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Research Article

Max-min degree index of a graph and its mathematical relation with other topological indices

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Abstract: Among the defined 148 discrete Adriatic indices, the max-min degree index is one. Vukićević proposed some problems related to the upper and lower bounds on the max-min degree index. Here we determine the max-min degree index of some special graphs. We characterize the graphs extremal with respect to max-min degree index over connected graphs, trees and unicyclic graphs with a given number of vertices. Finally, we establish its mathematical relation with other topological indices.

Keywords: max-min degree index, tree, unicyclic graph, bound.

AMS Subject classification: 05C07, 05C09, 05C92

1. Introduction

Let $G = (V(G), E(G))$ be a simple connected graph with $|V(G)| = n$ and $|E(G)| = m$. A graph G is said to be r-regular iff $d_i = r$ for every $i \in V(G)$. A vertex of degree one is said to be a pendant vertex. A connected graph G is said to be a tree and a unicyclic graph if and only if $m = n-1$ and $m = n$, respectively. By P_n, C_n, K_n, and S_n, we denote path, cycle, complete graph, and star graph on n vertices, respectively. Also, by $K_{m,n}$ and $\mathcal{G}_{n,m,\delta,\Delta}$, we denote a complete bipartite graph with partite sets of cardinality m and n, and the set of all connected graph of n vertices and m edges with minimum and maximum degree δ and Δ, respectively. An edge $e = ab$ of a graph G is said to be a (p, q)-edge if $d_a = p$ and $d_b = q$, respectively. By Γ_Δ, we denote the family of graphs with edges that are $(1, \Delta)$-edge and (δ_1, Δ)-edge.

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A topological index of a graph G is a numerical quantity that is invariant under graph automorphism. Due to application in chemistry, pharmacology, especially in QSPR/QSAR research as molecular structure descriptors [3, 5, 7, 8, 16], topological indices have gained considerable popularity. Among the groups of all topological indices, one of the most investigated and widely used is the vertex degree-based topological indices [8, 12, 13]. The oldest vertex degree-based topological indices, the first and the second Zagreb indices [5, 7], are defined as

$$M_1(G) = \sum_{i\sim j}(d_i + d_j) \text{ and}$$
$$M_2(G) = \sum_{i\sim j} d_i d_j.$$

The harmonic index and inverse degree [2] is defined as

$$H(G) = \sum_{i\sim j} \frac{2}{d_i + d_j} \text{ and}$$

$$ID(G) = \sum_{i\sim j} \left(\frac{1}{d_i^2} + \frac{1}{d_j^2}\right).$$

Vukicević et al. first defined the max-min degree index of a graph in [16] as

$$Mm_{deg}(G) = \sum_{a\sim b} \frac{max\{d_a, d_b\}}{min\{d_a, d_b\}},$$

where $a \sim b$ means that the vertices a and b are adjacent and d_a represent the degree of the vertex a in G. Then in [15], Vukicević proposed some open problems related to upper and lower bounds on the max-min degree index. The max-min degree index can also be written as

$$Mm_{deg}(G) = \sum_{a\sim b} \frac{d_a}{d_b}, \text{ where } d_a \geq d_b.$$

In chemical graph theory, the most famous problem is characterizing the extremal graphs with respect to different topological indices. We refer [10–12] for recent advances. All these observations prompted me to consider the extremal problems with respect to the max-min degree index over trees and unicyclic graphs.
In this paper, in Section 2, we determine the max-min degree index of some special graphs like the bridge graph [1], grid graph [4] and the Sierpinski network [9]. Then, in Section 3, we study the extremal graphs with respect to the max-min degree index over connected graphs, trees and unicyclic graphs with a given number of vertices. Also, we establish its mathematical relation with other topological indices.

2. Max-min degree index of some special graphs

The proof of the following results is immediate, and hence we omit the proof.

