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Research Article

Multiset Dimension of Prisms

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Abstract: Given a subset W of the vertex set of a graph G, the multiset representation $r_m(v|W)$ of a vertex v is the multiset of distances between v and each vertex in W. The subset W is called an m-resolving set of G if distinct vertices have distinct multiset representations. The m-resolving sets of a graph can be used to uniquely identify its vertices and has been shown to be equivalent to identification colorings. More recently, Kono and Zhang have established that the prism $K_2 \,\square\, C_n$ has an m-resolving set (equivalently, identification coloring) if and only if $n \geq 6$. In this work, we extend their result by determining the multiset dimension of prisms; that is, we determine the minimum cardinality of their m-resolving sets.

Keywords: m-resolving set, identification coloring, multiset dimension, prism.

AMS Subject classification: 05C12, 05C15

1. Introduction

There has been a great deal of interest among mathematicians on the problem of uniquely identifying vertices in a graph using distances from a particular set of vertices. If G is a connected graph, then the distance $d(u, v)$ between two vertices u and v is the length of the shortest u-v path. Slater [17], Harary and Melter [6], and Chartrand et al [2] independently introduced the idea of a resolving set of a graph, a subset $W = \{w_1, w_2, \ldots, w_k\}$ of the set of vertices of G such that for any two distinct vertices u and v of the graph, $d(u, w_i) \neq d(v, w_i)$, for some $1 \leq i \leq k$. The authors

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likewise defined the metric dimension of G as the cardinality of a minimal resolving set.

Years later, Saenpholphat [15] and Simanjuntak et al. [16] independently introduced the concept of an m-resolving set of a graph. In this case, instead of distances to an ordered set W, vertices may be uniquely identified via a multiset consisting of the distances to each vertex in W. Let $G = (V, E)$ be a simple nontrivial connected graph and $W \subset V$. For $v \in V$, we define the *multiset representation or* m-code of v with respect to W, denoted by $r_m(v|W)$ (or simply $r_m(v)$), to be the multiset of distances between v and the vertices in W; that is $r_m(v) = \{d(v, w) \mid w \in W\}$. If every pair of distinct vertices in G have distinct m-codes, then W is called an *m-resolving set* of G. If G has an m-resolving set, then the cardinality of a smallest m-resolving set is called the *multiset dimension* of G, denoted by $\mathrm{md}(G)$.

Identification colorings (or ID-colorings), introduced recently by Chartrand et al. [3], has a similar objective. Suppose c is a red-white coloring of the graph G; that is, $c(v)$ is either red or white for any vertex v of G. Let d be the diameter of G. The code of a vertex v is the d-vector $\vec{d}(v) = (a_1, a_2, \ldots, a_d)$, where for each $i \in \{1, 2, \ldots, d\}$, a_i is equal to the number of red vertices of distance i from v. Then c is called an *ID-coloring* of G if $\vec{d}(v) \neq \vec{d}(w)$ for any pair of distinct vertices v, w. Hakanen and Yero [5] have shown that given a graph G, a red-white coloring of G is an ID-coloring if and only if the set of red vertices is an m-resolving set of G.

ID-colorings and m-resolving sets and have been the subject of different studies including those focusing on trees [4, 10], caterpillars [7, 9, 11], and other topics [1, 8, 13]. In [12], Kono and Zhang studied the existence of identification colorings (equivalently, m-resolving sets) of grids $P_m \square P_n$ and prisms $K_2 \square C_n$. For prisms, their result is as follows.

Theorem 1 ([12]). *The prism $K_2 \square C_n$ has an m-resolving set if and only if $n \geq 6$.*

In the proof of Theorem 1, Kono and Zhang exhibited an m-resolving set consisting of n vertices. In this paper, we continue the work done in [12] by determining the multiset dimension of prisms; that is, we determine the minimum cardinality of their m-resolving sets. In Section 2, we present existing and new properties that will be useful for this study. In Section 3, we present our main results on the multiset dimension of any prism $K_2 \square C_n$, for $n \geq 6$.

2. Preliminaries

We begin by presenting the following known results that are used in this study.

Proposition 1 ([3]). *If $G = (V, E)$ is a connected graph, W a subset of V, $w \in W$, and $v \in V - W$, then $r_m(w|W) \neq r_m(v|W)$.*

Proposition 2 ([3],[15],[16]). *No connected graph has multiset dimension 2.*

Theorem 2 ([3],[15],[16]). *A nontrivial connected graph G has $md(G) = 1$ if and only if G is a path.*

For the next result, let P_n be the path $(0, 1, 2, \ldots, n-1)$ of order $n \geq 4$. We define a **symmetric** subset W of $V(P_n)$ to be one with the property $i \in W$ if and only if $n-1-i \in W$, for each $i \in V(P_n)$.

