Communications in Combinatorics and Optimization
Vol. xx, No. x (xxxx), pp. 1-16
https://doi.org/10.22049/cco.2025.30392.2449

CCO
Commun. Comb. Optim.

***Research Article***

# Signed total Italian $k$-domination

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*Received: 2 March 2025; Accepted: 6 June 2025*
*Published Online: 13 June 2025*

**Abstract:** Let $k \geq 1$ be an integer. A signed total Italian $k$-dominating function (STIkDF) on a graph $G = (V, E)$ is a function $f : V \to \{-1, 1, 2\}$ satisfying the conditions that $\sum_{u \in N(v)} f(u) \geq k$ for each vertex $v \in V$, where $N(v)$ is the neighborhood of $v$, and each vertex $u$ with $f(u) = -1$ is adjacent to a vertex $v$ with $f(v) = 2$ or to two vertices $w$ and $z$ with $f(w) = f(z) = 1$. The weight of an STIkDF $f$ is $w(f) = \sum_{v \in V} f(v)$. The signed total Italian $k$-domination number of $G$, denoted by $\gamma^k_{stI}(G)$, is the minimum weight of an STIkDF on $G$. In this paper, we prove that the decision problem for the signed total $k$-domination is NP-complete for $k \in \{1, 2\}$. We present tight lower bound on $\gamma^2_{stI}(G)$, and characterize all extremal graphs. Using a discharging method, we also determine the value $\gamma^2_{stI}(C_3 \Box C_n)$ for all $n \geq 3$.

**Keywords:** signed total Italian $k$-domination number, signed total Italian $k$-dominating function, complexity.

**AMS Subject classification:** 05C69

## 1. Introduction

All graphs considered in this paper are finite, simple and undirected. Let $G = (V, E)$ be a graph with vertex set $V$ and edge set $E$. The order of $G$ is given by $n = |V|$ and its size by $m = |E|$. The *open neighborhood* of a vertex $v$ is the set of all neighbors of $v$, denoted $N(v)$, whereas the *closed neighborhood* of $v$ is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v$ is denoted $d(v) = |N(v)|$. A *leaf* is a vertex of degree 1, and its neighbor is called a *support vertex.* A *cubic graph* is a graph in which every vertex has degree 3. We denote the *cycle graph* and *complete graph* on $n$ vertices by $C_n$ and $K_n$, respectively. A complete graph $K_3$ is called a *triangle.* The complete bipartite graph $K_{1,t}$ is called a *star.* Let $H$ be a graph. If $G$ does not contain $H$ as an induced subgraph, then we say that $G$ is $H$*-free.* A *matching* in a graph $G$ is a set of independent edges in $G$. A *perfect matching* $M$ in $G$ is a matching such that every vertex of $G$ is incident to an edge of $M$. For disjoint subsets $U$ and $W$ of

vertices, we let $[U,W]$ denote the set of edges between $U$ and $W$. For a set of vertices $S \subseteq V$, the subgraph induced by $S$ is denoted by $G[S]$. We use the standard notation $[k] = \{1, 2, \ldots, k\}$.
For a pair of graphs $G$ and $H$, the *Cartesian product* of $G$ and $H$ is the graph $G \square H$ whose vertex set is $V(G) \times V(H)$, and two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. We restrict our attention to the Cartesian product of $C_n$ and $C_m$. Let $V(C_n) = \{1, 2, \ldots, n\}$, $E(C_n) = \{i(i+1), 1n : i = 1, 2, \ldots, n-1\}$. Hence we will denote vertices of $V(C_m \square C_n)$ by $(i,j)$ for $i \in [m]$ and $j \in [n]$.
A *signed total dominating function* (STDF) of a graph $G$ is defined as a function $f : V \to \{-1, 1\}$ such that $f(N(v)) = \sum_{u \in N(v)} f(u) \geq 1$ for every $v \in V$. The *signed total domination number*, denoted $\gamma_t^s(G)$, of $G$ is the minimum weight of an STDF on $G$. The signed total domination was introduced in [11] and has been studied by several authors [3, 5].
If $k$ is a positive integer, then the *signed total Roman $k$-dominating function* (STRkDF) on a graph $G$ is defined as a function $f : V \to \{-1, 1, 2\}$ such that $f(N(v)) \geq k$ for every $v \in V$ and vertex $u$ with $f(u) = -1$ is adjacent to a vertex $v$ for which $f(v) = 2$. The *weight* of $f$ is the value $w(f) = \sum_{v \in V} f(v)$. The signed total Roman $k$-domination number $\gamma_{stR}^k(G)$ of $G$ is the minimum weight of an STRkDF on $G$. The signed total Roman $k$-domination number was first defined in [8]. The special case $k = 1$ was introduced and investigated by Volkmann [7]. For further details on Roman domination and signed total Roman $k$-domination, we refer the reader to the book chapter and survey paper in [1, 2].
A generalization of signed total Roman $k$-domination was introduced by Volkmann [10]. A *signed total Italian $k$-dominating function* (STIkDF) is a function $f : V \to \{-1, 1, 2\}$ such that $f(N(v)) = \sum_{u \in N(v)} f(u) \geq k$ for every $v \in V$ and each vertex $u$ with $f(u) = -1$ is adjacent to a vertex $v$ for which $f(v) = 2$ or two vertices $w$ and $z$ with $f(w) = f(z) = 1$. The *signed total Italian $k$-domination number*, denoted by $\gamma_{stI}^k(G)$, is the minimum weight of an STIkDF on $G$. The special case $k = 1$ was introduced in [9], where it is called weak signed Roman domination and $\gamma_{stI}^1(G)$ is written as $\gamma_{stI}(G)$. The definitions lead to $\gamma_{stR}^k(G) \geq \gamma_{stI}^k(G)$. An STIkDF with minimum weight is called a $\gamma_{stI}^k$-function. For an STIkDF $f$, let $V_i = \{v \in V : f(v) = i\}$ for $i = -1, 1, 2$. Since these three sets determine $f$, we can equivalently write $f = (V_{-1}, V_1, V_2)$.
In this paper, we continue the study of signed total Italian $k$-domination. Some of our results are extensions of well-known properties of the signed total Roman $k$-domination given in [6, 7]. As pointed out in [10], the signed total Italian $k$-domination number exists when $\delta \geq \frac{k}{2}$. Since in this paper we restrict our attention to the cases $k \in \{1, 2\}$, this assumption requires only that graphs under consideration are isolate-free.
We proceed as follows. In Section 2, we show that the decision problem for the signed total Italian $k$-domination is NP-complete on bipartite graphs for $k \in \{1, 2\}$. In Section 3, we present a sharp lower bound on the signed total Italian 2-domination number. Finally, in Section 4 we investigate the signed total Italian $k$-domination

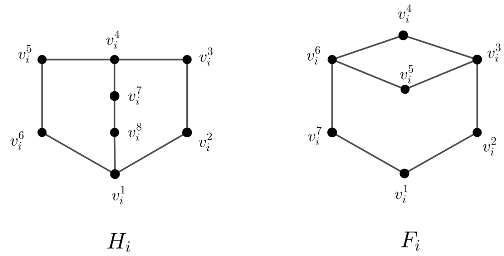


**Figure 1.** **The graphs $H_i$ and $F_i$.**

number for cubic graphs and determine $\gamma_{stI}^2(C_3\Box C_n)$ for all $n \geq 3$.