Proposition 1. *(a) The max-min degree index of the path graph P_n with $n \geq 3$ is $Mm_{deg}(P_n) = n+1$.*
(b) For a connected r-regular graph G with n vertices we have $Mm_{deg}(G) = \frac{rn}{2}$.
(c) Let m, n be natural numbers with $m \geq n$. Then $Mm_{deg}(K_{m,n}) = m^2$.
(d) For the wheel graph W_n with $n \geq 4$, we have $Mm_{deg}(W_n) = \frac{n^2+n-2}{3}$.
(e) For the n-friendship graph F_n with $n \geq 2$, we have $Mm_{deg}(F_n) = 2n^2 + n$.

Table 1. Edge set partition of $B_t(K_n)$ as per the degrees of end vertices of an edge.

Edge set	(d_a, d_b)	number of edges
E_1	$(n, n-1)$	$t(n-1)$
E_2	$(n-1, n-1)$	$\frac{tn(n-1)}{2} - t(n-1)$
E_3	(n, n)	$t-1$

The bridge graph $B_t(K_n)$ over the complete graph K_n is the simple graph obtained by connecting t copies of the complete graph K_n by $t-1$ bridges (see Figure 1).

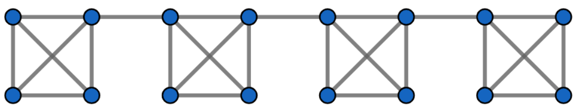

Figure 1. Bridge graph $B_4(K_4)$ on 4 copies of K_4.

Theorem 1. *Let $B_t(K_n)$ be the bridge graph over the complete graph K_n. Then*

$$Mm_{deg}(B_t(K_n)) = \frac{tn^2 - tn + 4t - 2}{2}.$$

Proof. The edge set partition of $B_t(K_n)$ as per the degrees of the end vertices of an edge is shown in Table 1. Therefore,

$$\begin{aligned} Mm_{deg}(B_t(K_n)) &= t(n-1)\frac{n}{n-1} + \frac{tn(n-1)}{2} - t(n-1) + t - 1 \\ &= \frac{tn^2 - tn + 4t - 2}{2}. \end{aligned}$$

□

Next, we consider the grid graph $P_n \square P_m$ (see Figure 2), and obtain its max-min degree index.

Theorem 2. *For the grid graph $P_n \square P_m$, we have $Mm_{deg}(P_n \square P_m) = \frac{6mn-m-n+4}{3}$.*

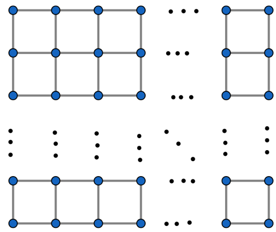

Figure 2. **The grid graph $P_n \square P_m$.**

Proof. The edge set partition of $P_n \square P_m$ as per the degrees of the end vertices of an edge is shown in Table 2. Therefore,

$$\begin{aligned} Mm_{deg}(P_n \square P_m) &= \frac{3}{2}.8 + \frac{4}{3}(2m+2n-8) + \frac{3}{3}(2m+2n-12) \\ &+ \frac{4}{4}(2mn-5m-5n+12) \\ &= \frac{6mn-m-n+4}{3}. \end{aligned}$$

□

Table 2. **Edge set partition of $P_n \square P_m$ as per the degrees of end vertices of an edge.**

Edge set	(d_a, d_b)	number of edges
E_1	$(3,2)$	8
E_2	$(4,3)$	$2m+2n-8$
E_3	$(3,3)$	$2m+2n-12$
E_4	$(4,4)$	$2mn-5m-5n+12$