Theorem 3 ([14]). *Let $n \geq 4$. If $W \subset V(P_n)$ contains 0 and $n-1$ and is not symmetric, then W is an m-resolving set of P_n.*

Now, let $G = (V, E)$ be a connected graph and let W be a subset of V with cardinality k. For any $v \in V$, we let $r_m(v) = \{d_1(v), d_2(v), \ldots, d_k(v)\}$ where $d_i(v) \leq d_{i+1}(v)$, for $i = 1, 2, \ldots, k-1$. For convenience, we also define $\text{sum}_2(v) = d_1(v) + d_2(v)$; that is, $\text{sum}_2(v)$ is the sum of the two shortest distances from v to the vertices in W. Consequently, we have the following.

Observation 4. Let $G = (V, E)$ be a simple connected graph and W a subset of V with cardinality 3. If $u, v \in V$ with $\text{sum}_2(u) \neq \text{sum}_2(v)$, then $r_m(u) \neq r_m(v)$.

In the next theorem we construct an m-resolving set of minimum cardinality for C_n, $n \geq 9$, that is different from the set used in [3], [16], and [15] to show that $\text{md}(C_n) = 3$. It can be proved using Theorem 3, Proposition 1, and Observation 4. The m-resolving set used here will be the basis for the more general result on prisms presented in this paper.

Theorem 5. *Let $n \geq 9$ and let G be the cycle C_n with $V(G) = \mathbb{Z}_n$ and $E(G) = \{\{i, i+1\} \mid 0 \leq i \leq n-1\}$, with addition done modulo n. Then $W = \{0, \lfloor n/4 \rfloor, 2\lfloor n/4 \rfloor + 1\}$ is an m-resolving set of G.*

Proof. Let $\ell = \lfloor n/4 \rfloor$. Then $\ell \geq 2$ and $W = \{0, \ell, 2\ell + 1\}$. Let $A = \{0, 1, \ldots, 2\ell + 1\} \subset V(G)$, and P the path $(0, 1, \ldots, 2\ell + 1)$. We note that for any $v \in A - \{2\ell + 1\}$, the m-code of v in G is equal to its m-code in P. Now,

$$\text{sum}_2(2\ell + 1) = \ell + 1 > \ell = \text{sum}_2(0) = \text{sum}_2(\ell).$$

By Theorem 3, Proposition 1, and Observation 4, it follows that the vertices in A have distinct m-codes.

Now, let $u, v \in V(G) - A$ with $u > v$. Suppose $r_m(u) = r_m(v)$ and $a = d_1(u) = d_1(v)$. Then $a \leq \ell + 1$. Let $a = d(u, 0) = d(v, 2\ell + 1)$. Then $u = n - a$ and $v = 2\ell + 1 + a$, and $d(v, 0) = d(u, 2\ell + 1)$. However, $d(u, \ell) = \ell + a \neq \ell + 1 + a = d(v, \ell)$, a contradiction. Finally, if $u \in A$ and $v \in V(G) - A$, then $\mathrm{sum}_2(u) = \ell$ or $\ell + 1$, while $\mathrm{sum}_2(v) \geq \ell + 2$. Therefore, all vertices of C_n have distinct m-codes and W is an m-resolving set of C_n. □

3. Multiset Dimension of Prisms

3.1. Multiset Dimension of Prisms: Special Cases

In this section, we will determine the multiset dimension of prisms $K_2 \,\square\, C_n$ for small values of n. We begin with the following result on $K_2 \,\square\, C_{13}$. Recall that the vertices of C_{13} are labeled using elements of $\mathbb{Z}_{13}$.

Lemma 1. *$md(K_2 \,\square\, C_{13}) \geq 4$.*

Proof. Suppose W is an m-resolving set for $G = K_2 \,\square\, C_{13}$ with $|W| = 3$. For simplicity, we label the vertices in G as in Figure 1. By symmetry, we may assume that W is one of the sets in Tables 1 and 2. It can be checked that for any such set W, the identified vertices u and v in the tables have equal m-codes. Therefore, the result follows. □

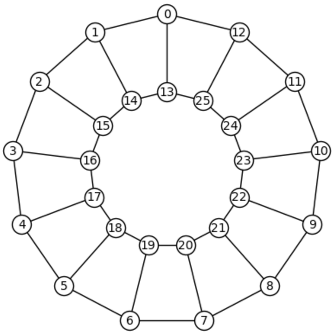

Figure 1. **The graph $K_2 \,\square\, C_{13}$.**

For almost all sets W in Tables 1 and 2, the existence of u and v can be proved by applying the following observation. From the conditions on u and v, it follows that these vertices have the same m-code.

Observation 6. Let G be a graph, H a subgraph of G, $W \subseteq V(G)$, $u, v \in V(H)$, and $u \neq v$ such that

- $W \subseteq V(H)$,
- $d_G(x, w) = d_H(x, w)$ for all elements $x \in \{u, v\}$ and $w \in W$,
- There is a graph automorphism of H that interchanges u and v.