## 2. Complexity result

In this section, we establish the NP-complete result for the signed total Italian $k$-domination problem on bipartite graphs for $k \in \{1, 2\}$.
**Signed total Italian $k$-domination problem (STIkDP)**
**Instance:** A graph $G$ and a positive integer $k \leq |V(G)|$.
**Question:** Does $G$ have an STIkDF with weight at most $k$ ?
We will prove that this problem is NP-complete by reducing it to the special case of Exact Cover by 3-sets, referred as X3C3. The completeness of X3C3 was proven in 2008 by Hickey et al [4].

**X3C3**
**Instance:** A set of elements $X$ with $|X| = 3q$ and a collection $C$ of $3q$ 3-element subsets of $X$, such that every element appears in exactly 3 members of $C$.
**Question:** Does there exist a subcollection $C' \subset C$ such that every element of $X$ occurs in exactly one member of $C'$ ?

**Theorem 1.** *Problem STI1DP is NP-complete for bipartite graphs.*

*Proof.* STI1DP is clearly in the class NP, since we can verify in polynomial time that a function $f : V \to \{-1, 1, 2\}$ has weight at most $k$ and is an STI1DF. In the following, let us transform any instance of X3C3 into an instance $G$ of STI1DP, so that one of them has a solution if and only if the other one has a solution. Assume that $I = (X, \mathcal{C})$ is an arbitrary instance of X3C3, with $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_{3q}\}$.
For each $x_i \in X$, we build a connected graph $H_i$ as shown in Figure 1. Let $W = \{v_1^1, v_2^1, \ldots, v_{3q}^1\}$. For each $C_j \in \mathcal{C}$, we create a vertex $c_j$ to which we associate two stars $K_{1,2}$ with centers $y_j$ and $z_j$ and two edges $c_jy_j$ and $c_jz_j$. Now to obtain a

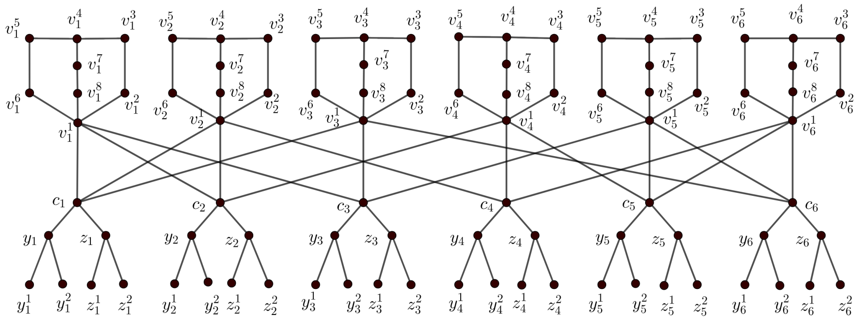

**Figure 2.** **NP-completeness of STI1D for bipartite graphs.**

bipartite graph $G$, we add edges $c_j v_i^1$ if $x_i \in C_j$. Set $k = 22q$. Figure 2 shows an example of the bipartite graph $G$ for $q = 2$.

Suppose that $\mathcal{C}'$ is a solution of the instance X3C3. Define an STI1DF $f$ on $G$ of weight $k$ as follows. For every $i \in \{1, 2, \ldots, 3q\}$, let $f(y_i) = f(z_i) = 2$, $f(y_i^1) = f(z_i^1) = -1$, $f(v_i^2) = f(v_i^6) = f(v_i^7) = f(v_i^8) = -1$, $f(v_i^1) = f(v_i^4) = 2$. For every $C_j$, if $C_j \in \mathcal{C}'$, let $f(c_j) = 2$, otherwise let $f(c_j) = 1$. Finally, assign 1 to all the remaining vertices of $G$. Since $\mathcal{C}'$ is a solution for X3C3, every vertex in $W$ is adjacent to a vertex assigned 2. It is straightforward to see that $f$ is a signed total Italian 1-dominating function with weight $2(3q) + 4(3q) + 4q = 22q = k$.

Conversely, suppose that $G$ has an STI1DF with weight at most $k$. Among all such functions, let $g$ be one that assigns smallest possible values to the leaves of $G$. Clearly, $g$ assigns a positive value to each support vertex. We claim that $g(c_i) \geq 1$ for all $1 \leq i \leq 3q$. If $g(c_i) = -1$ for some $i$, then $\sum_{j=1}^{2} g(y_i^j) \geq 2$ and $\sum_{j=1}^{2} g(z_i^j) \geq 2$. So $\sum_{j=1}^{2} g(y_i^j) + \sum_{j=1}^{2} g(z_i^j) + g(y_i) + g(z_i) + g(c_i) \geq 5$. Now we define $g' : V \to \{-1, 1, 2\}$ such that $g'(y_i^1) = g'(z_i^1) = -1$, $g'(y_i^2) = g'(z_i^2) = 1$, $g(y_i) = g(z_i) = 2$, $g'(c_i) = 1$ and $g = g'$ for the remaining vertices of $G$. Thus, $g'$ is an STI1DF of $G$ such that $w(g') \leq w(g)$, and the weight of leaves under $g'$ is less then the weight of leaves under $g$, contradicting the choice of $g$. So $g(c_i) \geq 1$ for $1 \leq i \leq 3q$. Therefore, $g(y_i^1) + g(y_i^2) = 0$ and $g(z_i^1) + g(z_i^2) = 0$.

Next, we shall show that $g(V(H_i)) \geq 2$ for $1 \leq i \leq 3q$, and if $g(V(H_i)) \in \{2, 3, 4\}$, then $g(v_i^2) = g(v_i^6) = g(v_i^8) = -1$. Observe that if $g(v_i^1) = -1$, then $g(v_i^3) = g(v_i^5) = g(v_i^7) = 2$, and if $g(v_i^1) = 1$, then $g(v_i^3) \geq 1$, $f(v_i^5) \geq 1$ and $g(v_i^7) \geq 1$. Similarly, if $g(v_i^4) = -1$, then $g(v_i^2) = g(v_i^6) = g(v_i^8) = 2$, and if $g(v_i^4) = 1$, then $g(v_i^2) \geq 1$, $g(v_i^6) \geq 1$ and $g(v_i^8) \geq 1$. It is easy to check that $g(V(H_i)) \geq 8$ when $g(v_i^4) = -1$. Suppose $g(v_i^4) = 2$. Since $g(N(v_i^4)) \geq 1$, then $g(V(H_i)) \geq 4$ when $g(v_i^1) = -1$, $g(V(H_i)) \geq 3$ when $g(v_i^1) = 1$ and $g(V(H_i)) \geq 2$ when $g(v_i^1) = 2$. Moreover, the equalities hold only if $g(v_i^2) = g(v_i^6) = g(v_i^8) = -1$. Finally, we assume $g(v_i^4) = 1$. Then $g(V(H_i)) \geq 9$ when $g(v_i^1) = -1$, $g(V(H_i)) \geq 8$ when $g(v_i^1) = 1$ and $g(V(H_i)) \geq 7$ when $g(v_i^4) = 2$.