The Sierpinski networks $S(n,k)$ [9] are defined on the vertex set $V(S(n,k)) = \{1,2,\ldots,k\}^n$. We have $|V(S(n,k))| = k^n$ for any $n \geq 1$ and $k \geq 1$. Any vertex of $S(n,k)$ can be written as $x = (x_1, x_2, \ldots, x_n)$, where $x_r \in \{1,2,\ldots,k\}$ and $r \in \{1,2,\ldots,n\}$. Two distinct vertices $x = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$, where $x_r, y_r \in \{1,2,\ldots,k\}$ and $r \in \{1,2,\ldots,n\}$ of $S(n,k)$ are adjacent if and only if there exists $t \in \{1,2,\ldots,n\}$ such that:

$$\begin{aligned} &x_h = y_h \text{ for } h = 1,\ldots,t-1, \\ &x_t = y_t, \text{ and} \\ &x_h = y_t \text{ and } y_h = x_t \text{ for } h = t+1,\ldots,n. \end{aligned}$$

Theorem 3. *For the Sierpinski network $S(n,k)$, we have*

$$Mm_{deg}(S(n,k)) = \frac{k^{n+2} - k^{n+1} - k^2 + 5k}{2(k-1)}.$$

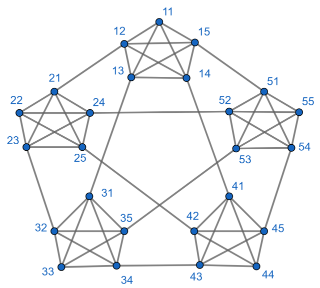

Figure 3. **The Sierpinski graph** $S(3,4)$.

Proof. The edge set partition of $S(n,k)$ as per the degrees of the end vertices of an edge is shown in Figure 3. Therefore,

$$\begin{aligned} Mm_{deg}(S(n,k)) &= 2k.\frac{k}{k-1} + \frac{k^{n+1} - 5k}{2} = \frac{2k^2}{k-1} + \frac{k^{n+1} - 5k}{2} \\ &= \frac{k^{n+2} - k^{n+1} - k^2 + 5k}{2(k-1)}. \end{aligned}$$

□

Table 3. **Edge set partition of $S(n,k)$ as per the degrees of end vertices of an edge.**

Edge set	(d_a, d_b)	number of edges
E_1	$(k, k-1)$	$2k$
E_2	(k, k)	$\frac{k^{n+1}-5k}{2}$

3. Sharp bounds on Max-min degree index

Theorem 4. *Let G be a graph with n vertices, p pendent vertices, m edges, maximum degree Δ and minimal non-pendent vertex degree δ_1. Then*

$$p\delta_1 + \frac{(m-p)\delta_1}{\Delta} \leq Mm_{deg}(G) \leq p\Delta + \frac{(m-p)\Delta}{\delta_1},$$

with right equality if and only if G is regular or biregular or $G \in \Gamma_\Delta$, and left equality if and only if G is regular or biregular.

Proof. By the definition of max-min degree index, we have

$$Mm_{deg}(G) = \sum_{a\sim b} \frac{max\{d_a, d_b\}}{min\{d_a, d_b\}}.$$

If G has p pendant edges, then we have

$$\begin{aligned} Mm_{deg}(G) &= \sum_{a\sim b, d_b=1} d_a + \sum_{a\sim b, d_b\geq 2} \frac{d_a}{d_b} \\ &\leq p\Delta + \sum_{a\sim b, d_b\geq 2, d_a\leq\Delta} \frac{d_a}{d_b} \\ &\leq p\Delta + \frac{(m-p)\Delta}{\delta_1} \quad \text{where} \quad \delta_1 \leq d_b \leq d_a \leq \Delta. \end{aligned}$$

