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Research Article

Robust optimality and duality for bilevel optimization problems under uncertain data

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Abstract: The exploration of robust bilevel programming problems is a relatively new development in optimization theory. In this study, we examine a bilevel optimization problem in which both the upper-level and lower-level constraints involve uncertainty. By reducing the problem to a single-level, nonlinear, and non-smooth program, we explore sufficient optimality conditions and duality theorems for robust optimal solutions of the considered non-smooth uncertain bilevel optimization problem, using Clarke subdifferentials. Leveraging the characteristics of Clarke subdifferentials, we propose Wolfe-type robust dual models. Additionally, we establish various duality theorems, including weak and strong robust duality, in terms of Clarke subdifferentials. Several illustrative examples are presented to confirm the applicability of the results developed.

Keywords: bilevel programming, robust optimization, sufficient optimality conditions, duality, robust convexity.

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1. Introduction

Bilevel programming is a prominent area of research in optimization theory, attracting considerable attention due to its diverse applications in fields such as engineering,

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finance, economics, and computer science, among others. These problems are inherently challenging, both theoretically and numerically. In a bilevel optimization problem, two nested optimization problems are defined, with one problem's solution constrained by the solution of the other. Specifically, the feasible solutions to the bilevel problem are the optimal lower-level solutions that satisfy the upper-level constraints. In recent years, bilevel programming has garnered significant attention from researchers [1, 6, 9–13, 15].
From both theoretical and practical perspectives, these papers have highlighted the potential of bilevel programming problems. The literature on this topic is extensive, covering a wide range of applications. For example, Zhang et al. [30] explored its use in addressing the watershed water trade problem. Wu and Chen [26] applied bilevel programming to the HTC smartphone product line design problem, demonstrating its practical utility. Ren [24] presented a case study on production planning as a real-world application of bilevel programming. Additionally, Xiao et al. [27] developed a dual-randomness bilevel interval multiobjective programming (DR-BIMP) model to facilitate water resource management across multiple sectors under conditions of complexity and uncertainty.
In real-world optimization problems, data are often uncertain due to prediction errors or incomplete information, meaning they are not precisely known at the time of solving the problem [3]. Robust optimization has become a prominent deterministic approach for addressing mathematical programming problems with uncertain data. Numerous researchers have extensively investigated both theoretical and practical aspects of robust optimization, as seen in studies such as [2, 4, 5, 14, 18–20, 25] and the references therein.
Recent studies have further expanded the applications of robust optimization across various domains. Zhang et al. [29] conducted a comprehensive survey of robust optimization approaches in inventory management, highlighting their applications and effectiveness in mitigating uncertainties inherent in supply chain operations. In the field of data privacy, Goseling and Lopuhaä-Zwakenberg [17] formulated robust optimization techniques to determine optimal data release protocols, ensuring resilience in local differential privacy scenarios. Lin et al. [21] present a detailed review of distributionally robust optimization, focusing on its theoretical foundations and diverse applications. They discuss various ambiguity set constructions and solution methodologies, illustrating the broad applicability of this approach in fields such as finance, logistics, and machine learning. These studies underscore the growing significance of robust optimization in effectively managing uncertainty across diverse disciplines.
For each $\mathfrak{r} \in \mathcal{R} = \{1, 2, \dots, p\}$, $\mathfrak{s} \in \mathcal{S} = \{1, 2, \dots, q\}$, let $\Omega_{\mathfrak{r}} \subseteq \mathbb{R}^{m_{\mathfrak{r}}}$ and $\Lambda_{\mathfrak{s}} \subseteq \mathbb{R}^{m_{\mathfrak{s}}}$ are nonempty convex, compact sets, where $m_{\mathfrak{r}}$ and $m_{\mathfrak{s}}$ are integers. This investigation focuses on the subsequent uncertain bilevel optimization problem $(\mathcal{H})$ of the form

$$(\mathcal{H}): \quad \begin{cases} \min\limits_{\mathfrak{z},\mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) \\ s.\ t. & \mathcal{T}_{\mathfrak{r}}(\mathfrak{z}, \xi_{\mathfrak{r}}) \leq 0, \quad \forall\ \mathfrak{r} \in \mathcal{R} = \{1, 2, \dots, p\}, \\ & \mathfrak{k} \in F_0(\mathfrak{z}), \end{cases}$$

where $\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}},\ \mathfrak{r} \in \mathcal{R} = \{1, 2, \ldots, p\}$ are uncertain parameters. For each $\mathfrak{z} \in \mathbb{R}^{n_1}$, $F_0(\mathfrak{z})$ represents the set of solutions to the following parametric optimization problem

$$(\mathcal{H}_{\mathfrak{z}}): \quad \begin{cases} \min\limits_{\mathfrak{k}} & \Upsilon(\mathfrak{z}, \mathfrak{k}) \\ s.\ t. & \zeta_{\mathfrak{s}}\big((\mathfrak{z}, \mathfrak{k}), \rho_{\mathfrak{s}}\big) \leq 0, \quad \forall\, \mathfrak{s} \in \mathcal{S} = \{1, 2, \ldots, q\}. \end{cases}$$

Where $\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}$ are uncertain parameters, $\Gamma,\ \Upsilon : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \longrightarrow \mathbb{R},\ \ \mathcal{T}_{\mathfrak{r}} : \mathbb{R}^{n_1} \times \Omega_{\mathfrak{r}} \longrightarrow \mathbb{R},\ \mathfrak{r} \in \mathcal{R}$ and $\zeta_{\mathfrak{s}} : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \Lambda_{\mathfrak{s}} \longrightarrow \mathbb{R},\ \mathfrak{s} \in \mathcal{S}$ are given functions. Here, p, q, n_1 and n_2 are integers. It is important to note that both the upper-level and lower-level constraints involve uncertainty.
The robust counterpart $(\mathcal{RH})$ of the uncertain bilevel optimization problem $(\mathcal{H})$ is defined as the following bilevel optimization problem

$$(\mathcal{RH}): \quad \begin{cases} \min\limits_{\mathfrak{z}, \mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) \\ s.\ t. & \mathcal{T}_{\mathfrak{r}}(\mathfrak{z}, \xi_{\mathfrak{r}}) \leq 0, \quad \forall\, \xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}} \ \ \forall\, \mathfrak{r} \in \mathcal{R}, \\ & \mathfrak{k} \in F(\mathfrak{z}), \end{cases}$$

where for each $\mathfrak{z} \in \mathbb{R}^{n_1}$, $F(\mathfrak{z})$ represents the set of solutions to the following parametric optimization problem

$$(\mathcal{RH}_{\mathfrak{z}}): \quad \begin{cases} \min\limits_{\mathfrak{k}} & \Upsilon(\mathfrak{z}, \mathfrak{k}) \\ s.\ t. & \zeta_{\mathfrak{s}}\big((\mathfrak{z}, \mathfrak{k}), \rho_{\mathfrak{s}}\big) \leq 0, \quad \forall\, \rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}} \ \ \forall\, \mathfrak{s} \in \mathcal{S}. \end{cases}$$

The robust counterpart model effectively handles worst-case uncertainty without directly depending on uncertain variables. Let

$$\mathcal{G} := \Big\{(\mathfrak{z}, \mathfrak{k}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} :\ \mathcal{T}_{\mathfrak{r}}(\mathfrak{z}, \xi_{\mathfrak{r}}) \leq 0, \quad \forall\, \xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}} \ \ \forall\, \mathfrak{r} \in \mathcal{R}, \quad \mathfrak{k} \in F(\mathfrak{z})\Big\}$$

is the feasible set of $(\mathcal{RH})$.
A vector $(\tilde{\mathfrak{z}}, \tilde{\mathfrak{k}})$ is a robust feasible solution of $(\mathcal{H})$ if it is a feasible solution of $(\mathcal{RH})$. On the other hand, a vector $(\tilde{\mathfrak{z}}, \tilde{\mathfrak{k}}) \in \mathcal{G}$ is said to be a robust optimal solution of $(\mathcal{H})$ if for every $(\mathfrak{z}, \mathfrak{k}) \in \mathcal{G}$

$$\Gamma(\mathfrak{z}, \mathfrak{k}) - \Gamma(\tilde{\mathfrak{z}}, \tilde{\mathfrak{k}}) \geq 0.$$