W	$u, v \in V(G)$ with $r_m(u) = r_m(v)$
$\{0,1,2\}$	$0, 2$
$\{0,1,c\}, 3 \le c \le 5$	$12, 13$
$\{0,1,6\}$	$12, 14$
$\{0,1,7\}$	$0, 1$
$\{0,2,4\}$	$0, 4$
$\{0,2,5\}$	$12, 13$
$\{0,2,6\}$	$5, 15$
$\{0,2,7\}$	$9, 25$
$\{0,3,6\}$	$0, 6$
$\{0,3,7\}$	$12, 16$
$\{0,3,8\}$	$0, 3$
$\{0,4,8\}$	$0, 8$

Table 1. Possible triples of vertices in $K_2 \,\square\, C_{13}$ where all vertices belong to the outer cycle.

W	$u, v \in V(G)$ with $r_m(u) = r_m(v)$
$\{0,1,13\}$	$1, 13$
$\{0,1,c\}, 15 \le c \le 20$	$2, 14$
$\{0,2,13\}$	$12, 15$
$\{0,2,14\}$	$0, 2$
$\{0,2,c\}, 16 \le c \le 20$	$3, 15$
$\{0,3,13\}$	$2, 14$
$\{0,b,14\}, b \in \{3,4,5,6\}$	$0, 14$
$\{0,3,c\}, 17 \le c \le 21$	$4, 16$
$\{0,4,13\}$	$7, 19$
$\{0,4,c\}, 18 \le c \le 21$	$5, 17$
$\{0,5,13\}$	$3, 15$
$\{0,5,15\}$	$4, 16$
$\{0,5,c\}, 19 \le c \le 22$	$6, 18$
$\{0,6,13\}$	$12, 14$
$\{0,6,15\}$	$8, 20$
$\{0,6,16\}$	$0, 6$
$\{0,6,20\}$	$3, 17$
$\{0,6,21\}$	$6, 21$
$\{0,6,22\}$	$4, 18$

Table 2. Possible triples of vertices in $K_2 \,\square\, C_{13}$ where exactly two vertices belong to the outer cycle.

Then $r_m(u \mid W) = r_m(v \mid W)$, and so W is not an m-resolving set of G.

To illustrate, when $W = \{0,1,2\}$ or $\{0,1,13\}$, then we can take H to be the path P_3 induced by W, so the endvertices of H have equal m-codes. When $W = \{0,1,20\}$, then we can form the subgraph H shown in Fig. 2, and the vertices 2 and 14 satisfy all the conditions in Observation 6. A similar graph construction can be made for $W = \{0,2,c\}$, $16 \le c \le 20$; $W = \{0,3,c\}, 17 \le c \le 21$; $W = \{0,4,c\}, 18 \le c \le 21$; and $W = \{0,5,c\}, 19 \le c \le 22$.

As in Lemma 1, it can be shown that $\mathrm{md}(K_2 \,\square\, C_n) \ge 5$ if $n = 6$ or 7, and $\mathrm{md}(K_2 \,\square\, C_n) \ge 4$ if $n = 8$, 9, or 10. Additionally, the sets W in Table 3 can

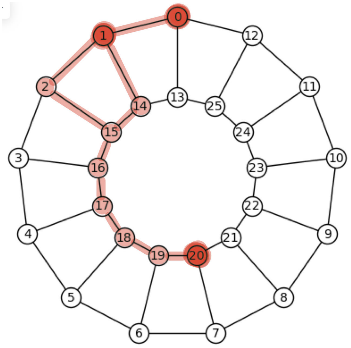

Figure 2. The subgraph H that satisfies the conditions in Observation 6 when $W = \{0, 1, 20\}$.

be shown to be m-resolving sets for the given prisms. Here, we follow the pattern used in Figures 1 and 2 in labeling the vertices of $K_2 \,\Box\, C_n$.

G	W
$K_2 \,\Box\, C_6$	$\{0, 1, 2, 4, 6\}$
$K_2 \,\Box\, C_7$	$\{0, 1, 2, 3, 11\}$
$K_2 \,\Box\, C_n, n = 8$ or 10	$\{0, 1, 3, 5\}$
$K_2 \,\Box\, C_9$	$\{0, 1, 2, 4\}$
$K_2 \,\Box\, C_n, n = 12$ or 16	$\{0, 4, n\}$
$K_2 \,\Box\, C_{13}$	$\{0, 1, 3, 6\}$

Table 3. Minimal m-resolving sets for $K_2 \,\Box\, C_n$ where $n \in \{6, 7, 8, 9, 10, 12, 13, 16\}$.