For equalities, one must have $d_a = \Delta$ and $d_b = 1$ for each pendent edge $ab \in E(G)$, and $d_a = \Delta$, $d_b = \delta_1$ for each non-pendent edge $ab \in E(G)$. If $m = p$, then $G \cong S_n$. Let $m > p$. Now, if $p = 0$, then $d_a = \Delta$ and $d_b = \delta_1$ for each edge $ab \in E(G)$. Thus, G is a biregular graph. If $p > 0$, then the edges in G are $(1, \Delta)$ and (Δ, δ_1). Hence $G \in \Gamma_\Delta$. Conversely, if G is regular or biregular or $G \in \Gamma_\Delta$, then equality holds.
Similarly, we have

$$\begin{aligned} Mm_{deg}(G) &= \sum_{a\sim b, d_b=1} d_a + \sum_{a\sim b, d_b\geq 2} \frac{d_a}{d_b} \\ &\geq p\delta_1 + \sum_{a\sim b, d_b\geq 2, d_a\leq\Delta} \frac{d_a}{d_b} \\ &\geq p\Delta + \frac{(m-p)\delta_1}{\Delta} \quad \text{where} \quad \delta_1 \leq d_b \leq d_a \leq \Delta. \end{aligned}$$

Clearly, equality holds if and only if G is regular or biregular. □

Lemma 1. *Let $g(x,y) = \frac{y}{x}$, and real number a, b satisfied that $0 < a \leq x \leq y \leq b$. Then $1 \leq g(x,y) \leq \frac{b}{a}$, with left equality if and only if $x = y$ and right equality if and only if $x = a$ and $y = b$.*

Proof. For the binary function $g(x,y) = \frac{y}{x}$, where $0 \le a \le x \le y \le b$, let $f(t) = t$ for $t = \frac{y}{x}$. Since $f'(t) = 1 > 0$, $g(t)$ is a monotonically incresing function. Hence $1 = f(1) \le g(x,y) = f(t) \le f(\frac{b}{a}) = \frac{b}{a}$, with left equality if and only if $x = y$ and right equality if and only if $x = a$ and $y = b$. □

Theorem 5. *Let $G \in \mathcal{G}_{n,m,\delta,\Delta}$. Then*

$$m \le Mm_{deg}(G) \le \frac{m\Delta}{\delta},$$

with left equality if and only if G is regular with $m \neq 2$, and right equality if and only if G is a regular or biregular.

Proof. Since for any edge $e = ab$ of G with $d_a \ge d_b$, one have $\frac{d_a}{d_b} \ge 1$ and hence $Mm_{deg}(G) \ge m$. Also from Lemma 1, one has

$$Mm_{deg}(G) = \sum_{a \sim b} \frac{d_a}{d_b} \ge m,$$

with equality if and only if G is regular with $m \neq 2$. Suppose $1 \le \delta \le d_b \le d_a \le \Delta$. Then by Lemma 1, one has

$$Mm_{deg}(G) = \sum_{a \sim b} \frac{d_a}{d_b} \le \sum_{a \sim b} \frac{\Delta}{\delta} = \frac{m\Delta}{\delta},$$

with equality if and only if $d_a = \Delta$ and $d_b = \delta$ for all $a \sim b$, i.e., G is regular or biregular. □

The proofs of the following results are trivial and hence we omit the proof.

Corollary 1. *Let $G \in \mathcal{G}_{n,m}$. Then*
(i) $Mm_{deg}(G) \ge 1$, with equality if and only if $G \cong K_2$.
(ii) There does not exists a graph such that $Mm_{deg}(G) = 2$.
(iii) For any natural number $n \ge 3$, there exists a graph on n vertices such that $Mm_{deg}(G) = n$.

Theorem 6. *For a simple connected graph G with $n \ge 3$ vertices, we have*

$$n \le Mm_{deg}(G) \le (n-1)^2,$$

with left equality if and only if $G \cong C_n$ and right equality if and only if $G \cong S_n$.

Proof. To establish the lower bound, we consider the following three cases.

Case 1. G is a unicyclic graph.

Since the contribution by each edge is at least 1 and the number of edges of a unicyclic graph with n vertices is n, we have $Mm_{deg} \geq n$ (*cf*. Theorem 5). The equality holds if and only if G is a connected 2-regular unicyclic graph if and only if $G \cong C_n$.

Case 2. G is a tree.