Now assume that $r = |V_2 \cap c_j|$, then $w(g) = 2r + (3q - r) + 4(3q) + 2(3q) = r + 21q \leq$

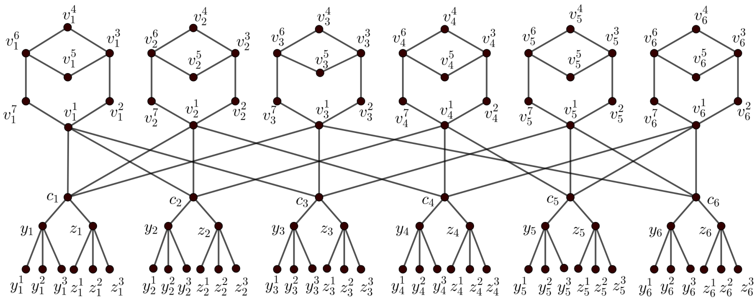

**Figure 3.** **NP-completeness of STI2D for bipartite graphs.**

$k = 22q$ and so $r \leq q$. On the other hand, each $c_j$ has exactly three neighbors in $W$, then $3r \geq 3q$. Hence, $r = q$ and $\mathcal{C}' = \{C_j : g(c_j) = 2\}$ is an exact cover for $C$. □

**Theorem 2.** *Problem STI2DP is NP-complete for bipartite graphs.*

*Proof.* Similar as the proof of Theorem 1, STI2DP belongs to NP, since we can check in polynomial time that a function $f : V \to \{-1, 1, 2\}$ has weight at most $k$ and is an STI2DF. In the following, let us transform any instance of X3C3 into an instance $G$ of STI2DP, so that one of them has a solution if and only if the other one has a solution. Assume that $I = (X, \mathcal{C})$ is an arbitrary instance of X3C3, with $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $\mathcal{C} = \{C_1, C_2, \ldots, C_{3q}\}$.

For each $x_i \in X$, we build a connected graph $F_i$ as shown in Figure 1. For each $C_i \in \mathcal{C}$, we create a vertex $c_j$ to which we associate two stars $K_{1,3}$ with centers $y_j$ and $z_j$ and two edges $c_jy_j$ and $c_jz_j$. Now to obtain a bipartite graph $G$, we add edges $c_jv_i^1$ if $x_i \in C_j$. Set $k = 34q$. Figure 3 shows an example of the bipartite graph $G$ for $q = 2$.

Suppose that $\mathcal{C}'$ is a solution of X3C3. Define a signed total Italian 2-domination function $f$ on $G$ of weight $k$ as follows. For every $i \in \{1, 2, \ldots, 3q\}$, let $f(y_i) = f(z_i) = 2$, $f(y_i^1) = f(z_i^1) = -1$, $f(v_i^2) = f(v_i^7) = -1$, $f(v_i^5) = 2$. For every $C_j$, if $C_j \in \mathcal{C}'$, let $f(c_j) = 2$, otherwise let $f(c_j) = 1$. Assign 1 to all the remaining vertices of $G$. It is straightforward to see that $f$ is a signed total Italian 2-dominating function with weight $4(3q) + 6(3q) + 4q = 34q = k$.

Conversely, suppose that $G$ has an STI2DF with weight at most $k$. Among all such functions, let $g$ be one that assigns smallest possible values to the leaves of $G$. Clearly, $g$ assigns value 2 to each support vertex. We claim that $g(c_i) \geq 1$ for all $1 \leq i \leq 3q$. If $g(c_i) = -1$ for some $i$, then $\sum_{j=1}^{3} g(y_i^j) \geq 3$ and $\sum_{j=1}^{3} g(z_i^j) \geq 3$. Now we define $g' : V \to \{-1, 1, 2\}$ such that $g'(y_i^1) = g'(z_i^1) = -1$ and $g'(x) = 1$ for any other leaf of $y_i$ and $z_i$, $g'(c_i) = 1$ and $g = g'$ for the remaining vertices of $G$. Thus, $g'$ is an

STI2DF of $G$ such that $w(g') < w(g)$, contradicting the choice of $g$. So $g(c_i) \geq 1$ for $1 \leq i \leq 3q$.
Notice that $g(v_i^1) \geq 1$, otherwise $\sum_{u\in N(v_i^2)} f(u) \leq 1$. With a similar reasoning, we conclude that $g(v_i^3) \geq 1$ and $g(v_i^6) \geq 1$. Since $\sum_{u\in N(v_i^6)} f(u) = g(v_i^4) + g(v_i^5) + g(v_i^7) \geq 2$, we get $g(V(F_i)) \geq 4$. If $g(V(F_i)) = 4$, then $g(v_i^1) = g(v_i^3) = g(v_i^6) = 1$, $g(v_i^2) = g(v_i^7) = -1$ and $g(v_i^4) + g(v_i^5) = 3$.
Now assume that $r = |V_2 \cap c_j|$, then $2r + (3q - r) + 4(3q) + 6(3q) = r + 33q \leq k = 34q$ and so $r \leq q$. On the other hand, each $c_j$ has exactly three neighbors in $W = \{v_1^1, v_2^1, \ldots, v_{3q}^1\}$, then $3r \geq 3q$. Hence, $r = q$ and hence, $\mathcal{C}' = \{C_j : g(c_j) = 2\}$ is an exact cover for $C$. □

## 3. Lower bounds

It was proved in [9] that for every connected graph $G$ of order $n$ and size $m$, $\gamma_{stI}(G) \geq \frac{11n-12m}{4}$. This section is dedicated to presenting a general lower bound on $\gamma_{stI}^2(G)$. If $f = (V_{-1}, V_1, V_2)$ is an STIkDF of $G$, then for notational convenience, we let $G_i = G[V_i]$ and $|V(G_i)| = n_i$, $|E(G_i)| = m_i$ for $i \in \{-1, 1, 2\}$. Let $V_{12} = V_1 \cup V_2$ and $G_{12} = G[V_{12}]$. Further, we let $|V(G_{12})| = n_{12}$ and $|E(G_{12})| = m_{12}$. Thus, $n_{12} = n_1 + n_2$ and $m_{12} = m_1 + m_2 + |[V_1, V_2]|$. Let $V_{-1}^2 \subseteq V_{-1}$ be the maximum set such that each vertex of $V_{-1}^2$ is adjacent to at least one vertex of $V_2$ and let $V_{-1}^1 = V_{-1} \setminus V_{-1}^2$.
For $k \geq 1$, let $H_k$ be the graph obtained from a graph $H$ of order $k$ by adding exactly $2d_H(v)-2$ or at least $2d_H(v)$ pendent edges at each vertex of $H$. Let $\mathcal{H} = \{H_k | k \geq 1\}$. Note that $\mathcal{H}$ contains all stars $K_{1,n}$ $(n \geq 1)$.