3.2. Multiset Dimension of Prisms: General Case

In this section, we turn to the general case for determining the multiset dimension of prisms $K_2 \,\Box\, C_n$ for $n \notin \{6, 7, 8, 9, 10, 12, 13, 16\}$. We begin with some notations and observations. Let G be the prism $K_2 \,\Box\, C_n$, where $n \geq 6$. We let

$$\begin{aligned} V(G) &= \{(i,j) \mid 0 \leq i \leq 1, 0 \leq j \leq n-1\}, \text{ and} \\ E(G) &= \{\{(i,j), (i, (j+1) \bmod n)\} \mid 0 \leq i \leq 1, 0 \leq j \leq n-1\} \\ &\qquad \cup \{\{(0,j), (1,j)\} \mid 0 \leq j \leq n-1\}. \end{aligned}$$

For $i = 0$ or 1, we denote the cycle $((i,0), (i,1), \ldots, (i,n-1), (i,0))$ by $C_n(i)$.

Suppose $W \subset V(C_n(0))$ with k elements. Then, for any j, $0 \leq j \leq n-1$, let u and v be the vertices $(0,j)$ and $(1,j)$, respectively. Then $d_i(v) = d_i(u) + 1$ for any i, $1 \leq i \leq k$, and $\mathrm{sum}_2(v) = 2+ \mathrm{sum}_2(u)$. Hence, we can make the following observation.

Observation 7. Let $G = K_2 \,\Box\, C_n$, where $n \geq 6$, and $W \subset V(C_n(0))$ an m-resolving set of $C_n(0)$. Then the m-codes of the vertices of $C_n(1)$ are distinct.

We now determine the multiset dimension of $K_2 \,\Box\, C_n$, where $n \geq 17$. We proceed by considering four cases based on the congruence class of n modulo 4.

Lemma 2. *Let* $G = K_2 \,\Box\, C_n$*, where* $n \geq 20$ *and* $n \equiv 0 \pmod 4$*. Then* $W = \{(0,0),(0,\lfloor n/4 \rfloor),(0, 2\lfloor n/4 \rfloor + 1)\}$ *is an m-resolving set of* G*.*

Proof. Let $n = 4\ell$, where $\ell \geq 5$. Then $W = \{0, \ell, 2\ell+1\}$. By Theorem 5, the m-codes of the vertices in $C_n(0)$ are distinct. By Observation 7, it follows that the m-codes in $C_n(1)$ are distinct as well. Hence it is enough to show that no vertex u in $C_n(0)$ has the same m-code as a vertex v in $C_n(1)$. We classify the vertices of G according to their sum_2 values. Note that for any $u \in V(C_n(0))$, we have $\ell \leq \mathrm{sum}_2(u) \leq 2\ell - 1$ while for any $v \in C_n(1)$, we have $\ell + 2 \leq \mathrm{sum}_2(v) \leq 2\ell + 1$. Refer to Figure 3.

By Observation 4, we may assume that $\mathrm{sum}_2(u) = \mathrm{sum}_2(v)$. To show that $r_m(u) \neq r_m(v)$ it is enough to show that $d_3(u) \neq d_3(v)$, where $d_3(x)$ is the largest distance of a vertex $x \in V(G)$ from a vertex in W. Suppose $\mathrm{sum}_2(u) = \mathrm{sum}_2(v) = k$. We divide the possible values of k into cases.

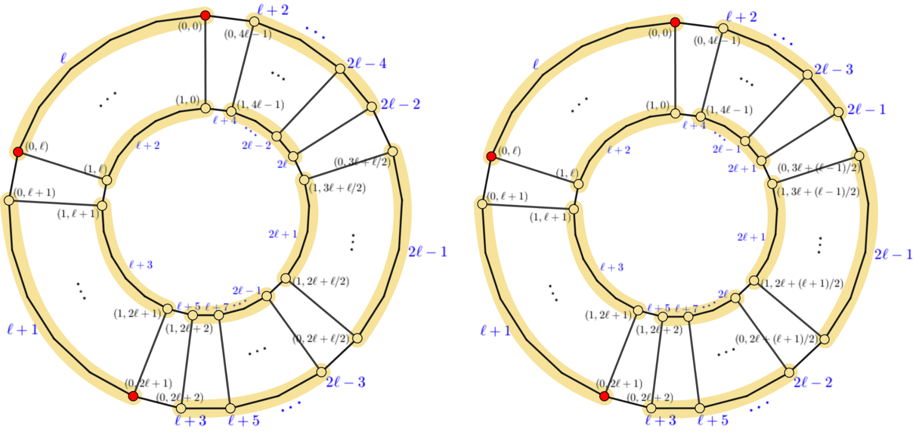

Figure 3. sum_2 values (in blue) for $K_2 \,\Box\, C_n$, where $n = 4\ell$, $\ell \geq 5$, and ℓ is even (left) or odd (right)

Case 1. $k = \ell + 2$.
It follows that u is the vertex $(0, n-1)$. Since $d_1(u) = 1$, we only consider $v = (1,0)$ or $(1,\ell)$. But we note: $d_3((1,\ell)) = \ell + 2 < d_3((0,n-1)) = 2\ell - 2 < d_3((1,0)) = 2\ell$.