Then G has at least two pendant edges and the contribution by each pendant edge to Mm_{deg} is at least 2. Also, the contribution by each other edge is at least 1. Therefore $Mm_{deg}(G) \geq 2.2 + (n-3) = n+1 > n$.

CASE (3): G has more than one cycle. Then $|E(G)| \geq n+1$ and hence $Mm_{deg}(G) \geq n+1 > n$.

To find the right bound it is enough to note that the maximum contribution to the max-min degree index of a graph G by an edge is $n-1$ (when $\Delta(G) = n-1$ and $\delta(G) = 1$). Then $Mm_{deg}(G) = (n-1)^2$. This will happen only if $G \cong S_n$. □

Corollary 2. *For a tree T with $n \geq 3$ vertices, we have*

$$n+1 \leq Mm_{deg}(G) \leq (n-1)^2,$$

with left equality if and only if $T \cong P_n$ and right equality if and only if $T \cong S_n$.

Lemma 2. *For real numbers x, y and z with $x \geq y \geq z \geq 0$, we have*

$$\frac{x}{y} + \frac{y}{z} \leq 1 + \frac{x}{z}.$$

Proof. Since $x - y \geq 0$ and $y - z \geq 0$, we have $(x-y)(y-z) \geq 0$ and so $xy + yz \geq y^2 + zx$. Thus $1 + \frac{x}{z} \geq \frac{y}{z} + \frac{x}{y}$. Hence, the result follows. □

Theorem 7. *For a connected unicyclic graph on $n \geq 3$ vertices, we have*

$$n \leq Mm_{deg} \leq n^2 - 3n + 3,$$

with left equality if and only if $U \cong C_n$ and right equality if and only if $U \cong \mathcal{U}_n$ (see Figure 4), where $\mathcal{U}_n$ is a unicylic graph on n vertices which contains a universal vertex.

Proof. The proof for the lower bound and the equality conditions follows from Theorem 6.

To prove the upper bound, we consider the following cases.

(1) $\Delta(G) = n-1$.

Then one can easily observe that $\mathcal{U}_n$ is the only such graph and

$$\begin{aligned} Mm_{deg}(\mathcal{U}_n) &= (n-3)(n-1) + 2.\frac{n-1}{2} + \frac{2}{2} \\ &= n^2 - 3n + 3. \end{aligned}$$

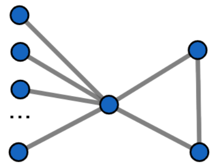

Figure 4. **Unicyclic graph $\mathcal{U}_n$ on n vertices with a universal vertex.**

(2) $\Delta(U) \leq n-2$ and the length of the cycle is at least four.
Then the contribution of each edge in the cycle is at most $\frac{n-2}{2}$ and the contribution of each other edge is at most $n-2$. Hence it is enough to prove that $(n-4)(n-2)+4.\frac{n-2}{2} < n^2-3n+3$, which is true if $n^2-3n+3-(n^2-4n+4) > 0$, i.e., if $n-1>0$. Since $n \geq 4$ by hypothesis, we have $Mm_{deg}(U) \leq Mm_{deg}(\mathcal{U}_n)$.
(3) $\Delta(U) \leq n-2$ and the length of the cycle is three.
Let the degrees of the vertices in the cycle be x, y and z such that $n-2 \geq x \geq y \geq z \geq 2$. Therefore, it is enough to prove the following inequality:

$$(n-3)(n-2) + \frac{x}{y} + \frac{y}{z} + \frac{x}{z} < n^2 - 3n + 3 = (n-3)(n-1) + n. \tag{3.1}$$

