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Research Article

On independent k-domination number of Hamming graphs

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Abstract: A subset S of the vertices of a graph is called a k-dominating set if every vertex outside S has at least k neighbors in S. If a k-dominating set is an independent subset of the vertices, then the set is called an independent k-dominating set. The size of the smallest such set is called the independent k-domination number of the graph. In this paper, we derive a lower bound on the independent k-domination number of Hamming graphs. For some sets of parameters, we show that this lower bound is exact.

Keywords: independent k-domination number, hamming graph, dominating sets, graph domination, covering problem.

AMS Subject classification: 05C69

1. Introduction

Let $G = (V, E)$ be a graph with the vertex set V and the edge set E. We say the vertex u *dominates* a vertex v if uv is an edge. For every subset $S \subseteq V$, we denote the subgraph of G induced on the set S by $G[S]$.

A subset $S \subseteq V$ is called a *k-dominating set* for G if every vertex $v \in V \setminus S$ is dominated by at least k distinct vertices of S. If, in addition, $G[S]$ has no edges, then S is called an *independent k-dominating set* for G. In [4], this concept is called a *k-dominating independent set*, and the existence and the number of such sets is studied.

The minimum size of an independent k-dominating set of G is called the *independent k-domination number* of G and is denoted by $\gamma_i(G, k)$.

Notice that for some graphs G and integer number k, there exist no independent k-dominating set. For instance, the complete graph K_n where $n \geq 2$ has no independent k-dominating set when $k \geq 2$. Thus, $\gamma_i(G, k)$ is not a well-defined parameter for all graphs G, and integer number $k \geq 2$. A classification of the pairs (G, k) in which G

is a graph and k is an integer number and $\gamma_i(G,k)$ is well-defined has been proposed in Claim 3.5 of [4].

For $k=2$, the notion of independent k-dominating set is equivalent to the notion of 2-dominating kernel that was first introduced by Włoch in [5].

For a broader introduction to graph theory and domination in graphs, the interested reader is referred to [3].

In this paper, we study the independent k-domination number of the *Hamming graphs.* Before defining Hamming graphs, we first introduce the notion of *Hamming distance* between two vectors. Throughout this paper, we denote the set $\{1,2,\ldots,n\}$ by $[n]$. Additionally, when referring to an alphabet set of size n, we mean the set $[n]$.

Let $\mathbf{a}=(a_1,\ldots,a_m)$ and $\mathbf{b}=(b_1,\ldots,b_m)$ be two vectors with entries from an arbitrary set. The Hamming distance between $\mathbf{a}$ and $\mathbf{b}$ is denoted by $d(\mathbf{a},\mathbf{b})$ and is defined as $d(\mathbf{a},\mathbf{b})=|\{j : a_j \neq b_j\}|$.

The Hamming graph of order n and dimension m, denoted by K_n^m, has the vertex set

$$V(K_n^m) := \big\{(a_1,a_2,\ldots,a_m) : a_i \in [n]\big\},$$

and for $\mathbf{u},\mathbf{v}\in V(K_n^m)$, the pair $\mathbf{uv}$ is an edge if and only if $d(\mathbf{u},\mathbf{v})=1$. When $n=2$, the Hamming graph K_2^m is called *Hypercube graph.* It is worth mentioning that the problem of independent 1-domination number of hypercubes was first introduced in [2].

Hypercube graphs, and more generally Hamming graphs K_n^m have a central role in the context of coding theory and theoretical computer science. In fact, a *code* of length m over the alphabet $[n]$ is just a subset of the vertices of K_n^m.

Some problems in coding theory can be directly translated in terms of questions about Hamming graphs. For instance, the authors of [1] considered the problem of the size of a covering of Hamming spaces with balls of radius 1 and the minimum distance 2. For a formal description of the problem, we need to define the notion of Hamming balls of certain radius and covering of the Hamming space.