In general, problem $(\mathcal{RH})$ is nonconvex, and the Karush-Kuhn-Tucker (KKT) optimality conditions derived by Gadhi and Ohda [16] (as outlined in Theorem 2) are only necessary. This raises the question: under what additional assumptions do the KKT conditions become sufficient to guarantee optimality for problem $(\mathcal{RH})$?
By applying an optimal value reformulation, we transform the problem $(\mathcal{RH})$ into a single-level optimization problem $(\mathcal{HP})$ that is fully equivalent to the original problem $(\mathcal{RH})$. Building on the necessary optimality conditions provided in [16], we derive

sufficient optimality conditions for the uncertain bilevel optimization problem $(\mathcal{H})$ in terms of Clarke subdifferentials for robust optimal solutions. Additionally, we formulate a Wolfe-type robust dual problem $(\mathcal{WH})$ and establish various duality theorems. Several examples are provided to illustrate our findings. To the best of our knowledge, no prior research has explored sufficient optimality conditions and duality results for uncertain bilevel optimization problems without assuming concavity in scenarios where uncertainties exist at both levels. Consequently, the results presented in this paper are novel and contribute significantly to the field.
The rest of the paper is organized as follows: Section 2 provides basic definitions and preliminaries. In Section 3, we present sufficient optimality conditions for robust optimal solutions to the uncertain bilevel optimization problem $(\mathcal{H})$. Section 4 derives weak and strong robust duality theorems for the Wolfe-type robust dual problem. Finally, Section 5 offers concluding remarks and explores potential directions for future research.

2. Preliminaries

In this section, we state a few definitions, notations and results, which we will refer to later in the paper. In what follows throughout this work $\mathbb{R}^n$ denotes the standard n-dimensional Euclidean space. We write the inner product as $\langle \cdot, \cdot \rangle$ and the closed line segment joining a and b in $\mathbb{R}^n$ is given by $[a, b] = \{\mu a + (1-\mu)b : 0 \leq \mu \leq 1\}$. Here we shall use the notation $(a, b) = \{\mu a + (1-\mu)b : 0 < \mu < 1\}$ to represent the open line segment from $a \in \mathbb{R}^n$ to $b \in \mathbb{R}^n$.
Let $\mathfrak{E}$ be a nonempty subset of $\mathbb{R}^n$, we denote the convex hull and closure of $\mathfrak{E}$ by $co\,\mathfrak{E}$ and $cl\,\mathfrak{E}$ respectively.
The negative polar cone and the strictly negative polar cone of $\mathfrak{E}$ are defined as follows:

(i) $\mathfrak{E}^- := \big\{\mathfrak{z} \in \mathbb{R}^n |\ \langle \mathfrak{z}, e\rangle \leq 0,\ \forall\ e \in \mathfrak{E}\big\}$.

(ii) $\mathfrak{E}^s := \big\{\mathfrak{z} \in \mathbb{R}^n | \langle \mathfrak{z}, e\rangle < 0,\ \ \forall\ e \in \mathfrak{E}\backslash\{0\}\big\}$.

A function $\Gamma : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz around a point $\bar{\mathfrak{z}} \in \text{dom}\Gamma := \{\mathfrak{z} \in \mathbb{R}^n |\, \Gamma(\mathfrak{z}) \in \mathbb{R}\}$ if there exist a neighbourhood $\mathcal{N} \subseteq \mathbb{R}^n$ of $\bar{\mathfrak{z}}$ and a constant $\mathcal{C} > 0$ such that

$$|\Gamma(\mathfrak{z}) - \Gamma(\mathfrak{k})| \leq \mathcal{C}\ \|\mathfrak{z} - \mathfrak{k}\| \quad \forall\, \mathfrak{z}, \mathfrak{k} \in \mathcal{N},$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$.
The generalized directional derivative of Γ at $\bar{\mathfrak{z}}$ in the direction $\delta \in \mathbb{R}^n$ and the Clarke subdifferential of Γ at $\bar{\mathfrak{z}}$ are defined by

$$\Gamma^\circ(\bar{\mathfrak{z}}, \delta) := \limsup_{\substack{\mathfrak{z} \to \bar{\mathfrak{z}} \\ \mu \downarrow 0}} \frac{\Gamma(\mathfrak{z} + \mu\delta) - \Gamma(\mathfrak{z})}{\mu}$$

and

$$\partial_c \Gamma(\bar{\mathfrak{z}}) := \big\{\mathfrak{z}^* \in \mathbb{R}^n : \big\langle \mathfrak{z}^*, \delta \big\rangle \leq \Gamma^\circ(\bar{\mathfrak{z}}, \delta) \quad \forall\ \delta \in \mathbb{R}^n\big\}.$$

Note that $\partial_c\Gamma(\bar{\mathfrak{z}})$ is a nonempty convex and compact set. Moreover, as established in [8, Proposition 2.1.2], we have

$$\Gamma^\circ(\bar{\mathfrak{z}}, \delta) = \max\left\{\langle \mathfrak{z}^*, \delta\rangle \mid \mathfrak{z}^* \in \partial_c\Gamma(\bar{\mathfrak{z}})\right\} \;\; \forall\, \delta \in \mathbb{R}^n.$$

If Γ is convex and continuous at $\bar{\mathfrak{z}}$, then Γ is locally Lipschitz and $\Gamma'(\bar{\mathfrak{z}}, \delta) = \Gamma^\circ(\bar{\mathfrak{z}}, \delta)$ for all $\delta \in \mathbb{R}^n$, where $\delta \to \Gamma'(\bar{\mathfrak{z}}, \delta)$ is the standard directional derivative defined by

$$\delta \to \Gamma'(\bar{\mathfrak{z}}, \delta) := \limsup_{\tau\downarrow 0} \frac{\Gamma(\bar{\mathfrak{z}} + \tau\delta) - \Gamma(\bar{\mathfrak{z}})}{\tau}.$$

Thus, $\partial_c\Gamma(\bar{\mathfrak{z}})$ is precisely the subdifferential of Γ in the context of convex analysis, commonly denoted as $\partial\Gamma(\bar{\mathfrak{z}})$.

Lemma 1. *[8] Let $\Gamma : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz, then for any $\bar{\mathfrak{z}} \in \mathbb{R}^n$ and scalar $a \in \mathbb{R}$, the Clarke subdifferential of $a\Gamma$ at $\bar{\mathfrak{z}}$ is given by*

$$\partial_c(a\Gamma)(\bar{\mathfrak{z}}) = a\partial_c\Gamma(\bar{\mathfrak{z}}).$$

Lemma 2. *[8] Let $\Gamma_k : \mathbb{R}^n \to \mathbb{R}$, for $k = 1, \ldots, m$, are locally Lipschitz functions, $\bar{\mathfrak{z}} \in \mathbb{R}^n$ be an arbitrary. Then*

$$\partial_c(\Gamma_1 + \cdots + \Gamma_m)(\bar{\mathfrak{z}}) \subset \partial_c\Gamma_1(\bar{\mathfrak{z}}) + \cdots + \partial_c\Gamma_m(\bar{\mathfrak{z}}).$$

3. Sufficient optimality conditions

In this section, we derive sufficient optimality conditions for a robust optimal solution to problem $(\mathcal{H})$ using the optimistic approach.
Let $(\mathfrak{z}, \mathfrak{k}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. For each $\mathfrak{r} \in \mathcal{R}$, we define

$$\Phi_{\mathfrak{r}}(\mathfrak{z}) := \max_{\xi_{\mathfrak{r}}\in\Omega_{\mathfrak{r}}} \mathcal{T}_{\mathfrak{r}}(\mathfrak{z}, \xi_{\mathfrak{r}}), \tag{3.1}$$

and for each $\mathfrak{s} \in \mathcal{S}$, we define

$$\Psi_{\mathfrak{s}}(\mathfrak{z}, \mathfrak{k}) := \max_{\rho_{\mathfrak{s}}\in\Lambda_{\mathfrak{s}}} \zeta_{\mathfrak{s}}\big((\mathfrak{z}, \mathfrak{k}), \rho_{\mathfrak{s}}\big). \tag{3.2}$$