Case 2. $k = \ell + 1 + 2h$ where $h \in \mathbb{N}$, $1 \leq h \leq (\ell - 3)/2$.
Then $u = (0, 2\ell + 1 + h)$ and $v = (1, 2\ell + h)$. But $d_3(v) = d_3(u) + 2$.

Case 3. $k = \ell + 2h$ where $h \in \mathbb{N}$, $2 \le h \le (\ell - 2)/2$.
Then $u = (0, n-h)$, while $v = (1, n-h+1)$. As in Case 2, $d_3(v) = d_3(u) + 2$.

Case 4. $k = 2\ell - 1$.
If ℓ is odd, then $v = (1, n - (\ell-3)/2)$. Since $d_1(v) = (\ell-1)/2$, we have two possibilities for u : $u_1 = (0, n - (\ell-1)/2)$ and $u_2 = (0, 2\ell + 1 + (\ell-1)/2)$. But $d_3(u_1) = (3\ell-1)/2$, $d_3(u_2) = (3\ell+1)/2$, and $d_3(v) = (3\ell+3)/2$.

If ℓ is even, then $v = (1, 2\ell + \ell/2 - 1)$. Since $d_1(v) = \ell/2 - 1$, we must have $u = (0, 2\ell + \ell/2)$. But, as in Case 2, we have $d_3(v) = d_3(u) + 2$. ☐

Lemma 3. *Let* $G = K_2 \,\square\, C_n$, *where* $n \ge 17$ *and* $n \equiv 1 \pmod 4$. *Then* $W = \{(0,0), (0, \lfloor n/4 \rfloor), (0, 2\lfloor n/4 \rfloor + 1)\}$ *is an m-resolving set of* G.

Proof. Let $n = 4\ell + 1$, where $\ell \ge 4$. Then $W = \{0, \ell, 2\ell+1\}$. As in the proof of Lemma 2, we need only to show that if $u \in C_n(0)$ and $v \in C_n(1)$ satisfy the equation $\mathrm{sum}_2(u) = \mathrm{sum}_2(v)$, then $d_3(u) \ne d_3(v)$. Note that for any $u \in C_n(0)$, we have $\ell \le \mathrm{sum}_2(u) \le 2\ell$; hence for any $v \in C_n(1)$, we have $\ell + 2 \le \mathrm{sum}_2(v) \le 2\ell + 2$. Refer to Figure 4.

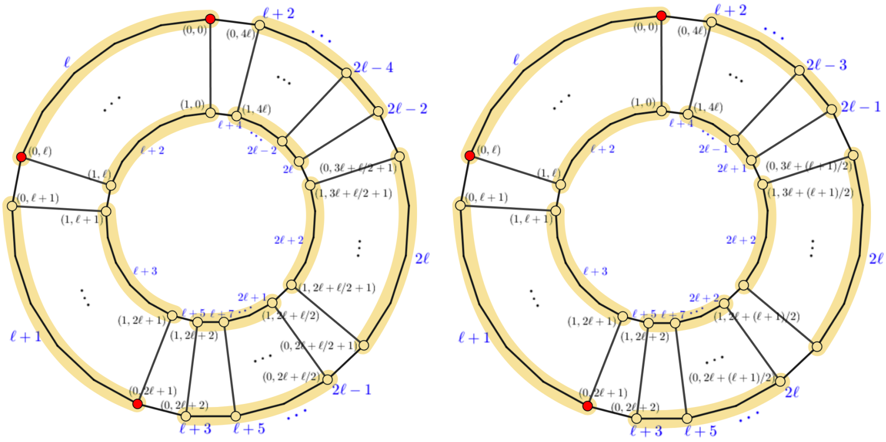

Figure 4. sum_2 values (in blue) for $K_2 \,\square\, C_n$, where $n = 4\ell + 1$, $\ell \ge 4$, and ℓ is even (left) or odd (right)

Suppose $\mathrm{sum}_2(u) = \mathrm{sum}_2(v) = k$. We proceed by considering cases based on the possible values of k.

Case 1. $k = \ell + 2$.
Then $u = (0, n-1)$, and since $d_1(u) = 1$, it suffices to consider only the following possibilities for the vertex v: $(1,0)$ and $(1,\ell)$. But $d_3(1,\ell) = \ell + 2 < 2\ell - 1 = d_3(0, n-1) < d_3(1,0) = 2\ell + 1$.

Case 2. $k = \ell + (2h+1)$ where $1 \le h \le (\ell-1)/2$.
Then $u = (0, (2\ell+1) + h)$ and $v = (1, (2\ell+1) + (h-1)) = (1, 2\ell + h)$. We note that

$d_3(u) = n-(2\ell+1+h) = 2\ell-h$ whereas $d_3(v) = (n-h+1)-(2\ell+1)+1 = 2\ell-h+2$. This implies $d_3(v) = d_3(u)+2$.