**Theorem 3.** *Let $G$ be a connected graph of order $n$ and size $m$. Then*

$$\gamma_{stI}^2(G) \geq 4n - 4m$$

*with equality if and only if $G \in \mathcal{H}$.*

*Proof.* Since $\gamma_{stI}^2(P_2) = \gamma_{stI}^2(P_3) = 4$, the result is immediate for $n = 2, 3$. Assume that $n \geq 4$ and let $f = (V_{-1}, V_1, V_2)$ be a $\gamma_{stI}^2$-function. If $V_{-1} = \emptyset$, then $\gamma_{stI}^2(G) \geq n \geq 4n - 4m$ since $G$ is connected. Suppose that $V_{-1} \neq \emptyset$. We have

$$2|V_{-1}^1| \leq |[V_{-1}^1, V_1]| = \sum_{v\in V_1} d_{V_{-1}^1}(v).$$

For each vertex $v \in V$, we have $2 \leq f(N(v)) = 2d_{V_2}(v) + d_{V_1}(v) - d_{V_{-1}}(v)$, and so

$d_{V_{-1}}(v) \leq 2d_{V_2}(v) + d_{V_1}(v) - 2$. Hence,

$$\begin{aligned} 2|V_{-1}^1| &\leq \sum_{v\in V_1} d_{V_{-1}^1}(v) \leq \sum_{v\in V_1} d_{V_{-1}}(v) \\ &\leq \sum_{v\in V_1} (2d_{V_2}(v) + d_{V_1}(v) - 2) \\ &= 2|[V_1, V_2]| + 2m_1 - 2n_1, \end{aligned}$$

and we obtain $|V_{-1}^1| \leq |[V_1, V_2]| + m_1 - n_1$.
Since each vertex of $V_{-1}^2$ is adjacent to at least one vertex in $V_2$, we have

$$\begin{aligned} |V_{-1}^2| &\leq |[V_{-1}^2, V_2]| = \sum_{v\in V_2} d_{V_{-1}^2}(v) \\ &\leq \sum_{v\in V_2} d_{V_{-1}}(v) \\ &= \sum_{v\in V_2} (2d_{V_2}(v) + d_{V_1}(v) - 2) \\ &= 4m_2 + |[V_1, V_2]| - 2n_2 \\ &= 4(m_{12} - m_1 - |[V_1, V_2]|) + |[V_1, V_2]| - 2n_2 \\ &= 4m_{12} - 4m_1 - 3|[V_1, V_2]| - 2n_2. \end{aligned}$$

Combining the corresponding inequalities, we deduce that

$$n_{-1} = |V_{-1}^1| + |V_{-1}^2| \leq 4m_{12} - 3m_1 - 2|[V_1, V_2]| - n_1 - 2n_2,$$

and thus $m_{12} \geq (n_{-1} + n_1 + 2n_2 + 3m_1 + 2|[V_1, V_2]|)/4$. This leads to

$$\begin{aligned} m &\geq m_{12} + |[V_{-1}, V_2]| + |[V_{-1}, V_1]| \\ &\geq \frac{1}{4}(n_{-1} + n_1 + 2n_2 + 3m_1 + 2|[V_1, V_2]|) + n_{-1} + \frac{1}{2}|[V_{-1}, V_1]| \\ &= \frac{1}{4}(5n_{-1} + 2n_{12} - n_1 + 3m_1 + 2|[V_1, V_2]|) + \frac{1}{2}|[V_{-1}, V_1]| \\ &= \frac{1}{4}(5n - 3n_{12} - n_1 + 3m_1 + 2|[V_1, V_2]|) + \frac{1}{2}|[V_{-1}, V_1]|. \end{aligned}$$

This yields

$$n_{12} \geq \frac{1}{3}(5n - n_1 + 3m_1 - 4m + 2|[V_1, V_2]| + 2|[V_{-1}, V_1]|).$$

Therefore,

$$\begin{aligned} \gamma_{stI}^2(G) &= 2n_2 + n_1 - n_{-1} \\ &= 3n_{12} - n - n_1 \\ &\geq 5n - n_1 + 3m_1 - 4m + 2|[V_1, V_2]| + 2|[V_{-1}, V_1]| - n - n_1 \\ &= 4n - 4m + 3m_1 - 2n_1 + 2|[V_1, V_2]| + 2|[V_{-1}, V_1]|. \end{aligned}$$

Let

$$\phi(n_1) = 3m_1 - 2n_1 + 2|[V_1, V_2]| + 2|[V_{-1}, V_1]|.$$

It suffices to show that $\phi(n_1) \geq 0$. If $n_1 = 0$, then $\phi(n_1) = 0$ and we are done. Next, we assume that $n_1 \geq 1$. Let $H_1, H_2, \ldots, H_t$ be the components of the induced subgraph $G[V_1]$ with order $h_1, h_2, \ldots, h_t$, respectively. Since $G$ is connected, each component $H_i$ contains a vertex adjacent to a vertex in $V_{-1} \cup V_2$. Therefore,

$$\begin{aligned} m_1 + |[V_1, V_2]| + |[V_1, V_{-1}]| &\geq (h_1 - 1) + (h_2 - 1) + \cdots + (h_t - 1) + t \\ &= h_1 + h_2 + \cdots + h_t = n_1. \end{aligned}$$

This yields that

$$\begin{aligned} \phi(n_1) &= 3m_1 - 2n_1 + 2|[V_1, V_2]| + 2|[V_{-1}, V_1]| \\ &\geq 2m_1 - 2n_1 + 2|[V_1, V_2]| + 2|[V_{-1}, V_1]| \\ &\geq 2n_1 - 2n_1 = 0. \end{aligned}$$

Suppose that $\gamma^2_{stI}(G) = 4n - 4m$. Then all the inequalities above must be equalities. In particular, $f(N(v)) = 2$ for each vertex $v \in V_1 \cup V_2$, $m_{-1} = 0$, $|V^2_{-1}| = |[V^2_{-1}, V_2]|$ and further, either $n_1 = 0$ or $|[V_{-1}, V_1]| = 0$, $m_1 = 0$. This implies that for each vertex $v \in V_{-1} \cup V_1$, we have $d_G(v) = 1$ and $v$ is adjacent to exactly one vertex in $V_2$. That is, each vertex of $V_{-1} \cup V_1$ is a leaf in $G$. Moreover, $d_{V_{-1}}(v) = 2d_{V_2}(v) + d_{V_1}(v) - 2$ for each vertex $v \in V_2$. This implies that $G \in \mathcal{H}$.
Conversely, suppose that $G \in \mathcal{H}$. Then $G$ is obtained from a graph $H$ of order $k$ by adding exactly $2d_H(v) - 2$ or at least $2d_H(v)$ pendent edges at each vertex of $H$. For any non-leaf vertex $v$ of $G$, let $v_1, v_2, \ldots, v_t$ be the leaves adjacent to $v$ in $G$. Denote $d'(v)$ the number of non-leaf neighbors of $v$. By our construction, $t \geq 2d'(v) - 2$ and $t \neq 2d'(v) - 1$. We define an STI2DF $f$ as follows. Let $f(v) = 2$ for every non-leaf vertex $v$ in $G$. For the leaf neighbors of $v$, define $f(v_i) = -1$ for $1 \leq i \leq 2d'(v) - 2$. If $t$ is even, then let $f(v_i) = (-1)^i$ for $2d'(v) - 1 \leq i \leq t$. If $t$ is odd, let $f(v_{2d'(v)-1}) = 2$, $f(v_{2d'(v)}) = f(v_{2d'(v)+1}) = -1$ and let $f(v_i) = (-1)^i$ for $2d'(v) + 2 \leq i \leq t$. So the total weight of all leaf neighbors of $v$ is $2 - 2d'(v)$. Let $G'$ be the subgraph of $G$ induced by all non-leaf vertices of $G$. Denote $|V(G')| = n'$ and $|E(G')| = m'$. Let $l$ be the number of leaves of $G$. Thus, $n = n' + l$ and $m = m' + l$. Then the weight of $f$ is

$$\begin{aligned} w(f) &= 2n' + \sum_{v \in V(G')} (2 - 2d'(v)) \\ &= 2n' + 2n' - 4m' \\ &= 4n' - 4(m - l) \\ &= 4n - 4m. \end{aligned}$$