Hence, it is enough to prove that the inequality 3.1 holds if $\frac{x}{y}+\frac{y}{z}+\frac{x}{z} < n$. Since $\frac{x}{y}+\frac{y}{z}+\frac{x}{z}$ is increasing in x and decreasing in z, it is enough to prove that $\frac{n-2}{b}+\frac{b}{2}+\frac{n-2}{2} \leq \frac{2(n-1)}{2}+1$, i.e., $\frac{n-2}{b}+\frac{b}{2} \leq \frac{(n-2)}{2}+1$, since $\frac{n-2}{2} \leq \frac{n-1}{2}$. The last inequality directly follows from Lemma 2. □

Theorem 8. *Let $G \in \mathcal{G}_{n,m.\delta,\Delta}$. Then*

$$\frac{M_1}{\Delta} - m \leq Mm_{deg}(G) \leq \frac{M_1}{\delta} - m,$$

with left equality if and only if G is regular and right equality if and only if G is regular or bi regular, and

$$\frac{M_2}{\Delta^2} \leq Mm_{deg}(G) \leq \frac{M_2}{\delta^2},$$

with left equality if and only if G is regular, and right equality if and only if G is regular or biregular.

Proof. By the definition of the max-min degree index, we have

$$Mm_{deg}(G) = \sum_{a\sim b} \frac{d_a}{d_b} = \sum_{a\sim b} \frac{d_a+d_b}{d_b} - m \leq \frac{M_1}{\delta} - m,$$

with equality if and only if $d_a = \Delta$ and $d_b = \delta$ i.e., G is regular or bi regular.
Similarly, we can show that $Mm_{deg}(G) \geq \frac{M_1}{\Delta} - m$ with equality if and only if G is

regular.
Again, we have

$$Mm_{deg}(G) = \sum_{a\sim b} \frac{d_a}{d_b} = \sum_{a\sim b} \frac{d_a d_b}{d_b^2} \geq \frac{M_2}{\Delta^2},$$

with equality if and only if $d_a = \Delta = d_b = \delta$ i.e., G is regular.
Similarly, one can show that $Mm_{deg}(G) \leq \frac{M_2}{\delta^2}$ with equality if and only if G is regular or biregular. □

Theorem 9. *Let $G \in \mathcal{G}_{n,m,\delta,\Delta}$. Then*

$$Mm_{deg}(G) \geq \frac{M_1^2}{M_2} - 2m - \frac{m\Delta}{\delta},$$

with equality if and only if G is regular or biregular.

Proof.

$$\begin{aligned} Mm_{deg}(G) &= \sum_{a\sim b} \frac{d_a}{d_b} \\ &= \sum_{a\sim b} \frac{d_a^2 + d_b^2 - d_b^2}{d_a d_b} \\ &= \sum_{a\sim b} \frac{(d_a + d_b)^2 - 2d_a d_b - d_b^2}{d_a d_b} \\ &= \sum_{a\sim b} \frac{(d_a + d_b)^2}{d_a d_b} - 2m - \sum_{a\sim b} \frac{d_b}{d_a} \\ &\geq \frac{(\sum_{a\sim b} d_a + d_b)^2}{\sum_{a\sim b} d_a d_b} - 2m - \sum_{a\sim b} \frac{d_b}{d_a} \\ &\geq \frac{M_1^2}{M_2} - 2m - \frac{m\Delta}{\delta}, \end{aligned}$$

with equality if and only if G is regular or biregular. □

The forgotten and harmonic topological indices [2] are defined as

$$F(G) = \sum_{a\sim b} d_a^2 + d_b^2 \quad \text{and} \quad H(G) = \sum_{a\sim b} \frac{2}{d_a + d_b}. \tag{3.2}$$

Theorem 10. *Let $G \in \mathcal{G}_{n,m,\delta,\Delta}$. Then*

$$Mm_{deg}(G) \geq 2\delta H(G) - m$$

and

$$Mm_{deg}(G) \leq \frac{F(G)}{\delta^2} - m.$$

In both cases, the equality holds if and only if G is regular.

