs have been well-studied. The *converse* $\overleftarrow{D}$ of an oriented digraph D is obtained when all the arcs in D are reversed. The digraph D is *self-converse* if $\overleftarrow{D}$ is isomorphic to D.

Eplett [15] determined necessary and sufficient conditions for the existence of self-converse tournaments in terms of scores. If $S = [s_1, s_2, \ldots, s_n]$ is the score list (in nondecreasing order) of a tournament T, then clearly the score list of the converse of T is $[n-1-s_n, n-1-s_{n-1}, \ldots, n-1-s_1]$.

Theorem 25. *If* $S = [s_1, s_2, \ldots, s_n]$ *is the score list of a tournament, then there is a self-converse tournament with score list* S *if and only if* $s_i + s_{n+1-i} = n-1$ *for* $i \le i \le n$.

Bagga [3] obtained conditions for self-converse bipartite tournaments. We first discuss the 0-1 matrix formulation of these results. Given lists $A = [a_1, a_2, \ldots, a_r]$ and $B = [b_1, b_2, \ldots, b_s]$ of nonnegative integers, we say that the pair (A, B) is *self-dual* if $a_i + a_{r+1-i} = s$ and $b_j + b_{s+1-j} = r$ for $1 \le i \le r$ and $1 \le j \le s$.

We define the *complement* of an $r \times s$ matrix $M = [\alpha_{ij}]$ of zeros and ones to be the $r \times s$ matrix in which the (i, j)-entry is $1 - \alpha_{(r+1-i)(s+1-j)}$, and M is called *self-complementary* if $M^c = M$.

Clearly, if M is self-complementary, then it has an equal number of zeros and ones, which means that rs is even. With notation as above, it is also easy to see that the pair (A, B) is self-dual. Now suppose that $r = 2r'$ and $s = 2s'$ are both even. Let x be the number of ones in the top left $r' \times s'$ submatrix of M, $x + y$, the number of ones in the left half of M, and $x + z$, that in the top half of M. Since M is self-complementary, $y + z = r's'$. Also, $x + z = \sum_{i=1}^{r'} a_i$ and $x + y = \sum_{j=1}^{s'} b_j$. Hence $\sum_{i=1}^{r'} a_i + \sum_{j=1}^{s'} b_j - r's'$ is even. We denote this quantity by $\sigma(A, B)$.

Bagga [3] proved the following results.

Theorem 26. *With notation as above, if M is a self-complementary matrix with the row sum vector $A = [a_1, a_2, \ldots, a_r]$ and the column sum vector $B = [b_1, b_2, \ldots, b_s]$, then (A, B) is self-dual and, either $r + s$ is odd or r, s, and $\sigma(A, B)$ are all even.*
Conversely, if M is a 0-1 matrix with A and B as above, and if the other facts also apply, then there is a self-complementary matrix with pair (A, B).

The conditions on scores of a bipartite tournament to be self-converse are somewhat different, as the next result shows.

Theorem 27. *With notation as above, let $A = [a_1, a_2, \ldots, a_r]$ and $B = [b_1, b_2, \ldots, b_s]$ be the score lists of a bipartite tournament. Then there exists a self-converse bipartite tournament with score lists A and B if and only if*
(1) (A, B) is self-dual,
(2) r or s is even,
(3) whenever r and s are both even and $a_{r'} < a_{r'+1}$ and $b_{s'} < b_{s'+1}$, then $\sigma(A, B)$ is even.

We conclude this section with an example that illustrates the difference between the conditions for matrices and bipartite tournaments. Figure 14 shows a self-converse bipartite tournament with score lists $A = [1, 3]$ and $B = [1, 1, 1, 1]$. We observe that the vertex y_1 is "fixed" in any isomorphism of the bipartite tournament.

On the other hand, the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

has a self-dual pair (A, B), $r = 2$ and $s = 4$ are both even, but $\sigma(A, B)$ is odd. There is no self-complementary matrix with the pair (A, B), since the column with entries 1 and 0 cannot be fixed.

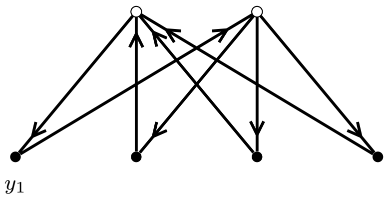

Figure 14. **A self-converse bipartite tournament**

8. Upsets in Bipartite Tournaments

Several methods of ranking players in round-robin tournaments have been proposed. The *transitive* round-robin tournament of order n has score list $[n-1, n-2, \ldots, 0]$ and the players are naturally ranked in terms of their scores, so that every player beats all players of lower scores. When a player in a round-robin tournament beats a player with equal or higher score, an *upset* occurs. For descriptions of rankings and upsets in robin-robin tournaments, see [17, 22, 32, 37].