These functions will act as a crucial tool to help us manage the uncertainties within both the upper and lower constraints of the uncertain bilevel optimization problem $(\mathcal{H})$. It is easy to see that the bilevel optimization problem $(\mathcal{RH})$ can be equivalently reformulated as follows

$$\begin{cases} \min\limits_{\mathfrak{z},\mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) \\ s.\ t. & \Phi_{\mathfrak{r}}(\mathfrak{z}) \le 0, \quad \forall\, \mathfrak{r} \in \mathcal{R}, \\ & \mathfrak{k} \in F(\mathfrak{z}), \end{cases}$$

where, for each $\mathfrak{z} \in \mathbb{R}^{n_1}$, $F(\mathfrak{z})$ represents the set of solutions to the following parametric optimization problem

$$\begin{cases} \underset{\mathfrak{k}}{min} & \Upsilon(\mathfrak{z},\mathfrak{k}) \\ s.\ t. & \Psi_{\mathfrak{s}}(\mathfrak{z},\mathfrak{k}) \leq 0, \ \ \forall\ \mathfrak{s} \in \mathcal{S}. \end{cases}$$

The following assumption was used by Gadhi and Ohda [16] to derive the Karush-Kuhn-Tucker type necessary optimality conditions.

- The following assumption $(\mathcal{U})$ holds for $\bar{\mathfrak{z}} \in \Delta := \{\mathfrak{z} \in \mathbb{R}^{n_1} |\ \mathcal{T}_{\mathfrak{r}}(\mathfrak{z},\xi_{\mathfrak{r}}) \leq 0,\ \forall\ \xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}\ \ \forall\ \mathfrak{r} \in \mathcal{R}\}$, if there is an open neighborhood $\aleph_{\bar{\mathfrak{z}}}$ of $\bar{\mathfrak{z}}$ such that:
  - $(\mathcal{U}_1)$: For each $\mathfrak{z} \in \aleph_{\bar{\mathfrak{z}}}$, the function $t \in \Omega_{\mathfrak{r}} \longmapsto \mathcal{T}_{\mathfrak{r}}(\mathfrak{z},t) \in \mathbb{R}$ is upper semicontinuous and the function $\mathcal{T}_{\mathfrak{r}}$ is Lipschitz continuous with respect to the first argument on $\aleph_{\bar{\mathfrak{z}}}$ with constant $\mathcal{C}_{\mathfrak{r}} > 0$, i.e
$$\big|\mathcal{T}_{\mathfrak{r}}(\mathfrak{z}_0,\xi_{\mathfrak{r}}) - \mathcal{T}_{\mathfrak{r}}(\mathfrak{z}_1,\xi_{\mathfrak{r}})\big| \leq \mathcal{C}_{\mathfrak{r}}\|\mathfrak{z}_0 - \mathfrak{z}_1\| \quad \forall \mathfrak{z}_0,\mathfrak{z}_1 \in \aleph_{\bar{\mathfrak{z}}},\ \forall \xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}. \tag{3.3}$$
  - $(\mathcal{U}_2)$: The multifunction $(\mathfrak{z},\xi_{\mathfrak{r}}) \in \aleph_{\bar{\mathfrak{z}}} \times \Omega_{\mathfrak{r}} \rightrightarrows \partial_c\mathcal{T}_{\mathfrak{r}}(\cdot,\xi_{\mathfrak{r}})(\mathfrak{z}) \subset \mathbb{R}^{n_1}$ is closed at $(\bar{\mathfrak{z}},\bar{\xi_{\mathfrak{r}}})$, for each $\bar{\xi_{\mathfrak{r}}} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$, where
$$\Omega_{\mathfrak{r}}(\bar{\mathfrak{z}}) := \big\{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}} \mid \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}},\xi_{\mathfrak{r}}) = \Phi_{\mathfrak{r}}(\bar{\mathfrak{z}})\big\}. \tag{3.4}$$
- Assumption $(\mathcal{V})$ holds for the pair $(\bar{\mathfrak{z}},\bar{\mathfrak{k}}) \in \mathcal{G}$, if there exist open neighbourhoods $\aleph_{\bar{\mathfrak{z}}}$ and $\aleph_{\bar{\mathfrak{k}}}$ of $\bar{\mathfrak{z}}$ and $\bar{\mathfrak{k}}$, respectively, such that:
  - $(\mathcal{V}_1)$: For each $(\mathfrak{z},\mathfrak{k}) \in \aleph_{\bar{\mathfrak{z}}} \times \aleph_{\bar{\mathfrak{k}}}$, the function $\alpha \in \Lambda_{\mathfrak{s}} \longmapsto \zeta_{\mathfrak{s}}\big((\mathfrak{z},\mathfrak{k}),\alpha\big) \in \mathbb{R}$ is upper semicontinuous and the function $\zeta_{\mathfrak{s}}$ is Lipschitz continuous with respect to the first argument on $\aleph_{\bar{\mathfrak{z}}} \times \aleph_{\bar{\mathfrak{k}}}$ with constant $\mathcal{D}_{\mathfrak{s}} > 0$, i.e
$$\begin{aligned} \big|\zeta_{\mathfrak{s}}\big((\mathfrak{z}_0,\mathfrak{k}_0),\rho_{\mathfrak{s}}\big) - \zeta_{\mathfrak{s}}\big((\mathfrak{z}_1,\mathfrak{k}_1),\rho_{\mathfrak{s}}\big)\big| \leq \mathcal{D}_{\mathfrak{s}}\|(\mathfrak{z}_0,\mathfrak{k}_0) - (\mathfrak{z}_1,\mathfrak{k}_1)\| \\ \forall\ (\mathfrak{z}_0,\mathfrak{k}_0),(\mathfrak{z}_1,\mathfrak{k}_1) \in \aleph_{\bar{\mathfrak{z}}} \times \aleph_{\bar{\mathfrak{k}}},\ \forall\ \rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}. \end{aligned} \tag{3.5}$$
  - $(\mathcal{V}_2)$:The multifunction $\big((\mathfrak{z},\mathfrak{k}),\rho_{\mathfrak{s}}\big) \in \big(\aleph_{\bar{\mathfrak{z}}} \times \aleph_{\bar{\mathfrak{k}}}\big) \times \Lambda_{\mathfrak{s}} \rightrightarrows \partial_c\zeta_{\mathfrak{s}}(\cdot,\rho_{\mathfrak{s}})(\mathfrak{z},\mathfrak{k}) \subset \mathbb{R}_1 \times \mathbb{R}_2$ is closed at $\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\bar{\rho_{\mathfrak{s}}}\big)$, for each $\bar{\rho_{\mathfrak{s}}} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$, where
$$\Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}},\bar{\mathfrak{k}}) := \big\{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}} \mid \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}}\big) = \Psi_{\mathfrak{s}}(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\big\}. \tag{3.6}$$

According to Outrata [23], the problem $(\mathcal{RH})$ is equivalent to the following single-level optimization problem

$$(\mathcal{HP}):\begin{cases} \underset{\mathfrak{z},\mathfrak{k}}{min} & \Gamma(\mathfrak{z},\mathfrak{k}) \\ s.\ t. & \Phi_{\mathfrak{r}}(\mathfrak{z}) \le 0, \quad \forall\ \mathfrak{r}\in\mathcal{R}, \\ & \Psi_{\mathfrak{s}}(\mathfrak{z},\mathfrak{k}) \le 0, \quad \forall\ \mathfrak{s}\in\mathcal{S}, \\ & \varphi(\mathfrak{z},\mathfrak{k}) \le 0, \end{cases}$$

where

$$\varphi(\mathfrak{z},\mathfrak{k}) := \Upsilon(\mathfrak{z},\mathfrak{k}) - v(\mathfrak{z}) \qquad \forall\ (\mathfrak{z},\mathfrak{k}) \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2},$$

such that for any $\mathfrak{z}\in\mathbb{R}^{n_1}$,

$$v(\mathfrak{z}) = \min_{\mathfrak{k}} \left\{ \Upsilon(\mathfrak{z},\mathfrak{k}) : \Psi_{\mathfrak{s}}(\mathfrak{z},\mathfrak{k}) \le 0, \quad \forall\ \mathfrak{s}\in\mathcal{S} \right\}$$

is the optimal value function of the lower-level problem $(\mathcal{RH}_{\mathfrak{z}})$.
Note that the optimal value function v is non-smooth, and common constraint qualifications like Slater's and Mangasarian-Fromovitz do not hold at any feasible point of $(\mathcal{HP})$ [28, Proposition 3.1]. To address this challenge, we adopt the partial calmness approach proposed by Ye and Zhu [28].