Proof. For every edge ab of (G), we have $(d_a + d_b)^2 \geq 4d_a d_b$ which implies $\frac{2}{d_a+d_b} \leq \frac{1}{2}(\frac{1}{d_a} + \frac{1}{d_b})$ with equality if and only if $d_a = d_b$. From Equation 3.2, we have

$$\begin{aligned} H(G) &= \sum_{a\sim b} \frac{2}{d_a + d_b} \\ &\leq \frac{1}{2} \sum_{a\sim b} \Big(\frac{1}{d_a} + \frac{1}{d_b}\Big) \\ &= \frac{1}{2} \sum_{a\sim b} \frac{1}{d_a} \Big(1 + \frac{d_a}{d_b}\Big) \\ &\leq \frac{1}{2\delta} \Big(\sum_{a\sim b} \frac{d_a}{d_b} + m\Big) \\ &= \frac{1}{2\delta} (Mm_{deg}(G) + m). \end{aligned}$$

Hence, $Mm_{deg}(G) \geq 2\delta H(G) - m$ with equality if and only if G is regular.
From the definition of max-min degree index, we have

$$\begin{aligned} Mm_{deg}(G) &= \sum_{a\sim b} \frac{d_a}{d_b} \\ &\leq \sum_{a\sim b} \Big(\frac{d_a^2}{d_b^2} + 1 - 1\Big) \\ &= \sum_{a\sim b} \Big(\frac{d_a^2 + d_b^2}{d_b^2} - 1\Big) \\ &= \sum_{a\sim b} \Big(\frac{d_a^2 + d_b^2}{d_b^2} - 1\Big) \\ &\leq \frac{F(G)}{\delta^2} - m. \end{aligned}$$

Clearly, equality holds if and only if G is regular. □

The next result relates to the max-min degree and forgotten topological indices.

Theorem 11. *Let $G \in \mathcal{G}_{n,m,\delta,\Delta}$. Then*

$$\frac{F(G)}{\Delta^2} - \frac{m\Delta}{\delta} \leq Mm_{deg}(G) \leq \frac{F(G)}{\delta^2} - \frac{m\delta}{\Delta},$$

with equality on both sides if and only if G is regular.

Proof. By the definition of the max-min degree index of a graph, we have

$$\begin{aligned}
Mm_{deg}(G) &= \sum_{a\sim b} \frac{d_a}{d_b} \quad \text{where} \quad d_a \geq d_b \\
&= \sum_{a\sim b} \frac{d_a^2 + d_b^2 - d_b^2}{d_a d_b} \\
&= \sum_{a\sim b} \frac{d_a^2 + d_b^2}{d_a d_b} - \sum_{a\sim b} \frac{d_b}{d_a} \\
&\leq \frac{F(G)}{\delta^2} - \sum_{a\sim b} \frac{d_b}{d_a} \\
&\leq \frac{F(G)}{\delta^2} - \frac{m\delta}{\Delta},
\end{aligned}$$

with equality if and only if G is regular.
Similarly, one can prove that $Mm_{deg}(G) \geq \frac{F(G)}{\Delta^2} - \frac{m\Delta}{\delta}$ with equality if and only if G is regular. □

The inverse degree index [14] of a graph G is defined as

$$ID(G) = \sum_{a\sim b} \left(\frac{1}{d_a^2} + \frac{1}{d_b^2} \right).$$

The next result relates inverse degree and max-min degree indices.

Theorem 12. *Let $G \in \mathcal{G}_{n,m,\delta,\Delta}$. Then*

$$\delta^2 ID(G) - \frac{\Delta m}{\delta} \leq Mm_{deg}(G) \leq \Delta^2 ID(G) - \frac{\delta m}{\Delta},$$

with equality if and only if G is regular.