Case 3. $k = \ell + 2h$ where $2 \leq h \leq (\ell-1)/2$.
Then $u = (0, n-h)$ and $v = (1, n-h+1)$. But as in the previous case, $d_3(v) = d_3(u)+2$ since $d_3(u) = (n-h)-(2\ell+1) = 2\ell-h$ whereas $d_3(v) = (n-h+1)-(2\ell+1)+1 = 2\ell-h+2$.

Case 4. $k = 2\ell$.
The case when ℓ is odd is covered by Case 2. Hence, we may assume that ℓ is even. Then among the vertices of G having $k = 2\ell$, the only vertices that need to be examined are those vertices x with $d_1(x) = \ell/2$. In $C_n(0)$, these are the vertices $(0, n-\ell/2) = (0, 3\ell+\ell/2+1)$ and $(0, 2\ell+\ell/2+1)$; while in $C_n(1)$, there is only one such vertex, namely $(1, n-\ell/2+1) = (1, 3\ell+\ell/2+2)$.

Now, $d_3((0, n-\ell/2)) = (n-\ell/2)-(2\ell+1) = 3\ell/2$, $d_3((0, 2\ell+\ell/2+1)) = \ell+\ell/2+1 = 3\ell/2+1$, and $d_3((1, n-\ell/2+1)) = (n-\ell/2+1)-(2\ell+1)+1 = 3\ell/2+2$. Thus, as in Case 2, we have $d_3(v) = d_3(u)+2$. □

Lemma 4. *Let* $G = K_2 \,\square\, C_n$*, where* $n \geq 14$ *and* $n \equiv 2 \pmod 4$*. Then* $W = \{(0,0), (0, \lfloor n/4 \rfloor), (0, 2\lfloor n/4 \rfloor + 1)\}$ *is an m-resolving set of* G*.*

Proof. Let $n = 4\ell+2$, where $\ell \geq 3$. Then $W = \{0, \ell, 2\ell+1\}$. Following the proofs of the previous lemmas, let $u \in C_n(0)$ and $v \in C_n(1)$. In this case, we have $\ell \leq \mathrm{sum}_2(u) \leq 2\ell+1$; hence, $\ell+2 \leq \mathrm{sum}_2(v) \leq 2\ell+3$. Refer to Figure 5.

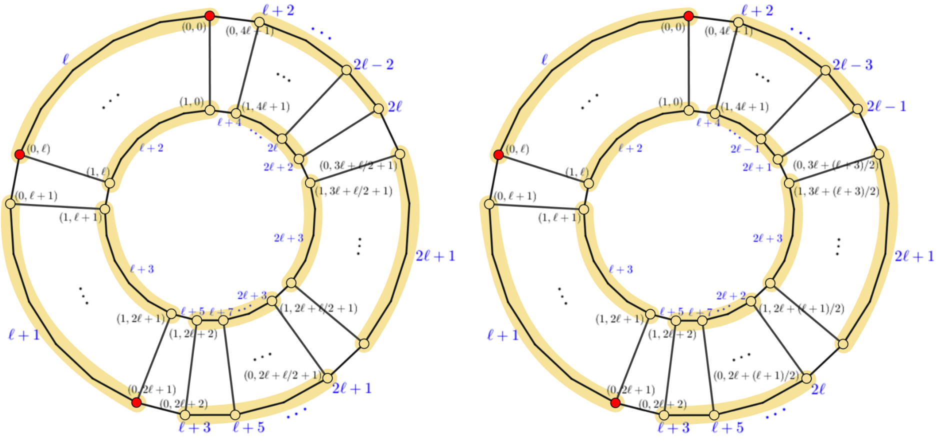

Figure 5. sum_2 values (in blue) for $K_2 \,\square\, C_n$, where $n = 4\ell+2$, $\ell \geq 3$, and ℓ is even (left) or odd (right)

Suppose $\mathrm{sum}_2(u) = \mathrm{sum}_2(v) = k$. We proceed by considering cases based on the possible values of k.

Case 1. $k = \ell + 2$.
Then $u = (0, n-1)$, and since $d_1(u) = 1$, we need only to consider $v = (1,0)$ and $(1,\ell)$. But $d_3(1,\ell) = \ell + 2 < 2\ell = d_3(0, n-1) < d_3(1,0) = 2\ell + 2$.

Case 2. $k = \ell + (2h+1)$ where $1 \le h \le \ell/2$.
Then $u = (0, (2\ell+1)+h)$ and $v = (1, (2\ell+1)+(h-1)) = (1, 2\ell+h)$. It follows that $d_3(u) = n - (2\ell+1+h) = 2\ell - h + 1$ whereas $d_3(v) = n - (2\ell+h) + 1 = 2\ell - h + 3$. Thus, $d_3(v) = d_3(u) + 2$.