Hence, $\gamma^2_{stI}(G) \leq w(f) = 4n - 4m$. Consequently, $\gamma^2_{stI}(G) = 4n - 4m$. □

Since $\gamma^2_{stI}(G) \le \gamma^2_{stR}(G)$, Theorem 3 leads to the following known lower bound.

**Corollary 1.** *([6]) Let $G$ be a connected graph of order $n$ and size $m$. Then*

$$\gamma^2_{stR}(G) \ge 4n - 4m.$$

## 4. Regular graphs

In this section, we investigate the signed total Italian $k$-domination on $r$-regular graphs with $r = 3, 4$. Volkmann [10] established a lower bound on the signed total Italian $k$-domination number for $r$-regular graphs.

**Lemma 1.** *([10]) If $G$ is an $r$-regular graph of order $n$ with $r \ge \frac{k}{2}$, then $\gamma^k_{stI}(G) \ge \frac{kn}{r}$.*

We note that if $\delta(G) \ge 2$, then any STDF of $G$ is also an STI1DF of $G$. Hence, the following result is immediate.

**Observation 4.** Let $G$ be a graph with $\delta(G) \ge 2$. Then $\gamma_{stI}(G) \le \gamma^s_t(G)$.

An *open packing* in a graph $G$ is a subset $S$ of vertices whose open neighborhoods are pairwise disjoint. The *open packing number*, $\rho^o(G)$, is the maximum cardinality of an open packing in $G$. It is shown in [3] that if $G$ is a cubic graph of order $n$, then $\gamma^s_t(G) = n - 2\rho^o(G)$. The following result shows that the signed total domination number and the signed total Italian domination number are equivalent for cubic graphs.

**Theorem 5.** *If $G$ is a cubic graph of order $n$, then $\gamma_{stI}(G) = \gamma^s_t(G) = n - 2\rho^o(G)$.*

*Proof.* Since $G$ is cubic, $\gamma_{stI}(G) \le \gamma^s_t(G) = n-2\rho^o(G)$. On the other hand, let $f$ be a $\gamma_{stI}$-function of $G$, $M$ the set of vertices of weight $-1$ under $f$. Then $|N(v) \cap M| \le 1$ for every $v \in V$. That is, $M$ is an open packing of $G$ and $|M| \le \rho^o(G)$. Thus, $w(f) \ge n - |M| - |M| \ge n - 2\rho^o(G)$ and then $\gamma_{stI}(G) = n - 2\rho^o(G)$. □

It is shown in [3] that $\gamma^s_t(G) \ge n/3$ is a sharp lower bound for a cubic graph $G$. Hosseini Moghaddam et al. [5] showed that $\gamma^s_t(G) \le 2n/3$ is a sharp upper bound for all connected cubic graphs $G$ different from the Heawood graph $G_{14}$. Note that $\rho^o(G_{14}) = 2$. As an immediate consequence, we have the following result.

**Corollary 2.** *If $G$ is a connected cubic graph different from the Heawood graph $G_{14}$, then $n/3 \le \gamma_{stI}(G) \le 2n/3$. And $\gamma_{stI}(G_{14}) = 10$ for the Heawood graph $G_{14}$.*

By Lemma 1, we establish lower and upper bounds on $\gamma^2_{stI}(G)$ for cubic graphs.

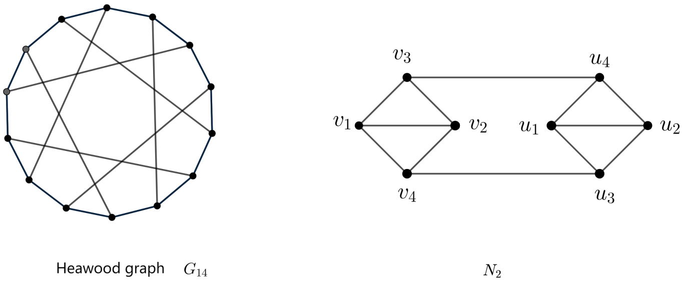


**Figure 4.** **Illustration for Corollary 2 and Example 1**

**Corollary 3.** *Let $G$ be a connected cubic graph of order $n$. Then $2n/3 \le \gamma^2_{stI}(G) \le n$.*

The following example will demonstrate that the upper bound in Corollary 3 is sharp.

**Example 1.** Let $N_2$ be a claw-free cubic graph of order $n = 8$, as shown in Figure 4. Then $\gamma^2_{stI}(N_2) = 8$.

*Proof.* Let $f$ be a $\gamma^2_{stI}$-function of $N_2$. Since at most one of $v_1$ and $v_2$ can be assigned value $-1$, we may assume $f(v_1) \ge 1$. Then $f(v_1) + f(N(v_1)) = \sum_{i=1}^{4} f(v_i) \ge 3$. If $\sum_{i=1}^{4} f(v_i) = 3$, then $f(v_1) = 1$ and $f(v_2) + f(v_3) + f(v_4) = 2$. We will show that $\sum_{i=1}^{4} f(u_i) \ge 5$ when $\sum_{i=1}^{4} f(v_i) = 3$. Since $f(v_1) + f(v_3) + f(v_4) = f(N(v_2)) \ge 2$, it follows that $f(v_2) \le 1$. If $f(v_2) = -1$, then $\{f(v_3), f(v_4)\} = \{1, 2\}$, $f(u_3) = f(u_4) = 2$ and $f(u_1) + f(u_2) \ge 1$. This implies $\sum_{i=1}^{4} f(u_i) \ge 5$. If $f(v_2) = 1$, then $\{f(v_3), f(v_4)\} = \{-1, 2\}$, $f(u_3) \ge 1$, $f(u_4) \ge 1$ and $f(u_1) + f(u_2) \ge 3$. This implies $\sum_{i=1}^{4} f(u_i) \ge 5$ again. Therefore, $\gamma^2_{stI}(N_2) \ge 8$ and then $\gamma^2_{stI}(N_2) = 8$ by the trivial upper bound of Corollary 3. □

To show that the lower bound in Corollary 3 is attained, we introduce some notations. Let $G$ be a connected $(K_{1,3}, K_4 - e, C_4)$-free cubic graph of order $n \ge 6$. Then every vertex of $G$ belongs to a unique triangle of $G$ and every two triangles in $G$ are joined by at most one edge. We define the *contraction graph* of $G$ to be the graph $G'$ whose vertices correspond to the triangles in $G$ and where two vertices of $G'$ are adjacent if and only if the corresponding triangles in $G$ are joined by an edge. Note that the order of $G'$ is $n(G') = n/3$, that is, the number of triangles in $G$.