Proof. From the definition of max-min degree index, we have

$$\begin{aligned}
Mm_{deg}(G) &= \sum_{a\sim b} \frac{d_a}{d_b} \\
&= \sum_{a\sim b} \left(\frac{d_a}{d_b} + \frac{d_b}{d_a} - \frac{d_b}{d_a} \right) \\
&= \sum_{a\sim b} \frac{d_a^2 + d_b^2}{d_a d_b} - \sum_{a\sim b} \frac{d_b}{d_a} \\
&= \sum_{a\sim b} \left(\frac{1}{d_a^2} + \frac{1}{d_b^2} \right) d_a d_b - \sum_{a\sim b} \frac{d_b}{d_a} \\
&\leq \Delta^2 ID(G) - \sum_{a\sim b} \frac{d_b}{d_a} \\
&\leq \Delta^2 ID(G) - \frac{m\delta}{\Delta},
\end{aligned}$$

with equality if and only if G is regular.
Similarly, one can prove that $Mm_{deg}(G) \geq \delta^2 ID(G) - \frac{\Delta m}{\delta}$ with equality if and only if G is regular. □

Recently, Gutman et al. [6] introduced the σ-index of a graph as a natural extension of the Albertson irregularity index as

$$\sigma(G) = \sum_{a\sim b}(d_a - d_b)^2.$$

The next result relates the max-min degree and the σ indices.

Theorem 13. *Let $G \in \mathcal{G}_{n,m,\delta,\Delta}$. Then*

$$\frac{\sigma(G)}{\Delta^2} + 2m - \frac{m\Delta}{\delta} \leq Mm_{deg}(G) \leq \frac{\sigma(G)}{\delta^2} + 2m - \frac{m\delta}{\Delta},$$

with both equality if and only if G is regular.
Also $Mm_{deg}(G) \geq m - \frac{\sigma(G)}{\delta^2}$, with equality iff G is regular.

Proof. By definition of the division degree index, one has

$$\begin{aligned} Mm_{deg}(G) &= \sum_{a\sim b}\frac{d_a}{d_b} \\ &= \sum_{a\sim b}\frac{d_a^2 + d_b^2 - d_b^2}{d_a d_b} \\ &= \sum_{a\sim b}\frac{(d_a - d_b)^2 + 2d_a d_b - d_b^2}{d_a d_b} \end{aligned}$$

$$\begin{aligned} &= \sum_{a\sim b}\frac{(d_a - d_b)^2}{d_a d_b} + 2m - \sum_{a\sim b}\frac{d_b}{d_a} \\ &\leq \frac{\sigma(G)}{\delta^2} + 2m - \frac{m\delta}{\Delta}, \end{aligned}$$

with equality if and only if G is regular.
In a similar way one can prove $Mm_{deg}(G) \geq \frac{\sigma(G)}{\Delta^2} + 2m - \frac{m\Delta}{\delta}$ with equality if and only if G is regular.
Again,

$$\begin{aligned} Mm_{deg}(G) &= \sum_{a\sim b}\frac{d_a}{d_b} \\ &\leq \sum_{a\sim b}\left(\frac{d_a^2}{d_b^2} + 1 - 1\right) \\ &= \sum_{a\sim b}\frac{(d_a - d_b)^2}{d_b^2} + 2\sum_{a\sim b}\frac{d_a}{d_b} - m \\ &\leq \frac{\sigma(G)}{\delta^2} + 2Mm_{deg}(G) - m. \end{aligned}$$

Therefore, $Mm_{deg}(G) \geq m - \frac{\sigma(G)}{\delta^2}$, with equality if and only if G is regular. □

Conclusion

In this paper, we have analyzed the max-min degree index of a graph G defined as $Mm_{deg}(G) = \sum_{a\sim b} \frac{max\{d_a,d_b\}}{min\{d_a,d_b\}}$. We have determined the max-min degree index of some special graphs, the grid graph, the bridge graph over the complete graph and Sierpinski networks. Also, we studied extremal values and graphs with respect to max-min degree index over connected graphs, trees and unicyclic graphs with the given number of vertices. Finally, we have established some mathematical relations of max-min degree index with other topological indices.

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