A *Hamming ball* of radius 1 with the center u over the alphabet $[n]$ consists of a vector $\mathbf{u}$ of length m with entries from $[n]$, together with all the vectors $\mathbf{u}'$ of the same length, and over the same alphabet, such that the Hamming distance between $\mathbf{u}$ and $\mathbf{u}'$ is equal to 1. A *covering* of the Hamming space with Hamming balls of radius 1 consists of a collection of radius 1 Hamming balls such that each element of Hamming space belongs to at least one of the balls in the family. A *Hamming space* of dimension m over the alphabet $[n]$ is the set of all the vectors of length m over the alphabet $[n]$.

If, in addition, non of these balls contain the center of another ball of the family, we call the family a *covering radius* 1 and minimum distance 2. Here, the *minimum distance* 2 reflects the fact that the Hamming distance between the centers of the balls is at least 2.

If we regard each vector of length m over $[n]$ as a vertex of K_n^m. then the centers of the balls in any covering of radius 1 and minimum distance 2 form an independent 1-dominating set in K_n^m. Thus, we are essentially generalizing the result of [1] which provides a lower bound for the size of codes with covering radius 1 and minimum distance 2. Our proof, when restricted to the case $k = 1$, is an alternative proof of the result of [1], which readily extends to larger values of k.

1.1. Our Contribution

The following theorem is the formal statement of this result.

Theorem 1. *For all integers n, k, m where $k \leq m$, we have*

$$\gamma_i(K_n^m, k) \geq \min\{n^{m-1}, \frac{kn^{m-1}}{m-1}\}.$$

Notice that the condition $k \leq m$ in the theorem is simply because there is no way to dominate a vertex of K_n^m with more than m vertices that form an independent set. Thus, the parameter $\gamma_i(K_n^m, k)$ only makes sense if $k \leq m$. Furthermore, according to our construction, it can be observed that for the set of parameters in Theorem 1, there exists a independent k-dominating set. Hence the parameter $\gamma_i(K_n^m, k)$ is well-defined provided that $k \leq m$.
We also find the exact value of $\gamma_i(K_n^m, k)$ for $k \in \{m-1, m\}$. Formally, we prove the following theorem.

Theorem 2. *For all integers n, m, k such that $k \in \{\, m-1, m\}$, we have*

$$\gamma_i(K_n^m, k) = n^{m-1}.$$

We postpone the proofs until the next section.

2. Technical Proofs

In this section, we present some technical lemmas in order to prove the main results of the paper, namely Theorem 1 and Theorem 2.

2.1. Lower Bound

To simplify the equations, we let $l := m-1$, and $V := V(K_n^m)$. For every subset $S \subseteq V$, we define the indicator function $f_S : V \to \{0,1\}$ as follows:

$$f_S(\mathbf{v}) = \begin{cases} 1 & \text{if } \mathbf{v} \in S \\ 0 & \text{otherwise.} \end{cases}$$

We also define the function $g_S : [n]^l \to \mathbb{Z}$ as follows:

$$g_S(v_1, v_2, \ldots, v_l) := \sum_{r=1}^{n} f_S(v_1, v_2, \ldots, v_l, r). \tag{2.1}$$

In other words, g_S takes $l = m-1$ coordinates as its argument and then counts the number of elements of S such that their initial l coordinates coincide with that argument.
The next lemma is the key observation which enables us to derive a lower bound for $|S|$. Notice that in the above, we took S as an arbitrary subset of V. In the next lemma, we only consider the subsets S that are independent k-dominating.

Lemma 1. *If S is an independent k-dominating set of K_n^m, then for every vector $\mathbf{a} \in [n]^l$, we have $g_S(\mathbf{a}) \in \{0,1\}$. Furthermore, if $g_S(\mathbf{a}) = 0$ then the following holds:*

$$\sum_{\mathbf{b}\in[n]^l, d(\mathbf{a},\mathbf{b})=1} g_S(\mathbf{b}) \geq nk.$$

In the above lemma, the summation is taken over all possible choices of $\mathbf{b} \in [n]^l$ with $d(\mathbf{a},\mathbf{b}) = 1$ and, g_S is defined as in (2.1).