Case 3. $k = \ell + 2h$ where $2 \le h \le \ell/2$.
Then $u = (0, n-h)$ and $v = (1, n-h+1)$. But as in the previous case, we have $d_3(v) = d_3(u) + 2$ since $d_3(u) = (n-h) - (2\ell+1) = 2\ell - h + 1$ whereas $d_3(v) = (n-h+1) - (2\ell+1) + 1 = 2\ell - h + 3$.
Case 4. $k = 2\ell + 1$.
The case when ℓ is even is covered by Case 2. Hence, we may assume that ℓ is odd. Then among the vertices of G satisfying $k = 2\ell+1$, we need to only examine those vertices x with $d_1(x) = (\ell+1)/2$. In $C_n(0)$, these vertices are $(0, n-(\ell+1)/2) = (0, 3l + (\ell+3)/2)$ and $(0, 2\ell+1+(\ell+1)/2) = (0, 2\ell + (\ell+3)/2)$; while in $C_n(1)$, there is only one such vertex, namely $v = (1, n - (\ell+1)/2 + 1) = (1, 3\ell + (\ell+5)/2)$.

Now, $d_3(0, 3\ell + (\ell+3)/2) = \ell + (\ell+1)/2 = (3\ell+1)/2$ and $d_3(0, 2\ell + (\ell+3)/2) = (5\ell+3)/2 - \ell = (3\ell+3)/2$ whereas $d_3(1, 3\ell + (\ell+5)/2) = (7\ell+5)/2 - (2\ell+1) + 1 = (3\ell+5)/2$. Thus, $d_3(0, 3\ell + (\ell+3)/2) < d_3(0, 2\ell + (\ell+3)/2) < d_3(1, 3\ell + (\ell+5)/2)$. ☐

Lemma 5. *Let $G = K_2 \square C_n$, where $n \ge 11$ and $n \equiv 3 \pmod 4$. Then $W = \{(0,0), (0, \lfloor n/4 \rfloor), (0, 2\lfloor n/4 \rfloor + 1)\}$ is an m-resolving set of G.*

Proof. Let $n = 4\ell + 3$, where $\ell \ge 2$. Then $W = \{0, \ell, 2\ell+1\}$. For any $u \in C_n(0)$, we have $\ell \le \mathrm{sum}_2(u) \le 2\ell+2$; hence, for any $v \in C_n(1)$, we have $\ell + 2 \le \mathrm{sum}_2(v) \le 2\ell + 4$. Refer Figure 6.
Suppose $\mathrm{sum}_2(u) = \mathrm{sum}_2(v) = k$. We proceed by considering cases based on the possible values of k.

Case 1. $k = \ell + 2$.
Then $u = (0, n-1)$, and since $d_1(u) = 1$, we need only to consider $v = (1,0)$ and $(1,\ell)$. But $d_3(1,\ell) = \ell + 2 < 2\ell + 1 = d_3(0, n-1) < d_3(1,0) = 2\ell + 2$.

Case 2. $k = \ell + (2h+1)$ where $1 \le h \le (\ell+1)/2$.
Then $u = (0, 2\ell+1+h)$ and $v = (1, (2\ell+1)+(h-1)) = (1, 2\ell+h)$. Now, $d_3(u) = n - (2\ell+1+h) = 2\ell - h + 2$ whereas $d_3(v) = n - (2\ell+h) + 1 = 2\ell - h + 4$. Then $d_3(v) = d_3(u) + 2$.
Case 3. $k = \ell + 2h$ where $2 \le h \le (\ell+2)/2$.
Then $u = (0, n-h)$ and $v = (1, n-h+1)$. But as in Case 2, we have $d_3(v) = d_3(u) + 2$ since $d_3(u) = (n-h) - (2\ell+1) = 2\ell - h + 2$ whereas $d_3(v) = (n-h+1) - (2\ell+1) + 1 = 2\ell - h + 4$. ☐

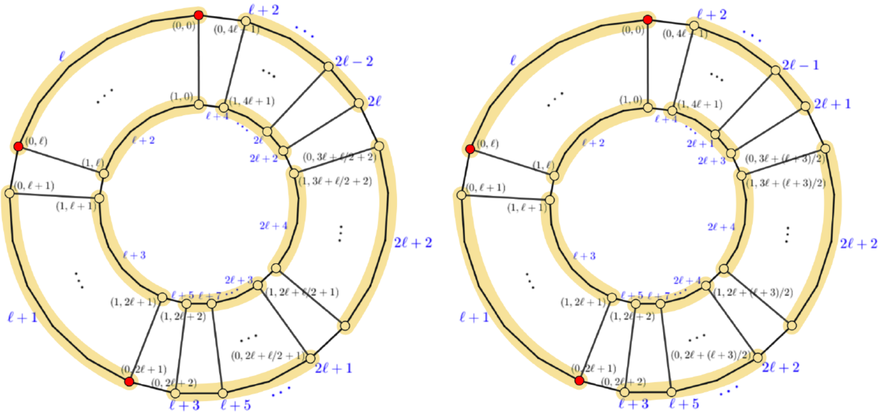

Figure 6. sum_2 values (in blue) for $K_2 \square C_n$, where $n = 4\ell + 3$, $\ell \geq 2$, and ℓ is even (left) or odd (right)

Combining the results from Subsection 3.1 and Lemmas 1-5, we obtain the following main result.