**Theorem 6.** *Let $G$ be a connected $(K_{1,3}, K_4 - e, C_4)$-free cubic graph of order $n \ge 6$. Then $\gamma^2_{stI}(G) = 2n/3$ if and only if the contraction graph $G'$ of $G$ contains three edge-disjoint perfect matchings.*

*Proof.* Since $G$ is a connected $(K_{1,3}, K_4 - e, C_4)$-free cubic graph, its contraction graph $G'$ is a connected cubic graph. Denote the $i$th triangle of $G$ by $\Delta_i$ and let $V(\Delta_i) = \{u_i, v_i, w_i\}$, $1 \leq i \leq n/3$. Let $f$ be a $\gamma^2_{stI}$-function of $G$. Then

$$2n \leq \sum_{v \in V} f(N(v)) = 3\sum_{i=1}^{n/3}(f(u_i) + f(v_i) + f(w_i)) = 3w(f) = 2n. \tag{4.1}$$

Thus, $f(N(v)) = 2$ for each vertex $v \in V$. Since at most one vertex in each triangle of $G$ can be assigned value $-1$, we have $f(u_i)+f(v_i)+f(w_i) \geq 1$. Suppose $f(u_i)+f(v_i)+f(w_i) = 1$ for some $i$, say $f(u_i) = -1$ and $f(v_i) = f(w_i) = 1$. Then $f(N(u_i)) \geq 3$, a contradiction. So $f(u_i) + f(v_i) + f(w_i) \geq 2$, and in fact $f(u_i) + f(v_i) + f(w_i) = 2$ for each $1 \leq i \leq n/3$ by (4.1). This implies that the vertices of $\Delta_i$ are assigned values $-1, 1, 2$, respectively. Then their corresponding neighbors outside $\Delta_i$ are assigned values $-1, 1, 2$, too. It is easy to check that the edges of $G$ whose endpoints assigned value $t$ induce a matching $M_t$ of $G$ for each $t \in \{-1, 1, 2\}$. These three matchings of $G$ corresponding to three edge-disjoint perfect matchings of $G'$.
Conversely, suppose that $M'_{-1}, M'_1$ and $M'_2$ are three edge-disjoint perfect matchings of $G'$. Each perfect matching $M'_t$ corresponding to a matching $M_t$ of $G$, whose edges connect two vertices in different triangles. We construct an STI2DF $f$ on $G$ of weight $2n/3$ as follows. For each $t \in \{-1, 1, 2\}$, assign value $t$ to the vertices covered by $M_t$ in $G$. Then the three vertices of each triangle are assigned distinct values and two endpoints of each edge connecting two triangles are assigned the same value. It is straightforward to check that $f$ is an STI2DF of $G$ with weight $\frac{2}{3}n$. So $\gamma^2_{stI}(G) \leq 2n/3$ and then $\gamma^2_{stI}(G) = 2n/3$ by Corollary 3. □

Finally in this section, we determine the signed total Italian 2-domination number of $C_3 \square C_n$, a family of 4-regular graph. For convenience, we introduce some notation. Denote the $i$th $C_3$-layer of $C_3 \square C_n$ by $C^i$, that is, $V(C^i) = \{(j, i) : j \in [3]\}$. If $f$ is an STI2DF of $C_3 \square C_n$, let $f_i = f(1, i) + f(2, i) + f(3, i)$, namely, $f_i$ stands for the weight of the $i$th $C_3$-layer with respect to $f$. Two $C_3$-layers $C^i$ and $C^j$ of $C_3 \square C_n$ are adjacent if $|i - j| = 1$.

**Lemma 2.** *Let $n \geq 3$ and let $f$ be an STI2DF of $C_3 \square C_n$. Then the following assertions hold, where all the indices are taken modulo $n$.*

*(i) If $f_{i+1} = 0$, then $f_i \geq 0$, $f_{i+2} \geq 0$, $f_i \neq 1$, $f_{i+2} \neq 1$, and $f_i + f_{i+2} \geq 6$.*

*(ii) If $f_{i+1} = -1$, then $f_i \geq 4$ and $f_{i+2} \geq 4$.*

*(iii) If $f_{i+1} = 1$, then $f_i + f_{i+2} \geq 4$ and $f_i \geq 1$, $f_{i+2} \geq 1$. Further, if $f_i + f_{i+2} = 4$, then $\{f_i, f_{i+2}\} = \{1, 3\}$.*

*(iv) If $f_{i+1} = -3$, then $f_i = f_{i+2} = 6$.*

*Proof.* (i) Suppose that $f_{i+1} = 0$. Then we may assume that $f(1, i+1) = f(2, i+1) = -1$ and $f(3, i+1) = 2$. Since $f(N(3, i+1)) \geq 2$, we have $f(3, i) =$

$f(3,i+2)=2$ and hence $f_i \geq 0$, $f_{i+2} \geq 0$, $f_i \neq 1$, $f_{i+2} \neq 1$. Moreover, since $f(N(1,i+1)) \geq 2$ and $f(N(2,i+1)) \geq 2$, it follows that $f(1,i)+f(1,i+2) \geq 1$ and $f(2,i)+f(2,i+2) \geq 1$, which implies that $f_i + f_{i+2} \geq 6$.

(ii) As $f_{i+1} = -1$, we may assume that $f(1,i+1) = f(2,i+1) = -1$ and $f(3,i+1) = 1$. Since $f(N(3,i+1)) \geq 2$, we have $f(3,i) = f(3,i+2) = 2$. If $f(2,i) = -1$ or $f(2,i+2) = -1$, then $f(N(2,i+1)) \leq 1$, a contradiction. Hence, $f(2,i) \geq 1$ and $f(2,i+1) \geq 1$. By a similar argument, we get $f(1,i) \geq 1$ and $f(1,i+1) \geq 1$. Consequently, $f_i \geq 4$ and $f_{i+2} \geq 4$.

(iii) Since $f_{i+1} = 1$, we may assume that $f(1,i+1) = f(2,i+1) = 1$ and $f(3,i+1) = -1$. As $f(N(1,i+1)) \geq 2$, we must have $f(1,i) \geq 1$ and $f(1,i+2) \geq 1$, and similarly, $f(2,i) \geq 1$ and $f(2,i+2) \geq 1$. Hence, $f_i \geq 1$ and $f_{i+2} \geq 1$. Meanwhile, $f(3,i)+f(3,i+2) \geq 0$ must hold as $f(N(3,i+1)) \geq 2$. So $f_i + f_{i+2} \geq 4$. Suppose further that $f_i + f_{i+2} = 4$. Then $f(1,i) = f(2,i) = 1$, $f(1,i+2) = f(2,i+2) = 1$ and $\{f(3,i), f(3,i+2)\} = \{1,-1\}$. Consequently, $\{f_i, f_{i+2}\} = \{1,3\}$.