Proof. Let $\mathbf{a}$ be an arbitrary member of $[n]^l$. Suppose $\mathbf{a} = (a_1,\ldots,a_l)$. For the first part of the claim, note that $g_S(\mathbf{a})$ counts the number of elements $r \in [n]$ such that the m-length vector $(a_1,a_2,\ldots,a_l,r)$ is a vertex in S. If for at least two values of r, namely r_1 and r_2, the vectors $\mathbf{v} = (a_1,a_2,\ldots,a_l,r_1)$ and $\mathbf{u} = (a_1,a_2,\ldots,a_l,r_2)$ belong to S, then $\mathbf{u},\mathbf{v}$ differ from each other exclusively in their last coordinate. Thus, $\mathbf{uv}$ must be an edge in the graph K_n^m. This contradicts the fact that S is an independent subset of $V(K_n^m)$. Therefore, the first claim is concluded.
For the second part of the lemma, suppose that $g_S(\mathbf{a}) = 0$. This means that there is no element of S with the first $m-1$ coordinates being equal to $\mathbf{a}$. Let t be a fixed but arbitrary element of $[n]$. Define $\mathbf{v}_t := (a_1,a_2,\ldots,a_l,t)$. Notice that $\mathbf{v}_t$ is an element of $V \setminus S$. Since S is a k-dominating set, there must exist at least k vertices $\mathbf{u}_1,\mathbf{u}_2,\ldots,\mathbf{u}_k \in S$ each of which dominates $\mathbf{v}_t$. This means that $d(\mathbf{u}_j,\mathbf{v}_t) = 1$ for all $j \in [k]$. Thus, each $\mathbf{u}_j$ differs from $\mathbf{v}_t$ in exactly one coordinate. This coordinate must be one of the first $m-1$ positions since we argued earlier that there is no element of S with the first $m-1$ coordinates being equal to $\mathbf{a}$.
Notice that there are n possibilities for the choice of t. For each choice, there exist at least k distinct elements of S that contribute in the summation. Therefore, Therefor the claim will be proved once we verify that no vertex of S can simultaneously dominate $\mathbf{v}_t, \mathbf{v}_{t'}$ for two different values of t and t'.
For a contradiction, suppose $\mathbf{u}$ dominates both $\mathbf{v}_t, \mathbf{v}_{t'}$ and $t \neq t'$. Thus, $d(\mathbf{u},\mathbf{v}_t) = d(\mathbf{u},\mathbf{v}_{t'}) = 1$. Consider the last coordinate of $\mathbf{u}$. This coordinate is different from at least one of t or t' (or possibly both). Without loss of generality, let us assume that the last coordinate of $\mathbf{u}$ is different from t. Since $d(\mathbf{u},\mathbf{v}_t) = 1$, the other coordinates of $\mathbf{u}$ are identical with that of $\mathbf{v}_t$. This contradicts the fact that $g_S(\mathbf{a}) = 0$ simply because in this case $f_S(\mathbf{u}) = 1$ and this term contributes in $g_S(\mathbf{a})$.

□

We need one more technical lemma before we prove Theorem 1.

Lemma 2. *Let l, k, n be integer numbers such that $k \leq l+1$. Let $h : [n]^l \to \{0,1\}$ be a function. Suppose that for every $\mathbf{c} \in [n]^l$ with $h(\mathbf{c}) = 0$ we have*