Definition 1. [22] Let $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ be a local optimal solution of $(\mathcal{RH})$. We say that $(\mathcal{RH})$ is partially calm at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ if there exist $d>0$ and $a^*>0$ such that for each $(\mathfrak{z},\mathfrak{k},z)\in\mathcal{B}_d(\bar{\mathfrak{z}},\bar{\mathfrak{k}},0)$ satisfying

$$\begin{cases} \Phi_{\mathfrak{r}}(\mathfrak{z}) \le 0, & \mathfrak{r}\in\mathcal{R}, \\ \Psi_{\mathfrak{s}}(\mathfrak{z},\mathfrak{k}) \le 0, & \mathfrak{s}\in\mathcal{S}, \\ \varphi(\mathfrak{z},\mathfrak{k}) \le z, \end{cases}$$

we have $\Gamma(\mathfrak{z},\mathfrak{k}) - \Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}}) + a^*|z| \ge 0$.

Remark 1. [22] Partial calmness of $(\mathcal{RH})$ at one of its local minimizers $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ is equivalent to $(\mathfrak{z},\mathfrak{k}) \longmapsto \varphi(\mathfrak{z},\mathfrak{k})$ being a locally exact penalty function for the problem $(\mathcal{HP})$ at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$. For more details, see [22, Lemma 3.1].

The concept of partial calmness is strongly related to partial exact penalization, as shown in the following result, from [28].

Theorem 1. *[28] Let $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ be a local optimal solution to $(\mathcal{RH})$. Problem $(\mathcal{RH})$ is called partially calm at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ if and only if there exists $a^*>0$ such that $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ is a local optimal solution to the partially penalized problem*

$$(\mathcal{HP}_1):\begin{cases} \underset{\mathfrak{z},\mathfrak{k}}{min} & \Gamma(\mathfrak{z},\mathfrak{k}) + a^*\varphi(\mathfrak{z},\mathfrak{k}) \\ s.\ t. & \Phi_{\mathfrak{r}}(\mathfrak{z}) \le 0, \quad \mathfrak{r}\in\mathcal{R}, \\ & \Psi_{\mathfrak{s}}(\mathfrak{z},\mathfrak{k}) \le 0, \quad \mathfrak{s}\in\mathcal{S}. \end{cases}$$

Let $\mathbf{R} := \{1, \ldots, p+q\}$. Consider the functions $\phi : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and $\Theta : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{p+q}$ defined by

$$\phi(\mathfrak{z}, \mathfrak{k}) := \Gamma(\mathfrak{z}, \mathfrak{k}) + a^* \varphi(\mathfrak{z}, \mathfrak{k})$$

and

$$\Theta_{\mathfrak{r}}(\mathfrak{z}, \mathfrak{k}) := \begin{cases} \Phi_{\mathfrak{r}}(\mathfrak{z}), & \mathfrak{r} = 1, \ldots, p, \\ \Psi_{\mathfrak{r}-p}(\mathfrak{z}, \mathfrak{k}), & \mathfrak{r} = p+1, \ldots, p+q. \end{cases}$$

To derive the necessary optimality conditions, Gadhi and Ohda [16] proposed the following constraint qualification.

Definition 2. [16] We say that the Extended Non-smooth Mangasarian-Fromovitz constraint qualification (ENMFCQ) holds at the point $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathcal{G}$ if

$$\exists\, \delta \in \mathbb{R}^{n_1+n_2} \backslash \{0\} \;\text{ such that }\; \Theta^{\circ}_{\mathfrak{r}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \delta\big) < 0, \quad \forall\, \mathfrak{r} \in \mathbf{R}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}),$$

where

$$\mathbf{R}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) := \big\{\mathfrak{r} \in \mathbf{R} : \Theta_{\mathfrak{r}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = 0\big\}.$$

The following result, established by Gadhi and Ohda [16], provides KKT-type necessary optimality conditions.

Theorem 2. *[16] Let $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathcal{G}$ be a locally robust optimal solution for $(\mathcal{H})$. Assume that the functions Γ and Υ are locally Lipschitz continuous, that $(\mathcal{RH})$ is partial calm at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, that (ENMFCQ) is satisfied at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, and assumptions $(\mathcal{U})$ and $(\mathcal{V})$ hold for $\bar{\mathfrak{z}}$ and $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, respectively. Then, there exist $\mathfrak{x} > 0,\; a^* > 0, \mathfrak{y}_{\mathfrak{r}} \geq 0, \mathfrak{r} \in \mathcal{R}$ and $\mathfrak{f}_{\mathfrak{s}} \geq 0, \mathfrak{s} \in \mathcal{S}$, such that*

$$(0,0) \in \left\{ \begin{array}{r} \mathfrak{x}\partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) + \mathfrak{x} a^* \partial_c \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}}\; co\Bigg(\bigcup\limits_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})} \partial_c \mathcal{T}_{\mathfrak{r}}(\cdot, \xi_{\mathfrak{r}})(\bar{\mathfrak{z}}) \times \{0\} \Bigg) \\ + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}}\; co\Bigg(\bigcup\limits_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})} \partial_c \zeta_{\mathfrak{s}}\big(\cdot, \rho_{\mathfrak{s}}\big)(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Bigg) \end{array} \right\} \tag{3.7}$$

and

$$\begin{cases} \mathfrak{y}_{\mathfrak{r}} \max\limits_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \; \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}) = 0, & \mathfrak{r} \in \mathcal{R}, \\ \mathfrak{f}_{\mathfrak{s}} \max\limits_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \; \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}}\big) = 0, & \mathfrak{s} \in \mathcal{S}. \end{cases} \tag{3.8}$$

We introduce the definition of ∂_c-robust convexity within our framework, which aligns closely with the concept proposed by Chong [7].