(iv) As $f_{i+1} = -3$, $f(j,i+1) = -1$ holds for any $j \in [3]$. Therefore, $f(j,i) = f(j,i+2) = 2$ for any $j \in [3]$, and hence $f_i = f_{i+2} = 6$.

□

Since $C_3 \square C_n$ is 4-regular, by Lemma 1, $\gamma^2_{stI}(C_3 \square C_n) \geq \frac{2}{4} \times 3n = \frac{3}{2}n$. To obtain a tight lower bound of $\gamma^2_{stI}(C_3 \square C_n)$, we apply the discharging method to prove this lower bound again.

**Lemma 3.** *If $n \geq 3$, then $\gamma^2_{stI}(C_3 \square C_n) \geq \frac{3}{2}n$.*

*Proof.* Let $f$ be an STI2DF of $C_3 \square C_n$. We define the initial charge of each column $C^i$ of $C_3 \square C_n$ by $ch(C^i) = f_i$. Let $ch'(C^i)$ be the the final charge after the following discharging process is finished.

**R:** Every $C^i$-layer with $ch(C^i) \geq 2$ gives $\frac{ch(C^i)-1.5}{2}$ charge to each adjacent $C^j$-layer with $ch(C^j) \leq 1$.

We now distinguish the cases based on the value of $ch(C^i)$ to show that $ch'(C^i) \geq \frac{3}{2}$ for any $i \in [n]$.

**Case 1.** $ch(C^i) \geq 2$.

Since $C^i$ sends at most $\frac{ch(C^i)-1.5}{2}$ charge to two adjacent columns, $ch'(C^i) \geq ch(C^i) - 2 \times \frac{ch(C^i)-1.5}{2} = \frac{3}{2}$.

**Case 2.** $ch(C^i) = 1$.

By Lemma 2(iii), $C^i$ must be adjacent to a column $C^j$ with $ch(C^j) \geq 3$. So $C^i$ receives $\frac{ch(C^j)-1.5}{2} \geq \frac{3}{4}$ charge from $C^j$. Thus, the final charge $ch'(C^i) \geq 1 + \frac{3}{4} = \frac{7}{4}$.

**Case 3.** $ch(C^i) = 0$.

By Lemma 2(i), $C^i$ is either adjacent to a column $C^j$ with $ch(C^j) \geq 5$ or adjacent to two columns $C^{i-1}$ and $C^{i+1}$ with $ch(C^{i-1}) \geq 2$, $ch(C^{i+1}) \geq 2$ and $ch(C^{i-1}) + ch(C^{i+1}) \geq 6$. Hence, $C^i$ receives $\frac{ch(C^j)-1.5}{2} \geq \frac{7}{4}$ charge from $C^j$

or $\frac{ch(C^{i-1})+ch(C^{i+1})-3}{2} \geq \frac{3}{2}$ charge from $C^{i-1}$ and $C^{i+1}$. Thus, the final charge $ch'(C^i) \geq \frac{3}{2}$.

**Case 4.** $ch(C^i) = -1$.
In this case, Lemma 2(ii) asserts that $ch(C^{i-1}) \geq 4$ and $ch(C^{i+1}) \geq 4$. Hence, $C^i$ receives a charge at least $\frac{5}{4}$ from each of $C^{i-1}$ and $C^{i+1}$. Thus, $ch'(C^i) \geq -1+\frac{5}{2} = \frac{3}{2}$.

**Case 5.** $ch(C^i) = -3$.
In this case, Lemma 2(iv) asserts that $ch'(C^i) = -3 + \frac{ch(C^{i-1})+ch(C^{i+1})-3}{2} = \frac{3}{2}$.

In summary, we have that $ch'(C^i) \geq \frac{3}{2}$ for any $i \in [n]$. Since the discharging procedure preserves the total value of initial charge, we conclude that $w(f) = \sum_{i=1}^{n} ch(C^i) = \sum_{i=1}^{n} ch'(C^i) \geq \frac{3}{2}n$. This completes the proof. □

**Lemma 4.** *If $n \equiv 1 \pmod 2$ and $n \geq 3$, then $\gamma_{stI}^2(C_3 \Box C_n) \geq \lceil \frac{3n}{2} \rceil + 1$.*

*Proof.* Suppose, to the contrary that, $f$ is an STI2DF of $C_3 \Box C_n$ with weight $w(f) = \frac{3n+1}{2}$. Let $ch(C^i) = f_i$ be the initial charge of the column $C^i$ and $ch'(C^i)$ the final charge of $C^i$ after the discharging process is finished according to **R**. By the analysis in the proof of Lemma 3, we obtain that $ch'(C^i) \leq 2$ for each $i \in [n]$, and if $ch'(C^i) = 2$ for some $i$, then $ch'(C^j) = \frac{3}{2}$ for any $j \neq i$. Due to the symmetry, we consider the initial charge on $C^2$.

**Case 1.** $ch(C^2) = -3$.

By Lemma 2(iv), $ch(C^1) = ch(C^3) = 6$. Since $C^3$ sends $\frac{9}{4}$ charge to $C^4$, $ch(C^4)+\frac{9}{4} \leq ch'(C^4) \leq 2$. So $ch(C^4) \leq -1$. Suppose $ch(C^4) = -1$, say $f(1,4) = f(2,4) = -1$ and $f(3,4) = 1$. Then $f(1,5) \geq 1$, $f(2,5) \geq 1$ and $f(3,5) = 2$, which indicates $ch(C^5) \geq 4$. But then $ch'(C^4) \geq -1 + \frac{6+4-3}{2} = \frac{5}{2}$, a contradiction. Hence, $ch(C^4) = -3$ and similarly, $ch(C^n) = -3$. Repeat above process, we can see that $ch(C^i) = -3$ for even $i$ and $ch(C^i) = 6$ for odd $i$. But then $ch(C^n) = 6$, a contradiction again.

**Case 2.** $ch(C^2) = -1$.
We may assume that $f(1,2) = f(2,2) = -1$ and $f(3,2) = 1$. By Lemma 2(ii) we know that $ch(C^1) \geq 4$ and $ch(C^3) \geq 4$. Since $ch'(C^2) = -1+\frac{ch(C^1)+ch(C^3)-3}{2} \leq 2$, we conclude that $ch(C^1)+ch(C^3) \leq 9$. So at least one of $ch(C^1)$ and $ch(C^3)$ equals to 4, say $ch(C^3) = 4$. Then $f(1,3) = f(2,3) = 1$ and $f(3,3) = 2$, which yields $f(1,4) \geq 1$ and $f(2,4) \geq 1$. Hence $ch(C^4) \geq 1$ and $C^4$ receives a charge $\frac{5}{4}$ from $C^3$. But then $ch'(C^4) \geq 1 + \frac{5}{4} = \frac{9}{4} > 2$, a contradiction.
From Cases 1 and 2, we deduce that $ch(C^i) \geq 0$ for any $i \in [n]$.