Theorem 8. *Let* $G = C_n \square K_2$, *where* $n \geq 6$. *Then*

$$md(G) = \begin{cases} 3, & \text{if } n \geq 11, n \neq 13, \\ 4, & \text{if } n = 8, 9, 10, 13, \\ 5, & \text{if } n = 6, 7. \end{cases}$$

Moreover, for all $n \geq 11$, *except* $n = 12, 13, 16$, *the set* $\{(0,0), (0, \lfloor n/4 \rfloor), (0, 2\lfloor n/4 \rfloor + 1)\}$ *is an m-resolving set of* G.

4. Conclusion and Future Direction

In this study, we have determined the multiset dimension of all prisms $K_2 \square C_n$, $n \geq 6$. This extends a previous result of Kono and Zhang, who have constructed m-resolving sets for such prisms but whose cardinalities are not necessarily minimum. Particularly, we have shown that for $n \geq 6$ and $n \notin \{6, 7, 8, 9, 10, 13\}$, the set $\{(0,0), (0, \lfloor n/4 \rfloor), (0, 2\lfloor n/4 \rfloor + 1)\}$ is an m-resolving set of $K_2 \square C_n$ with minimum cardinality.

The techniques and arguments in this work might find applications in further work on this topic. Particularly, it is worthwhile to consider the multiset dimension of *cylindrical graphs* $P_m \square C_n$, where $m \geq 3$ and $n \geq 3$. We use Observation 4 in proving the next result.

Observation 9. Let $G = (V, E)$ be a connected graph, $W \subseteq V(G)$, and H a subgraph of G that satisfies the following:

1. $W \subseteq V(H)$
2. $d_G(u,w) = d_H(u,w)$ for any $u \in V(H)$ and $w \in W$.

Then for each $u \in V(H)$ the m-code of u in H is equal to the m-code of u in G.

Let $m \geq 3$ and $n \geq 3$ be positive integers, and $G = P_m \square C_n$. Suppose

$$\begin{aligned} V(G) &= \{(i,j) \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\}, \text{ and} \\ E(G) &= \{\{(i,j), (i,(j+1) \bmod n)\} \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\} \\ &\qquad \cup \{\{(i,j),(i+1,j)\} \mid 0 \leq i \leq m-2, 0 \leq j \leq n-1\}. \end{aligned}$$

Theorem 10. *Let $m \geq 3$. Then $W = \{(0,0),(0,1),(2,0)\}$ is a m-resolving set of $P_m \square C_3$. Hence, $md(P_m \square C_3) = 3$.*

Proof: (By induction on m)
If $m = 3$, then Fig. 7 shows that the m-codes of the vertices are distinct, so the result holds. In the figure, the elements of W are shaded.

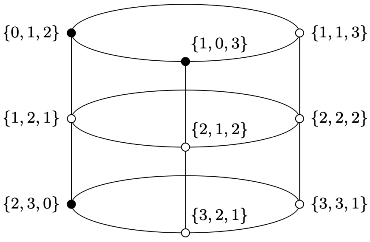

Figure 7. **The m-codes of the vertices in $P_3 \square C_3$ where $W = \{(0,0),(0,1),(2,0)\}$.**

Suppose the result holds for $m \geq 3$, and $G = P_{m+1} \square C_3$. Let $H = P_m \square C_3$ be the subgraph of G, induced by the vertices $\{(i,j) \mid 0 \leq i \leq m-1, 0 \leq j \leq n-1\}$. For each $u \in V(H)$, observe that the m-code of u does not change if we look at u as a vertex of G, by Observation 9. Hence, by inductive assumption, it is enough to show that the m-codes of the vertices (m,i), where $0 \leq i \leq 2$ are (a) distinct from each other and are (b) each distinct from the m-codes of other vertices of G. First, (a) is true since each element of $r_m(m,i)$ is one more than the corresponding element of $r_m(m-1,i)$. Second, (b) holds because the diameter of $P_m \square C_3$ is m, while each $r_m(m,i)$ has an element $m+1$. ☐

Theorem 6 can serve as a starting point for further future work, particularly on the determination of the multiset dimension of all cylindrical graphs $P_m \square C_n$, where $m \geq 3$ and $n \geq 3$. Aside from the multiset dimension, we can also consider the ID spectrum, as defined by Chartrand et al. in [3], of such graphs; that is, we consider the problem of finding all values of r for which $P_m \,\Box\, C_n$ has an m-resolving set with cardinality r.

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Data Availability: Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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