Definition 3. We say that $\Gamma(\cdot,\cdot)$, $\varphi(\cdot,\cdot)$, $\zeta_{\mathfrak{s}}\big((\cdot,\cdot),\; \rho_{\mathfrak{s}}\big)$ and $\mathcal{T}_{\mathfrak{r}}(\cdot, \xi_{\mathfrak{r}})$ are ∂_c-robust convex at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \in \mathcal{G}$, if for all $(\mathfrak{z}, \mathfrak{k}) \in \mathcal{G}$,

$$\Gamma(\mathfrak{z}, \mathfrak{k}) - \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \geq \Big\langle \vartheta,\; (\mathfrak{z}, \mathfrak{k}) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big\rangle,\; \forall\, \vartheta \in \partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}),$$

$$\varphi(\mathfrak{z},\mathfrak{k}) - \varphi(\bar{\mathfrak{z}},\bar{\mathfrak{k}}) \geq \Big\langle \eta,\ (\mathfrak{z},\mathfrak{k}) - (\bar{\mathfrak{z}},\bar{\mathfrak{k}}) \Big\rangle,\ \forall\, \eta \in \partial_c \varphi(\bar{\mathfrak{z}},\bar{\mathfrak{k}}),$$

$$\zeta_{\mathfrak{s}}\big((\mathfrak{z},\mathfrak{k}),\rho_{\mathfrak{s}}\big) - \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}}\big) \geq \Big\langle \lambda_{\mathfrak{s}},\ (\mathfrak{z},\mathfrak{k}) - (\bar{\mathfrak{z}},\bar{\mathfrak{k}}) \Big\rangle,\ \forall\, \lambda_{\mathfrak{s}} \in \partial_c \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}}\big),$$
$$\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\ \forall\ \mathfrak{s} \in \mathcal{S},$$

$$\mathcal{T}_{\mathfrak{r}}(\mathfrak{z},\xi_{\mathfrak{r}}) - \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}},\xi_{\mathfrak{r}}) \geq \Big\langle \beta_{\mathfrak{r}},\ (\mathfrak{z},0) - (\bar{\mathfrak{z}},0) \Big\rangle,\ \forall\, \beta_{\mathfrak{r}} \in \partial_c \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}},\xi_{\mathfrak{r}}),$$
$$\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}}),\quad \forall\ \mathfrak{r} \in \mathcal{R}.$$

Now, we derive and prove the sufficient optimality conditions for a robust optimal solution of the uncertain bilevel optimization problem $(\mathcal{H})$ under ∂_c-robust convexity assumptions.

Theorem 3. *Let* $(\bar{\mathfrak{z}},\bar{\mathfrak{k}}) \in \mathcal{G}$. *Further, assume that* $\Gamma(\cdot,\cdot),\ \varphi(\cdot,\cdot),\ \zeta_{\mathfrak{s}}\big((\cdot,\cdot),\rho_{\mathfrak{s}}\big), \rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ *and* $\mathcal{T}_{\mathfrak{r}}(\cdot,\xi_{\mathfrak{r}}),\ \xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$ *are* ∂_c*-robust convex at* $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ *on* $\mathcal{G}$. *Moreover, suppose there exist* $\mathfrak{x} > 0,\ a^* > 0, \mathfrak{y}_{\mathfrak{r}} \geq 0, \mathfrak{r} \in \mathcal{R}$ *and* $\mathfrak{f}_{\mathfrak{s}} \geq 0, \mathfrak{s} \in \mathcal{S}$, *satisfying (3.7) and (3.8). Then,* $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ *is a robust optimal solution of* $(\mathcal{H})$.

Proof. Contrary to the result, suppose that $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ is not a robust optimal solution of $(\mathcal{H})$, then there exists other $(\mathfrak{z}_0,\mathfrak{k}_0) \in \mathcal{G}$ such that

$$\Gamma(\mathfrak{z}_0,\mathfrak{k}_0) - \Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}}) < 0.$$

As $\mathfrak{x} > 0$, we get

$$\mathfrak{x}\Big[\Gamma(\mathfrak{z}_0,\mathfrak{k}_0) - \Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\Big] < 0. \tag{3.9}$$

Since $(\bar{\mathfrak{z}},\bar{\mathfrak{k}}) \in \mathcal{G}$ satisfies (3.7) and (3.8), there exist $\mathfrak{x} > 0,\ \vartheta \in \partial_c \Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\ a^* > 0,\ \eta \in \partial_c \varphi(\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\ \mathfrak{y}_{\mathfrak{r}} \geq 0,\ \mathfrak{r} \in \mathcal{R},\ \bar{u}_{\mathfrak{r}i} \geq 0,\ \beta_{\mathfrak{r}i} \in \partial_c \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}},\xi_{\mathfrak{r}i}),\ \xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}}), i \in I_{\mathfrak{r}} := \{1,\ldots,n_{\mathfrak{r}}\}, n_{\mathfrak{r}} \in \mathbb{N}$, and $\mathfrak{f}_{\mathfrak{s}} \geq 0,\ \mathfrak{s} \in \mathcal{S},\ \bar{w}_{\mathfrak{s}j} \geq 0,\ \lambda_{\mathfrak{s}j} \in \partial_c \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}j}\big),\ \rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}},\bar{\mathfrak{k}}), j \in J_{\mathfrak{s}} := \{1,\ldots,n_{\mathfrak{s}}\}, n_{\mathfrak{s}} \in \mathbb{N}$, such that

$$\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} = 1, \qquad \sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} = 1,$$

$$0 = \mathfrak{x}\,\vartheta + \mathfrak{x}a^*\eta + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\,\beta_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\,\lambda_{\mathfrak{s}j} \right) \tag{3.10}$$

and

$$\mathfrak{y}_{\mathfrak{r}} \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}},\xi_{\mathfrak{r}}) = 0, \qquad \mathfrak{r} \in \mathcal{R}, \tag{3.11a}$$

$$\mathfrak{f}_{\mathfrak{s}} \max_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}}\big) = 0, \qquad \mathfrak{s} \in \mathcal{S}. \tag{3.11b}$$

Since $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$,

$$\mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) = \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}), \qquad \forall\, i \in I_{\mathfrak{r}}, \ \ \forall\, \mathfrak{r} \in \mathcal{R}.$$

Thus, it follows by (3.11a) that

$$\mathfrak{y}_{\mathfrak{r}} \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) = 0, \qquad \forall\, i \in I_{\mathfrak{r}}, \ \ \forall\, \mathfrak{r} \in \mathcal{R}. \tag{3.12}$$

And since $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$,

$$\zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) = \max_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}}\big) \qquad \forall\, j \in J_{\mathfrak{s}}, \ \forall\, \mathfrak{s} \in \mathcal{S}.$$

Thus, it follows by (3.11b) that

$$\mathfrak{f}_{\mathfrak{s}} \ \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) = 0 \qquad \forall\, j \in J_{\mathfrak{s}}, \ \forall\, \mathfrak{s} \in \mathcal{S}. \tag{3.13}$$

Since $\Gamma(\cdot,\cdot)$, $\varphi(\cdot,\cdot)$, $\zeta_{\mathfrak{s}}\big((\cdot,\cdot),\rho_{\mathfrak{s}}\big)$, $\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ and $\mathcal{T}_{\mathfrak{r}}(\cdot, \xi_{\mathfrak{r}})$, $\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$ are ∂_c-robust convex functions at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ on G, by Definition 3, for any $\vartheta \in \partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, $a^* > 0$, $\eta \in \partial_c \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, $\mathfrak{y}_{\mathfrak{r}} \geq 0$, $\mathfrak{r} \in \mathcal{R}$, $\bar{u}_{\mathfrak{r}i} \geq 0$, $\beta_{\mathfrak{r}i} \in \partial_c \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i})$, $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}}), i \in I_{\mathfrak{r}} := \{1, \ldots, n_{\mathfrak{r}}\}, n_{\mathfrak{r}} \in \mathbb{N}$, and $\mathfrak{f}_{\mathfrak{s}} \geq 0$, $\mathfrak{s} \in \mathcal{S}$, $\bar{w}_{\mathfrak{s}j} \geq 0$, $\lambda_{\mathfrak{s}j} \in \partial_c \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big)$, $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), j \in J_{\mathfrak{s}} := \{1, \ldots, n_{\mathfrak{s}}\}, n_{\mathfrak{s}} \in \mathbb{N}$,

$$\begin{aligned}
0 &= \mathfrak{x}\Big\langle \vartheta,\ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big\rangle + \mathfrak{x} a^* \Big\langle \eta,\ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big\rangle \\
&+ \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \Big\langle \beta_{\mathfrak{r}i},\ (\mathfrak{z}_0, 0) - (\bar{\mathfrak{z}}, 0) \Big\rangle \right) \\
&+ \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \Big\langle \lambda_{\mathfrak{s}j},\ (\mathfrak{z}_0, \mathfrak{k}_0) - (\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big\rangle \right) \\
&\leq \mathfrak{x}\Big[\Gamma(\mathfrak{z}_0, \mathfrak{k}_0) - \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big] + \mathfrak{x} a^* \Big[\varphi(\mathfrak{z}_0, \mathfrak{k}_0) - \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big] \\
&+ \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \Big[\mathcal{T}_{\mathfrak{r}}(\mathfrak{z}_0, \xi_{\mathfrak{r}i}) - \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) \Big] \right) \\
&+ \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \Big[\zeta_{\mathfrak{s}}\big((\mathfrak{z}_0, \mathfrak{k}_0), \rho_{\mathfrak{s}j}\big) - \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) \Big] \right).
\end{aligned}$$