**Case 3.** $ch(C^2) = 1$.
We may assume that $f(1,2) = -1$, $f(2,2) = f(3,2) = 1$. By Lemma 2(iii), $ch(C^1) + ch(C^3) \geq 4$. If $C^2$ is adjacent to a column $C^i$ with $ch(C^i) \geq 4$, then $ch'(C^2) \geq 1 + \frac{4-1.5}{2} > 2$, a contradiction. Hence, we may assume that $ch(C^1) \leq 3$ and $ch(C^3) \leq 3$. If $ch(C^1) + ch(C^3) \geq 5$ and $ch(C^1) \geq 2$, $ch(C^3) \geq 2$, then $ch'(C^2) \geq 1 + \frac{5-3}{2} = 2$. Thus, $ch'(C^i) = \frac{3}{2}$ for each $i \neq 2$ and $\{ch(C^1), ch(C^3)\} = \{2, 3\}$. Suppose $ch(C^3) = 2$.

Since $f(2,3) \geq 1$ and $f(3,3) \geq 1$, we may assume $f(1,3) = -1$, $f(2,3) = 1$ and $f(3,3) = 2$. In this case, $f(i,4) \geq 1$ for $i \in [3]$. Then $ch(C^4) \geq 3$. Since $C^4$ sends at most $\frac{ch(C^4)-1.5}{2}$ charge to $C^5$, $ch'(C^4) \geq ch(C^4) - \frac{ch(C^4)-1.5}{2} > \frac{3}{2}$, a contradiction.
Therefore, $ch(C^1) + ch(C^3) = 4$. By Lemma 2(iii) again, $\{ch(C^1), ch(C^3)\} = \{1,3\}$. Assume that $ch(C^1) = 3$ and $ch(C^3) = 1$. Since $f(2,1)+f(2,3) \geq 2$, $f(3,1)+f(3,3) \geq 2$ and $f(1,1) + f(1,3) \geq 0$, we have $f(2,1) = f(2,3) = f(3,1) = f(3,3) = 1$ and $f(1,1) = 1$, $f(1,3) = -1$. Then $f(j,4) \geq 1$ for each $j \in [3]$ and hence $ch(C^4) \geq 3$. If $ch(C^4) \geq 4$, then $ch'(C^3) \geq 1 + \frac{5}{4} > 2$, a contradiction. So $ch(C^4) = 3$ and $f(j,4) = 1$ for each $j \in [3]$. In this case, $ch'(C^2) = ch'(C^3) = 1 + \frac{3}{4} = \frac{7}{4}$. Hence, $ch'(C^2) + ch'(C^3) = \frac{7}{2} = 2 \times \frac{3}{2} + \frac{1}{2}$. Then $ch'(C^i) = \frac{3}{2}$ for any $i \notin \{2,3\}$. This implies $ch(C^5) = 0$. By the values assigned on $C^3$ and $C^4$, we conclude that $f(1,5) = 2$ and $f(2,5) = f(3,5) = -1$. However, $f(N(1,5)) \leq 1$ which yields a contradiction.

**Case 4.** $ch(C^2) = 0$.
By Lemma 2(i), $ch(C^1) \geq 0$, $ch(C^3) \geq 0$ and $ch(C^1) + ch(C^3) \geq 6$. If $ch(C^3) = 0$, then $ch(C^1) = 6$. Hence, $ch'(C^2) = \frac{ch(C^1)-\frac{3}{2}}{2} = \frac{6-\frac{3}{2}}{2} = \frac{9}{4} > 2$, a contradiction. So by Lemma 2(i) again, we have $ch(C^1) \geq 2$ and $ch(C^3) \geq 2$. As $ch(C^2) = 0$, we may assume that $f(1,2) = f(2,2) = -1$ and $f(3,2) = 2$. Hence, $f(3,3) = f(1,3) = 2$.
Suppose $ch(C^1) = 2$, and assume that $f(1,1) = -1$ and $f(2,1) = 1$. Now $f(1,3) = 2$ and $f(2,3) \geq 1$, which implies $ch(C^3) \geq 5$. Since $ch(C^4) \geq 0$, then $ch'(C^2) \geq \frac{2+5-3}{2} = 2$ and $ch'(C^4) \geq \frac{5-\frac{3}{2}}{2} = \frac{7}{4} > \frac{3}{2}$, a contradiction. Therefore, $ch(C^1) \geq 3$ and $ch(C^3) \geq 3$. Meanwhile, since $ch'(C^2) = \frac{ch(C^1)+ch(C^3)-3}{2} \leq 2$, we have $ch(C^1) + ch(C^3) \leq 7$. Hence, at least one of $ch(C^1)$ and $ch(C^3)$ equals to 3, say $ch(C^3) = 3$. If $ch(C^1) = 4$, then $ch'(C^2) = 2$ and so $ch'(C^i) = \frac{3}{2}$ for each $i \neq 2$. This forces $ch(C^{2l+1}) = 3$ and $ch(C^{2l}) = 0$ for $1 \leq l \leq \frac{n-1}{2}$. Especially, $ch(C^n) = 3$. However, in this case, $ch'(C^1) = 4 - \frac{4-1.5}{2} > \frac{3}{2}$, a contradiction. Hence, $ch(C^1) = ch(C^3) = 3$. To ensure $ch'(C^3) \leq 2$, it must have $ch(C^4) \leq 1$. By Case 3, $ch(C^4) = 0$. Repeat the above argument, we have that the weight sequence on the columns of $C_3 \square C_n$ is $3030 \cdots 0$, but this is impossible since $n$ is odd.
By the above analysis and symmetry, we have $ch(C^i) \geq 2$ for each $i \in [n]$, then $w(f) \geq 2n > \frac{3n+1}{2}$, a contradiction. Therefore, $\gamma^2_{stI}(C_3 \square C_n) \geq \lceil \frac{3n}{2} \rceil + 1$ for odd $n$. □

We are now in a position to provide the formula for $\gamma^2_{stI}(C_3 \square C_n)$.

**Theorem 7.** *Let $n \geq 3$. Then* $\gamma^2_{stI}(C_3 \square C_n) = \begin{cases} \frac{3}{2}n, & n \equiv 0 \pmod 2; \\ \lceil \frac{3n}{2} \rceil + 1, & n \equiv 1 \pmod 2. \end{cases}$

*Proof.* It suffices to provide an STI2DF $f$ with the desired bounds. We distinguish two cases depending on the parity of $n$. For even $n$, assign value 2 to each vertex of column $C^i$ if $i$ is odd, and assign value $-1$ to each vertex of column $C^i$ if $i$ is even. For odd $n$, assign value 2 to each vertex of column $C^i$ if $i$ is odd and $i \leq n-2$, and assign value $-1$ to each vertex of column $C^i$ if $i$ is even and $i \leq n-3$. Assign value 2 to vertices $(2, n-1)$ and $(2, n)$, value $-1$ to $(1, n-1)$, $(3, n-1)$, $(1, n)$ and $(3, n)$.

It is straightforward to check that $f$ is an STI2DF with $w(f) = \frac{3}{2}n$ for even $n$ and $w(f) = \lceil \frac{3n}{2} \rceil + 1$ for odd $n$. $\square$

**Acknowledgements:** This work was supported by National Nature Science Foundation of China under Grant 11701542.

**Conflict of Interest:** The authors declare that they have no conflict of interest.

**Data Availability:** Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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