For an (r,s) bipartite tournament T on teams $X = \{x_1, x_2, \ldots, x_r\}$ and $Y = \{y_1, y_2, \ldots, y_s\}$ with corresponding scores $A = [a_1, a_2, \ldots, a_r]$ and $B = [b_1, b_2, \ldots, b_s]$ (in nondecreasing order), we rank the players in each team according to their scores, with players of the same score ranked in some fixed order. An upset is said to have occurred in team X if there exist i, j, k with $1 \leq i < j \leq r$ and $1 \leq k \leq s$, such that $x_i y_k$ and $y_k x_j$ are arcs in T. Thus an upset results when a player "beats" a player of equal or higher score in the same team "via" a player of the other team. We call a bipartite tournament that has no cycles *consistent* (or *acyclic*). The following results were proved by Bagga [1].

Theorem 28. *The following statements are equivalent for a bipartite tournament T.*
(1) T has no upsets.
(2) T has no 4-cycles.
(3) T is consistent.

Due to the above result, we are interested in the minimum number of upsets in strong bipartite tournaments. The next result provides the answer.

Theorem 29. *The minimum number of upsets in the class of all (r,s) strong bipartite tournaments is $r+s-2$.*

The (r,s) bipartite tournament with score lists $[s-1, 1, \ldots, 1]$ and $[r-1, r-1, \ldots, r-$

1, 1] and matrix

$$M = \begin{bmatrix} 0 & 1 & \dots & 1 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ & & . & & \\ & & . & & \\ & & . & & \\ 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

attains the minimum.

On the other hand, the maximum number of upsets in all (r, s) bipartite tournaments is given by the following result.

Theorem 30. *The number of upsets in an (r, s) bipartite tournament is bounded above by*

$$\frac{s}{2}\left\lfloor \frac{r^2}{4} \right\rfloor + \frac{r}{2}\left\lfloor \frac{s^2}{4} \right\rfloor.$$

We observe that the above bound is sharp. When r and s are both even, it is easy to check that the bound is attained by an appropriate bi-quadratic bipartite tournament.

In the last result of this section, it is shown that the number of upsets can be determined from the score lists.

Theorem 31. *For an (r, s) bipartite tournament on teams $X = \{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ with corresponding scores $A = [a_1, a_2, \dots, a_r]$ and $B = [b_1, b_2, \dots, b_s]$ (in nondecreasing order), the number of upsets in X is*

$$U_X = \frac{1}{2}\Big[\sum_{k=1}^{s} b_k(r - b_k) - \sum_{1 \le i < j \le r} (a_j - a_i)\Big].$$

As a corollary of this result, it has been shown in [1] that the upper bound above is also attained when $r = s$, and that it is not attained in any other cases.

9. Automorphism Groups of Bipartite Tournaments

An automorphism of a graph is a mapping of its set of vertices to the same set for which edges of the graph are isomorphic. The automorphism group of a graph is the set of automorphisms of the graph (and is a group). Not surprisingly, digraphs have corresponding concepts, but naturally the undirected graphs case came first. What can be called the fundamental theorem of groups of graphs is the 1938 result of Roberto Frucht [16], that every finite group is the automorphism group of some graph. For example, the group $C_2 \times C_2$ has the graph with four vertices and five edges as its automorphism group.

However, not every finite group is the automorphism group of a tournament. Moon [31] proved in 1964 that there are groups that are not the automorphism group of any tournament. It was later shown by Beineke and Lipman [10] that every group belongs to a bipartite tournament; in fact, special features can be required.

Theorem 32. *(a) A finite group is the automorphism group of some tournament if and only if it is a group of odd order.*
(b) Every finite group is the automorphism group of some bipartite tournament. In fact, the partite sets can be chosen to be of equal size.

Figure 15 shows such an example with the additional requirement that there is no mapping between the two partite sets.

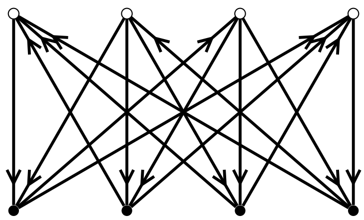

Figure 15. **An automorphism group of order 2**

10. Conclusion

One major aspect of our focus has been on how bipartite tournaments differ from tournaments themselves. There are of course other partite sets that have an interest, with the number of partite sets being three or more. In 2007, Lutz Volkmann [42] published a survey of multipartite tournaments. It is impressive that the number of articles in his paper approached the two-hundred mark. Here are some of the interesting examples of such digraphs that appeared (including their authors).