This implies that

$$\begin{aligned}
0 &\leq \mathfrak{x}\Big[\Gamma(\mathfrak{z}_0, \mathfrak{k}_0) - \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big] + \mathfrak{x} a^* \Big[\varphi(\mathfrak{z}_0, \mathfrak{k}_0) - \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \Big] \\
&+ \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \Big[\mathcal{T}_{\mathfrak{r}}(\mathfrak{z}_0, \xi_{\mathfrak{r}i}) - \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) \Big] \right) \\
&+ \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \Big[\zeta_{\mathfrak{s}}\big((\mathfrak{z}_0, \mathfrak{k}_0), \rho_{\mathfrak{s}j}\big) - \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) \Big] \right).
\end{aligned}$$

Thus

$$\begin{aligned}\mathfrak{x}\Big[\Gamma(\mathfrak{z}_0,\mathfrak{k}_0)-\Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\Big] \geq &-\mathfrak{x}a^*\Big[\varphi(\mathfrak{z}_0,\mathfrak{k}_0)-\varphi(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\Big]\\ &-\sum_{\mathfrak{r}=1}^{p}\mathfrak{y}_{\mathfrak{r}}\left(\sum_{i=1}^{n_{\mathfrak{r}}}\bar{u}_{\mathfrak{r}i}\,\Big[\mathcal{T}_{\mathfrak{r}}(\mathfrak{z}_0,\xi_{\mathfrak{r}i})-\mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}},\xi_{\mathfrak{r}i})\Big]\right)\\ &-\sum_{\mathfrak{s}=1}^{q}\mathfrak{f}_{\mathfrak{s}}\left(\sum_{j=1}^{n_{\mathfrak{s}}}\bar{w}_{\mathfrak{s}j}\,\Big[\zeta_{\mathfrak{s}}\big((\mathfrak{z}_0,\mathfrak{k}_0),\rho_{\mathfrak{s}j}\big)-\zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}j}\big)\Big]\right).\end{aligned}$$

Consequently,

$$\begin{aligned}\mathfrak{x}\Big[\Gamma(\mathfrak{z}_0,\mathfrak{k}_0)-\Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\Big] \geq &-\mathfrak{x}a^*\Big[\varphi(\mathfrak{z}_0,\mathfrak{k}_0)-\varphi(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\Big]\\ &-\sum_{\mathfrak{r}=1}^{p}\left(\sum_{i=1}^{n_{\mathfrak{r}}}\bar{u}_{\mathfrak{r}i}\mathfrak{y}_{\mathfrak{r}}\,\Big[\mathcal{T}_{\mathfrak{r}}(\mathfrak{z}_0,\xi_{\mathfrak{r}i})-\mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}},\xi_{\mathfrak{r}i})\Big]\right)\\ &-\sum_{\mathfrak{s}=1}^{q}\left(\sum_{j=1}^{n_{\mathfrak{s}}}\bar{w}_{\mathfrak{s}j}\mathfrak{f}_{\mathfrak{s}}\,\Big[\zeta_{\mathfrak{s}}\big((\mathfrak{z}_0,\mathfrak{k}_0),\rho_{\mathfrak{s}j}\big)-\zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_{\mathfrak{s}j}\big)\Big]\right).\end{aligned}$$

According to (3.12) and (3.13), since $(\mathfrak{z}_0,\mathfrak{k}_0)\in\mathcal{G}$, it follows that

$$\mathfrak{x}\Big[\Gamma(\mathfrak{z}_0,\mathfrak{k}_0)-\Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\Big]\geq 0,$$

which contradicts (3.9). Therefore, the proof is complete. □

Algorithm 1 An algorithm for finding robust optimal solution of the problem $(\mathcal{H})$

Step 1. Input $\Gamma(\cdot,\cdot),\mathcal{T}_{\mathfrak{r}}(\cdot,\xi_{\mathfrak{r}}),\Upsilon,\zeta_{\mathfrak{s}}\big((\cdot,\cdot),\rho_{\mathfrak{s}}\big),\mathfrak{r}\in\mathcal{R},\xi_{\mathfrak{r}}\in\Omega_{\mathfrak{r}},\mathfrak{s}\in\mathcal{S}$ and $\rho_{\mathfrak{s}}\in\Lambda_{\mathfrak{s}}$ in $(\mathcal{H})$.
Step 2. Compute the value function $v(\mathfrak{z})$ of the lower level problem $(\mathcal{H}_{\mathfrak{z}})$.
Step 3. Choose a pair of point $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})\in\mathcal{G}$ in the feasible region of $(\mathcal{RH})$.
Step 4. Check the functions $\Gamma,\mathcal{T}_{\mathfrak{r}}(\cdot,\xi_{\mathfrak{r}}),\Upsilon,\zeta_{\mathfrak{s}}(\cdot,\mathfrak{k}),\mathfrak{r}\in\mathcal{R},\xi_{\mathfrak{r}}\in\Omega_{\mathfrak{r}},\mathfrak{s}\in\mathcal{S}$ and $\rho_{\mathfrak{s}}\in\Lambda_{\mathfrak{s}}$ are locally Lipschitz continuous at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$.
Step 5. Check the ∂_c-robust convexity of $\Gamma(\cdot,\cdot),\ \varphi(\cdot,\cdot),\ \zeta_{\mathfrak{s}}\big((\cdot,\cdot),\rho_{\mathfrak{s}}\big),\ \rho_{\mathfrak{s}}\in\Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ and $\mathcal{T}_{\mathfrak{r}}(\cdot,\xi_{\mathfrak{r}}),\ \xi_{\mathfrak{r}}\in\Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$ at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$.
Step 6. Choose the multipliers $\mathfrak{x}>0,\ a^*>0,\mathfrak{y}_{\mathfrak{r}}\geq 0,\mathfrak{r}\in\mathcal{R}$ and $\mathfrak{f}_{\mathfrak{s}}\geq 0,\mathfrak{s}\in\mathcal{S}$.
Step 7. The point $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ will be robust optimal solution of $(\mathcal{H})$.

Now, we present an example of a non-smooth uncertain bilevel optimization problem. This example illustrates the sufficient optimality conditions outlined in Theorem 3.

Example 1. Let $\Omega_1 = [0,1], \Lambda_1 = [-\frac{1}{2}, 1], \Gamma(\mathfrak{z}, \mathfrak{k}) = \frac{1}{3}\mathfrak{z} + \frac{1}{3}|\mathfrak{k}|, \mathcal{T}_1(\mathfrak{z}, \xi_1) = -\mathfrak{z} - \xi_1, \Upsilon(\mathfrak{z}, \mathfrak{k}) = \sqrt{\mathfrak{z}} + \mathfrak{k} + 2$, and $\zeta_1\big((\mathfrak{z}, \mathfrak{k}), \rho_1\big) = \mathfrak{k}^2 - \mathfrak{k} + ln(1+\rho_1)$. Consider the following uncertain bilevel optimization problem

$$(\mathcal{EH}): \quad \begin{cases} \min\limits_{\mathfrak{z},\mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) \\ s.\ t. & \mathcal{T}_1(\mathfrak{z}, \xi_1) \le 0, \\ & (\mathfrak{z}, \mathfrak{k}) \in \mathbb{R} \times \mathbb{R}, \quad \mathfrak{k} \in F_0(\mathfrak{z}), \end{cases}$$

where for each $\mathfrak{z} \in \mathbb{R}^{n_1}$, $F_0(\mathfrak{z})$ represents the set of solutions to the following parametric optimization problem

$$(\mathcal{EH}_{\mathfrak{z}}): \quad \begin{cases} \min\limits_{\mathfrak{k}} & \Upsilon(\mathfrak{z}, \mathfrak{k}) \\ s.\ t. & \zeta_1\big((\mathfrak{z}, \mathfrak{k}), \rho_1\big) \le 0, \end{cases}$$

with $\xi_1 \in \Omega_1$ and $\rho_1 \in \Lambda_1$.