$$\sum_{\mathbf{b}\in[n]^l, d(\mathbf{c},\mathbf{b})=1} h(\mathbf{b}) \geq kn.$$

Then,

$$\sum_{\mathbf{a}\in[n]^l} h(\mathbf{a}) \geq \min\{n^l, \frac{kn^l}{l}\}.$$

Proof. First, consider the case that $h(\mathbf{a}) = 1$ for all $\mathbf{a} \in [n]^l$. In this case the assertion is trivial. Thus, in the rest of the proof we assume that $h(\mathbf{a}) = 0$ for at least one value of $\mathbf{a} \in [n]^l$. Let $s := \sum_{\mathbf{a}\in[n]^l} h(\mathbf{a})$. Therefore, $s < n^l$. For every $\mathbf{a} \in [n]^l$ and $j \in [l]$, let $\hat{h}(\mathbf{a}, j) := \sum_{\mathbf{b}} h(\mathbf{b})$ where the summation is taken over all $\mathbf{b} \in [n]^l$ which differ from $\mathbf{a}$ in precisely its j-th coordinate.
Based on the assumption of the lemma, if $h(\mathbf{a}) = 0$ then we have

$$\sum_{j=1}^{l} \hat{h}(\mathbf{a}, j) = \sum_{\mathbf{b}\in[n]^l, d(\mathbf{a},\mathbf{b})=1} h(b) \geq kn. \tag{2.2}$$

We add the inequalities (2.2) for all possible $\mathbf{a}$'s with $h(\mathbf{a}) = 0$. By the choice of s, there are precisely $n^l - s$ such $\mathbf{a}$'s. Thus, the right hand sides will add up to $kn(n^l - s)$. For the left hand side, we calculate the sum of the terms $\sum_{j=1}^{l} \hat{h}(\mathbf{a}, j)$ over all possible choices of $\mathbf{a}$ with $h(\mathbf{a}) = 0$. Thus, the overall summation of the left hand side is equal to

$$\sum_{\mathbf{a}:h(\mathbf{a})=0} \sum_{j=1}^{l} \hat{h}(\mathbf{a}, j) = \sum_{j=1}^{l} \sum_{\mathbf{a}:h(\mathbf{a})=0} \hat{h}(\mathbf{a}, j). \tag{2.3}$$

The above equality is obtained by changing the order of the summations. We claim that

$$\sum_{j=1}^{l} \sum_{\mathbf{a}:h(\mathbf{a})=0} \hat{h}(\mathbf{a}, j) = \sum_{\mathbf{b}\in[n]^l} \Big(\sum_{j=1}^{l} \big(n - \hat{h}(\mathbf{b}, j)\big)\Big) h(\mathbf{b}). \tag{2.4}$$

To prove this equality, note that both sides can be written as a summation of terms $h(\mathbf{a})$ for $\mathbf{a} \in [n]^l$ with certain multiplicities. Thus, in order to prove Equation (2.4), it suffices to show that the corresponding multiplicities of non-zero terms in both sides are equal.
Let us pick an arbitrary $\mathbf{b} \in [n]^l$ satisfying $h(\mathbf{b}) \neq 0$. In the left hand side of Equation (2.4), the term $h(\mathbf{b})$ contributes in $\hat{h}(\mathbf{a}, j)$ when $h(\mathbf{a}) = 0$ and $\mathbf{b}, \mathbf{a}$ differ in their j-th coordinates. Observe that $\hat{h}(\mathbf{b}, j)$ counts the total number of 1's in $h(\mathbf{a})$'s where $\mathbf{a}$ is identical with $\mathbf{b}$ in all positions except possibly the j-th coordinate. Therefore,

among such entries, there are precisely $n - \hat{h}(\mathbf{b}, j)$ zero terms. This means that for a fixed value of j, the term $h(\mathbf{b})$ contributes $n - \hat{h}(\mathbf{b}, j)$ times in the summation

$$\sum_{\mathbf{a}:h(\mathbf{a})=0} \hat{h}(\mathbf{a}, j).$$

When j varies over the range $[l]$, the total contribution of $h(\mathbf{b})$ is equal to

$$\sum_{j=1}^{l} \big(n - \hat{h}(\mathbf{a}, j)\big).$$

This contribution is precisely the contribution of $h(\mathbf{b})$ in the right hand side of Equation (2.4).
In conclusion, the left hand side of Equation (2.3) is equal to

$$\sum_{\mathbf{b}\in[n]^l} \Big(\sum_{j=1}^{l} \big(n - \hat{h}(\mathbf{b}, j)\big)\Big) h(\mathbf{b}).$$