- On multipartite tournaments (W. Goddard, G. Kubicki, O. Oellermann, and S. Tian, 1991, [19]).
- On cycles in multipartite tournaments (G. Gutin, 1993, [21]).
- A complete solution of a problem of Bondy concerning multipartite tournaments (Y. Guo and L. Volkmann, 1994, [20]).
- Weakly Hamiltonian-connected ordinary multipartite tournaments (J. Bang-Jensen, G. Gutin, and J. Huang, 1995, [8]).
- The number of kings in a multipartite tournament (K. Koh and B. Tan, 1997, [28]).
- Strong subtournaments of multipartite tournaments (L. Volkmann, 1999, [41]).

- The average connectivity of regular multipartite tournaments (M. Henning and O. Oellermann, 2001, [23]).
- Vertex-pancyclic multipartite tournaments (G. Zhou, K. Zhang, and G. Xue, 2001, [44]).

It follows that there is a plentitude of variations beyond the standard tournaments and bipartite tournaments. We believe it would be worthwhile to explore extensions of results on decycling, unique realizability, self-converseness, and upsets, among others, to multipartite tournaments. The authors are not aware of any relevant work in this area.

Conflict of Interest: The authors declare that they have no conflict of interest.

Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

[1] K. Bagga, *On upsets in bipartite tournaments*, pp. 37–46, John Wiley & Sons, Inc., USA, 1985.

[2] K. Bagga and L. Beineke, *On superstrong bipartite tournaments*, Congr. Numer. **53** (1986), 113–119.

[3] K.S. Bagga, *Some structural properties of bipartite tournaments*, Ph.D. thesis, Purdue University, 1984.

[4] K.S. Bagga and L.W. Beineke, *Uniquely realizable score lists in bipartite tournaments*, Czech. Math. J. **37** (1987), no. 2, 323–333.

[5] ———, *On the numbers of some subtournaments of a bipartite tournament*, Ann. N.Y. Acad. Sci. **555** (1989), no. 1, 21–29.

[6] Y. Bai, B. Li, and H. Li, *Vertex-disjoint cycles in bipartite tournaments*, Discrete Math. **338** (2015), no. 8, 1307–1309. https://doi.org/10.1016/j.disc.2015.02.012.

[7] C. Balbuena, D. González-Moreno, and M. Olsen, *Vertex disjoint 4-cycles in bipartite tournaments*, Discrete Math. **341** (2018), no. 4, 1103–1108. https://doi.org/10.1016/j.disc.2017.10.023.

[8] J. Bang-Jensen, G. Gutin, and J. Huang, *Weakly Hamiltonian-connected ordinary multipartite tournaments*, Discrete Math. **138** (1995), no. 1-3, 63–74. https://doi.org/10.1016/0012-365X(94)00188-O.

[9] L. Beineke, *A tour through tournaments: A comparative study of bipartite and ordinary tournaments*, Combinatorics, Proceedings of the Eighth British Combinatorial Conference (H.N.V. Temperley ScD, ed.), Cambridge University Press, 1981, pp. 41–55.

[10] L. Beineke and M. Lipman, *The group-universality of bipartite tournaments*, Scientia: Math. Sci. **2** (1988), 11–12.

[11] J.C. Bermond and C. Thomassen, *Cycles in digraphs–A survey*, J. Graph Theory **5** (1981), no. 1, 1–43.

[12] B. Bollobés, O. Frank, and M. KarońSki, *On 4-cycles in random bipartite tournaments*, J. Graph Theory **7** (1983), no. 2, 183–194.
https://doi.org/10.1002/jgt.3190070207.

[13] R.A. Brualdi and J. Shen, *Disjoint cycles in Eulerian digraphs and the diameter of interchange graphs*, J. Comb. Theory B. **85** (2002), no. 2, 189–196.
https://doi.org/10.1006/jctb.2001.2094.

[14] P. Camion, *Chemins et circuits hamiltoniens des graphes complets*, C.R. Hebd. Seances Acad. Sci. **249** (1959), no. 21, 2151–2152.

[15] W.J.R. Eplett, *Self-converse tournaments*, Can. Math. Bull. **22** (1979), no. 1, 23–27.
https://doi.org/10.4153/CMB-1979-004-6.

[16] R. Frucht, *Herstellung von graphen mit vorgegebener abstrakter gruppe*, Compos. Math. **6** (1939), 239–250.

[17] D.R. Fulkerson, *Upsets in round robin tournaments*, Can. J. Math. **17** (1965), 957–969.
https://doi.org/10.4153/CJM-1965-091-7.

[18] D. Gale, *A theorem on flows in networks*, Pacific J. Math. **7** (1957), 1073–1082.

[19] W.D. Goddard, G. Kubicki, O.R. Oellermann, and S. Tian, *On multipartite tournaments*, J. Comb. Theory B **52** (1991), no. 2, 284–300.
https://doi.org/10.1016/0095-8956(91)90069-V.