⋆ The robust counterpart of $(\mathcal{EH})$ is the bilevel optimization problem

$$(\mathcal{REH}): \quad \begin{cases} \min\limits_{\mathfrak{z},\mathfrak{k}} & \Gamma(\mathfrak{z}, \mathfrak{k}) \\ s.\ t. & \mathcal{T}_1(\mathfrak{z}, \xi_1) \le 0, \quad \forall\ \xi_1 \in \Omega_1, \\ & \mathfrak{k} \in F(\mathfrak{z}), \end{cases}$$

where for each $\mathfrak{z} \in \mathbb{R}^{n_1}$, $F(\mathfrak{z})$ represents the set of solutions to the following parametric optimization problem

$$(\mathcal{REH}_{\mathfrak{z}}): \quad \begin{cases} \min\limits_{\mathfrak{k}} & \Upsilon(\mathfrak{z}, \mathfrak{k}) \\ s.\ t. & \zeta_1\big((\mathfrak{z}, \mathfrak{k}), \rho_1\big) \le 0, \quad \forall\ \rho_1 \in \Lambda_1. \end{cases}$$

In this case, we have $\mathcal{R} = \{1\}, \mathcal{S} = \{1\}, F(\mathfrak{z}) = \{0\}, v(\mathfrak{z}) = \sqrt{\mathfrak{z}} + 2$ and

$$\begin{aligned} \Phi_1(\mathfrak{z}) &= \max_{\xi_1 \in \Omega_1} \mathcal{T}_1(\mathfrak{z}, \xi_1) = -\mathfrak{z}, \\ \Psi_1(\mathfrak{z}, \mathfrak{k}) &= \max_{\rho_1 \in \Lambda_1} \zeta_{\mathfrak{s}}\big((\mathfrak{z}, \mathfrak{k}), \rho_1\big) = \mathfrak{k}^2 - \mathfrak{k}, \\ \varphi(\mathfrak{z}, \mathfrak{k}) &= \mathfrak{k}. \end{aligned}$$

As a consequence,

$$\mathcal{G} = \Big\{(\mathfrak{z}, 0) \in \mathbb{R} \times \mathbb{R} \mid \mathfrak{z} \ge 0\Big\},$$

and

$$\Theta_1(\mathfrak{z}, \mathfrak{k}) = -\mathfrak{z}, \quad \Theta_2(\mathfrak{z}, \mathfrak{k}) = \mathfrak{k}^2 - \mathfrak{k}.$$

Observe that $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = (0,0) \in \mathcal{G}$ and that assumption $(\mathcal{U})$ and $(\mathcal{V})$ are satisfied for $\bar{\mathfrak{z}}$ and $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$, respectively. Furthermore, we have

$$\mathbf{R}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = \{1, 2\}, \quad \Omega_1(\bar{\mathfrak{z}}) = \{0\}, \text{ and } \Lambda_1(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = \{0\}.$$

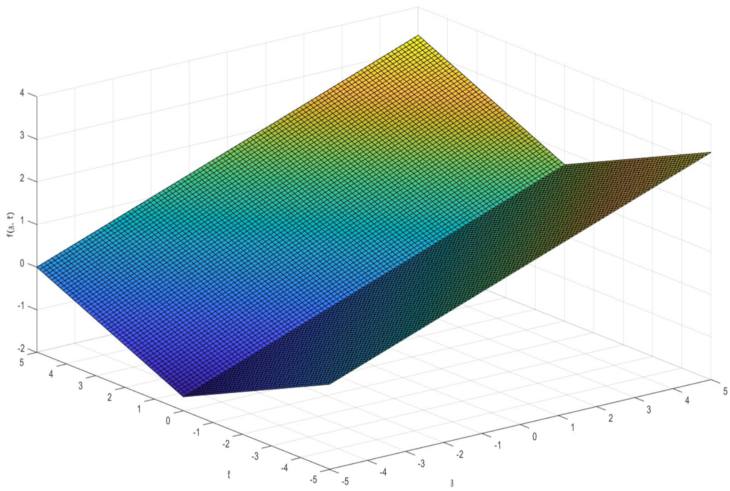

Figure 1. **The plot of the objective function $\Gamma(\mathfrak{z},\mathfrak{k})=\frac{1}{3}\mathfrak{z}+\frac{1}{3}|\mathfrak{k}|$ of the $(\mathcal{EH})$ in Example 1.**

The sets

$$\partial_c\Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})=\left\{\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{3},-\frac{1}{3}\right)\right\},\ \partial_c\varphi(\bar{\mathfrak{z}},\bar{\mathfrak{k}})=\{(0,1)\},$$

$$\partial_c\mathcal{T}_1(\cdot,0)(\bar{\mathfrak{z}})=\{-1\},\quad\text{and}\quad\partial_c\zeta_1\big((\cdot,\cdot),0\big)(\bar{\mathfrak{z}},\bar{\mathfrak{k}})=\{(0,-1)\}$$

are the Clarke subdifferentials of Γ, φ, $\mathcal{T}_1$ and ζ_1 at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$.

$\star$ Note that, by Definition 3 that $\Gamma(\cdot,\cdot)$, $\varphi(\cdot,\cdot)$, $\mathcal{T}_1(\cdot,\xi_1)$ and $\zeta_1\big((\cdot,\cdot),\rho_1\big)$ are ∂_c-robust convex at $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$.

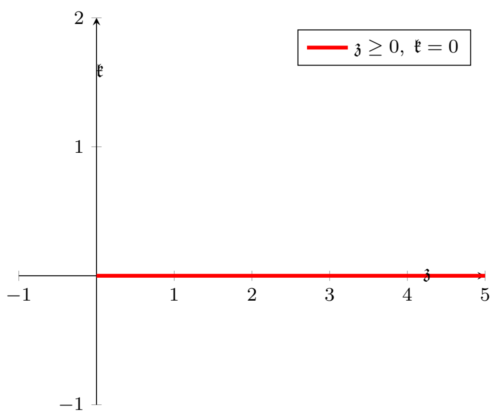

Figure 2. **The red line represent the feasible region of Example 1.**

The constraint qualification (ENMFCQ) is satisfied at the point $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$. Specifically, by selecting $\delta = (\delta_1, \delta_2) = (1, 2) \neq (0, 0)$, we obtain

$$\Theta_1^\circ\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \delta\big) = -\delta_1 = -1 < 0$$

and

$$\Theta_2^\circ\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \delta\big) = -\delta_2 = -2 < 0.$$

Since $(\mathcal{REH})$ is partially calm at $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$. Indeed, for $d = 1 > 0$ and $a^* = 2 > 0$ and $(\mathfrak{z}, \mathfrak{k}, z) \in \mathcal{B}_d(0, 0, 0)$ satisfying

$$\begin{aligned} \Phi_1(\mathfrak{z}) &= -\mathfrak{z} \leq 0, \\ \Psi_1(\mathfrak{z}, \mathfrak{k}) &= \mathfrak{k}^2 - \mathfrak{k} \leq 0, \\ \varphi(\mathfrak{z}, \mathfrak{k}) &= \mathfrak{k} \leq z, \end{aligned}$$

we have

$$\begin{aligned} \Gamma(\mathfrak{z}, \mathfrak{k}) - \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) + a^*|z| &= \frac{1}{3}\mathfrak{z} + \frac{1}{3}|\mathfrak{k}| + 2|z| \\ &= \frac{1}{3}\mathfrak{z} + \frac{1}{3}|\mathfrak{k}| + 2|\mathfrak{k}| \geq 0. \end{aligned}$$

⋆ For $\mathfrak{x} = \frac{1}{2}$, $a^* = 2, \mathfrak{y}_1 = \frac{1}{6}$ and $\mathfrak{f}_1 = \frac{7}{6}$. As a result, inclusion (3.7) and equality (3.8) are valid.