Hence, we have

$$\sum_{\mathbf{b}\in[n]^l} \Big(\sum_{j=1}^{l} \big(n - \hat{h}(\mathbf{b}, j)\big)\Big) h(\mathbf{b}) \geq nk(n^l - s).$$

After reversing the order of the summations in the left hand side, we get

$$\sum_{j=1}^{l} \sum_{\mathbf{b}\in[n]^l} \big(n - \hat{h}(\mathbf{b}, j)\big) h(\mathbf{b}) \geq nk(n^l - s).$$

For a fixed value of j, the term $\sum_{a\in[n^l]} h(\mathbf{a})\big(n - \hat{h}(\mathbf{a}, j)\big)$ counts the number of the pairs $(\mathbf{b}, \mathbf{c})$ such that $h(\mathbf{b}) = 1, h(\mathbf{c}) = 0$ and $\mathbf{b}, \mathbf{c}$ differ exclusively in their j-th coordinates.
Consider the following equivalence relation $\sim_j$ over the elements of $[n]^l$. We say $\mathbf{a} \sim_j \mathbf{b}$ if and only if $\mathbf{a}, \mathbf{b}$ are identical on every position except possibly on the j-th coordinate. One can easily observe that $\sim_j$ is indeed an equivalence relationship and there are precisely $[n]^{l-1}$ equivalence classes; namely $C_1, C_2, \ldots, C_{n^{l-1}}$. Each class contains n elements. Also, the term $\sum_{\mathbf{a}\in[n]^l} h(\mathbf{a})\big(n - \hat{h}(\mathbf{a}, j)\big)$ counts the number of pairs of two entries $b \sim_j c$ such that $h(\mathbf{b}) = 0, h(\mathbf{c}) = 1$. Suppose that the class C_i has r_i zeros and $n - r_i$ ones. Thus,

$$\begin{aligned}\sum_{\mathbf{a}\in[n^l]} h(\mathbf{a})\big(n-\hat{h}(\mathbf{a},j)\big) &= \sum_{i=1}^{n^{l-1}} r_i(n-r_i)\\ &= \sum_{i=1}^{n^{l-1}} nr_i - \sum_{i=1}^{n^{l-1}} r_i^2\\ &= ns - \sum_{i=1}^{n^{l-1}} r_i^2\\ &\le ns - \frac{1}{n^{l-1}}\Big(\sum_{i=1}^{n^{l-1}} r_i\Big)^2\\ &= s(n-\frac{s}{n^{l-1}}).\end{aligned}$$

In the above chain, the third equality is because $\sum_{i=1}^{n^{l-1}} r_i$ counts the total number of 1's which is defined to be s. The inequality in the chain is due to arithmetic-quadratic mean inequality.
Note that the above argument works for any choice of j. Thus, by adding all of these inequalities for all values of $j \in [l]$ we obtain the following:

$$\sum_{j=1}^{l}\sum_{\mathbf{a}\in[n]^l}\big(n-\hat{h}(\mathbf{a},j)\big) \le ls(n-\frac{s}{n^{l-1}}).$$

Combined with the summation of the right hand side of (2.2) we conclude that

$$ls(n-\frac{s}{n^{l-1}}) \ge \sum_{j=1}^{l}\sum_{\mathbf{a}\in[n]^l}\big(n-\hat{h}(\mathbf{a},j)\big) \ge nk(n^l-s).$$

Finally, using the fact that $s < n^l$ and from inequality $ls(n-\frac{s}{n^{l-1}}) \ge nk(n^l-s)$, we obtain $s \ge \frac{kn^l}{l}$. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let S be an independent k-dominating set for K_n^m. Let $l := m-1$ and $h := g_S$ where g_S is as defined in Equation (2.1). Lemma 1 guarantees that the assumptions of Lemma 2 are satisfied. Finally, Lemma 2 implies that

$$|S| = \sum_{\mathbf{a}\in[n]^l} h(a) \ge \min\{n^l, \frac{kn^l}{l}\} = \min\{n^{m-1}, \frac{kn^{m-1}}{m-1}\}.$$

Consequently, $\gamma_i(K_n^m, k) \ge \min\{n^{m-1}, \frac{kn^{m-1}}{m-1}\}$. □

2.2. Exact Value

This subsection is devoted to prove Theorem 2 which gives the exact value of $\gamma_i(K_n^m, k)$ when $k \in \{m-1, m\}$.