[20] Y. Guo and L. Volkmann, *A complete solution of a problem of Bondy concerning multipartite tournaments*, J. Comb. Theory B **66** (1996), no. 1, 140–145.
https://doi.org/10.1006/jctb.1996.0010.

[21] G. Gutin, *On cycles in multipartite tournaments*, J. Comb. Theory B **58** (1993), no. 2, 319–321.
https://doi.org/10.1006/jctb.1993.1047.

[22] F. Harary and L. Moser, *The theory of round robin tournaments*, Am. Math. Mon. **73** (1966), no. 3, 231–246.
https://doi.org/10.1080/00029890.1966.11970749.

[23] M.A. Henning and O.R. Oellermann, *The average connectivity of regular multipartite tournaments*, Australas. J. Comb. **23** (2001), 101–114.

[24] M.A. Henning and A. Yeo, *Vertex disjoint cycles of different length in digraphs*, SIAM J. Discrete Math. **26** (2012), no. 2, 687–694.
https://doi.org/10.1137/100802463.

[25] A. Holtkamp and L. Volkmann, *Solution of a conjecture of Vandell on decycling bipartite tournaments by deleting arcs*, J. Combin. Math. Combin. Comput. **87** (2013), 3–19.

[26] B. Jackson, *Cycles in bipartite graphs*, J. Comb. Theory B **30** (1981), no. 3, 332–342.
https://doi.org/10.1016/0095-8956(81)90050-2.

[27] ———, *Long paths and cycles in oriented graphs*, J. Graph Theory **5** (1981), no. 2, 145–157.

https://doi.org/10.1002/jgt.3190050204.

[28] K.M. Koh and B.P. Tan, *The number of kings in a multipartite tournament*, Discrete Math. **167** (1997), 411–418.
https://doi.org/10.1016/S0012-365X(96)00244-0.

[29] N. Lichiardopol, *Proof of a conjecture of Henning and Yeo on vertex-disjoint directed cycles*, SIAM J. Discrete Math. **28** (2014), no. 3, 1618–1627.
https://doi.org/10.1137/130922653.

[30] J.W. Moon, *On the score sequence of an n-partite tournament*, Can. Math. Bull. **5** (1962), no. 1, 51–58.
https://doi.org/10.4153/CMB-1962-008-9.

[31] ———, *Tournaments with a given automorphism group*, Can. J. Math. **16** (1964), 485–489.
https://doi.org/10.4153/CJM-1964-050-9.

[32] ———, *Topics on Tournaments*, Holt, Rinehart, Winston, 1968.

[33] J.W. Moon and L. Moser, *On the distribution of 4-cycles in random bipartite tournaments*, Can. Math. Bull. **5** (1962), no. 1, 5–12.
https://doi.org/10.4153/CMB-1962-002-0.

[34] L. Ŕedei, *Ein kombinatorischer satz*, Acta Litt. Szeged. **7** (1934), 39–43.

[35] K.B. Reid and L.W. Beineke, *Tournaments, ch. 7*, Selected Topics in Graph Theory (K.B. Reid and J. Wilson, eds.), Academic Press, 1978, pp. 169–204.

[36] H.J. Ryser, *Combinatorial properties of matrices of zeros and ones*, Can. J. Math. **9** (1957), 371–377.
https://doi.org/10.4153/CJM-1957-044-3.

[37] M. Stob, *Rankings from round-robin tournaments*, Manag. Sci. **31** (1985), no. 9, 1191–1195.

[38] C. Thomassen, *Hamiltonian-connected tournaments*, J. Comb. Theory B **28** (1980), no. 2, 142–163.
https://doi.org/10.1016/0095-8956(80)90061-1.

[39] R.C. Vandell, *Decycling bipartite tournaments by deleting arcs*, Ars Combin. **116** (2014), 331–342.

[40] R.D.V. Vega, *On the balanced case of the Brualdi-Shen conjecture on 4-cycle decompositions of Eulerian bipartite tournaments*, Electron. J. Graph Theory Appl. **3** (2015), no. 2, 191–196.
http://dx.doi.org/10.5614/ejgta.2015.3.2.7.

[41] L. Volkmann, *Strong subtournaments of multipartite tournaments*, Australas. J. Comb. **20** (1999), 189–196.

[42] ———, *Multipartite tournaments: A survey*, Discrete Math. **307** (2007), no. 24, 3097–3129.
https://doi.org/10.1016/j.disc.2007.03.053.

[43] K. Wayland, *Bipartite score sets*, Can. Math. Bull. **26** (1983), no. 3, 273–279.
https://doi.org/10.4153/CMB-1983-044-7.

[44] G.F. Zhou, K.M. Zhang, and G. Xue, *Vertex-pancyclic multipartite tournaments*, Nanjing Daxue Xuebao Ziran Kexue Ban **37** (2001), 477–485.