Proof of Theorem 2. First, notice that any independent k-dominating set is also an independent k'-dominating set for $k' < k$. Thus, for fixed values of m and n, the function $\gamma_i(K_n^m, k)$ is increasing as a function of k, while $k \leq m$. For $k > m$, as we pointed out earlier, there is no way to dominate any vertex with at least k independent vertices. Thus, we only consider the regime $k \in [m]$. By Theorem 1, for $k \in \{m-1, m\}$, the lower bound for $\gamma_i(K_n^m, k)$ is n^{m-1}. Therefore, if we construct an independent m-dominating set of size n^{m-1}, then we have

$$n^{m-1} \leq \gamma_i(K_n^m, m-1) \leq \gamma_i(K_n^m, m) \leq n^{m-1}.$$

A direct consequence of the above chain is that

$$\gamma_i(K_n^m, k) = n^{m-1},$$

for $k \in \{m-1, m\}$. This will prove Theorem 2.
We now present an independent m-dominating set of size n^{m-1}. Define the subset $W \subseteq V$ as follows.

$$W := \{(a_1, a_2, \ldots, a_m) \in V : \sum_{i=1}^{m} a_i \equiv 0 \pmod{n}\}.$$

We first claim that W is independent subset of the vertices of K_n^m. Suppose, for contradiction, that $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{a}' = (a_1', \ldots, a_m')$ are adjacent vertices in W. Then $d(\mathbf{a}, \mathbf{a}') = 1$. This means that $\mathbf{a}, \mathbf{a}'$ differ in precisely one position. This is not possible because for both of $\mathbf{a}, \mathbf{a}' \in W$, the sum of the entries are congruent modulo n. In fact, if they agree in $m-1$ positions, the remaining position is also uniquely determined.
We also show that W is an m-dominating set. For any vertex $\mathbf{b} = (b_1, b_2, \ldots, b_m) \in V$, if $\sum_{i=1}^{m} b_i \equiv 0 \pmod{n}$ then $b \in W$. Otherwise, for any index $j \in [m]$, we may change the j-th coordinate of $\mathbf{b}$ to obtain a vertex in W. This vertex has Hamming distance 1 from $\mathbf{b}$, hence dominates it. Thus, W contains at least m elements each of which dominates $\mathbf{b}$. This shows that W is an independent m-dominating set for K_n^m. Note that for every choice of $b_1, \ldots, b_{m-1} \in [n]$, there exists a unique $b_m \in [n]$ such that $\mathbf{b} = (b_1, \ldots, b_m) \in W$. Therefore, $|W| = n^{m-1}$. $\square$

3. Concluding Remarks

In this paper, inspired by the application of the covering problem in Hamming space, we study the independent k-domination number of Hamming graphs. For $k = 1$, the independent k-domination number of Hamming graphs has been previously studied in [1]. Thus, our result generalizes the previous result of [1]. Our proof is completely different from that in [1], even for $k = 1$. It is not clear if the proof of [1] for $k = 1$ can be extended to larger values of k. In contrast, our proof provides a lower bound for the independent k-domination number of Hamming graphs. For $k \in \{m-1, m\}$, we prove that the lower bound is exact. In particular, this result shows that while the function $\gamma_i(G, k)$ is an increasing function of k, it is not necessarily strictly increasing. The problem of finding the exact value of independent k-domination number for Hamming graphs for general value of k remains open.

Conflict of Interest: The authors declare that they have no conflict of interest.

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