Hence, by Theorem 3, it follows that $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ is robust optimal solution of $(\mathcal{EH})$.

4. Duality in robust bilevel optimization

In this section, we introduce a Wolfe-type robust dual problem for the primal problem $(\mathcal{RH})$. We then investigate weak and strong robust duality results under ∂_c-robust convexity assumptions. To begin with, we should define

$$\mathbb{R}_+^{\mathbb{N}_1} := \Big\{\mathfrak{y} := (\mathfrak{y}_\mathfrak{r}, \bar{u}_{\mathfrak{r}i}), \mathfrak{r} = 1, \ldots, p,\ i = 1, \ldots, n_\mathfrak{r} | n_\mathfrak{r} \in \mathbb{N}, \mathfrak{y}_\mathfrak{r} \geq 0, \bar{u}_{\mathfrak{r}i} \geq 0,\ \sum_{i=1}^{n_\mathfrak{r}} \bar{u}_{\mathfrak{r}i} = 1\Big\}$$

and

$$\mathbb{R}_+^{\mathbb{N}_2} := \Big\{\mathfrak{f} := (\mathfrak{f}_\mathfrak{s}, \bar{w}_{\mathfrak{s}j}), \mathfrak{s} = 1, \ldots, q,\ j = 1, \ldots, n_\mathfrak{s} | n_\mathfrak{s} \in \mathbb{N}, \mathfrak{f}_\mathfrak{s} \geq 0, \bar{w}_{\mathfrak{s}j} \geq 0, \sum_{j=1}^{n_\mathfrak{s}} \bar{w}_{\mathfrak{s}j} = 1\Big\}.$$

Let $(\mathfrak{g},\mathfrak{l}) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. We define $(\mathcal{WH})$ as the Wolfe-type robust dual of the primal problem $(\mathcal{RH})$, which is formulated as follows:

$$(\mathcal{WH}) \begin{cases} \max\limits_{\mathfrak{g},\mathfrak{l}} \Big\{ \Gamma(\mathfrak{g},\mathfrak{l}) + \mathfrak{x} a^* \varphi(\mathfrak{g},\mathfrak{l}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \Big(\sum_{i=1}^{n_\mathfrak{r}} \bar{u}_{\mathfrak{r}i}\ \mathcal{T}_{\mathfrak{r}}(\mathfrak{g},\xi_{\mathfrak{r}i}) \Big) \\ \qquad\qquad + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \Big(\sum_{j=1}^{n_\mathfrak{s}} \bar{w}_{\mathfrak{s}j}\ \zeta_{\mathfrak{s}}\big((\mathfrak{g},\mathfrak{l}),\rho_{\mathfrak{s}j}\big) \Big) \Big\} \\ s.\ t. \\ \quad (0,0) \in \partial_c \Gamma(\mathfrak{g},\mathfrak{l}) + \mathfrak{x} a^* \partial_c \varphi(\mathfrak{g},\mathfrak{l}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \Big(\sum_{i=1}^{n_\mathfrak{r}} \bar{u}_{\mathfrak{r}i}\ \beta_{\mathfrak{r}i} \Big) \\ \qquad\qquad + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \Big(\sum_{j=1}^{n_\mathfrak{s}} \bar{w}_{\mathfrak{s}j}\ \lambda_{\mathfrak{s}j} \Big), \\ \lambda_{\mathfrak{s}j} \in \Big\{ \cup\, \partial_c \zeta_{\mathfrak{s}}\big((\mathfrak{g},\mathfrak{l}),\rho_{\mathfrak{s}j}\big),\ \rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\mathfrak{g},\mathfrak{l}) \Big\},\ \beta_{\mathfrak{r}i} \in \Big\{ \cup\, \partial_c \mathcal{T}_{\mathfrak{r}}(\mathfrak{g},\xi_{\mathfrak{r}i}),\ \xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\mathfrak{g}) \Big\}, \\ \quad \Pi^* = \Big\{ \big(\mathfrak{x},\ \mathfrak{y},\ \mathfrak{f}\big) :\ \mathfrak{x} > 0,\ \mathfrak{y} \geq 0,\ \mathfrak{f} \geq 0 \Big\}, \end{cases}$$

where $\Omega_{\mathfrak{r}}(\mathfrak{g})$ is defined as in (3.4) by replacing $\bar{\mathfrak{z}}$ with $\mathfrak{g}$ and $\Lambda_{\mathfrak{s}}(\mathfrak{g},\mathfrak{l})$ is defined as in (3.6) by replacing $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})$ with $(\mathfrak{g},\mathfrak{l})$.
The set $\mathcal{G}_{\mathcal{WH}}$ of all feasible points of $(\mathcal{WH})$ is defined as:

$$\mathcal{G}_{\mathcal{WH}} = \Bigg\{ \Big((\mathfrak{g},\mathfrak{l}), \Pi^* \Big) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R} \times \mathbb{R}_+^{\mathbb{N}_1} \times \mathbb{R}_+^{\mathbb{N}_2} \Big|\ (0,0) \in \partial_c \Gamma(\mathfrak{g},\mathfrak{l}) + \mathfrak{x} a^* \partial_c \varphi(\mathfrak{g},\mathfrak{l}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_\mathfrak{r}} \bar{u}_{\mathfrak{r}i}\ \beta_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_\mathfrak{s}} \bar{w}_{\mathfrak{s}j}\ \lambda_{\mathfrak{s}j} \right), \Pi^* = \Big\{ \big(\mathfrak{x},\ \mathfrak{y},\ \mathfrak{f}\big) :\ \mathfrak{x} > 0,\ \mathfrak{y} \geq 0,\ \mathfrak{f} \geq 0 \Big\} \Bigg\}.$$

The following theorem presents a result on weak robust duality.

Theorem 4. ***(Weak Robust Duality):*** *Let* $(\mathfrak{z}_0,\mathfrak{k}_0) \in \mathcal{G}$ *and* $((\mathfrak{g},\mathfrak{l}),\ \Pi^*) \in \mathcal{G}_{\mathcal{WH}}$. *If* $\Gamma(\cdot,\cdot)$, $\varphi(\cdot,\cdot)$, $\zeta_{\mathfrak{s}}\big((\cdot,\cdot),\rho_{\mathfrak{s}}\big)$ *and* $\mathcal{T}_{\mathfrak{r}}(\cdot,\xi_{\mathfrak{r}})$ *are* ∂_c*-robust convex at* $(\mathfrak{g},\mathfrak{l})$ *on* $(\mathcal{WH})$, *then*

$$\Gamma(\mathfrak{z}_0,\mathfrak{k}_0) \geq \Gamma(\mathfrak{g},\mathfrak{l}) + \mathfrak{x} a^* \varphi(\mathfrak{g},\mathfrak{l}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_\mathfrak{r}} \bar{u}_{\mathfrak{r}i}\ \mathcal{T}_{\mathfrak{r}}(\mathfrak{g},\xi_{\mathfrak{r}i}) \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_\mathfrak{s}} \bar{w}_{\mathfrak{s}j}\ \zeta_{\mathfrak{s}}\big((\mathfrak{g},\mathfrak{l}),\rho_{\mathfrak{s}j}\big) \right). \tag{4.1}$$

Proof. Since $((\mathfrak{g},\mathfrak{l}),\ \Pi^*) \in \mathcal{G}_{\mathcal{WH}}$, there exist $\mathfrak{x} > 0$, $\vartheta \in \partial_c \Gamma(\mathfrak{g},\mathfrak{l})$, $a^* > 0$, $\eta \in \partial_c \varphi(\mathfrak{g},\mathfrak{l})$, $\mathfrak{y}_{\mathfrak{r}} \geq 0$, $\mathfrak{r} \in \mathcal{R}$, $\bar{u}_{\mathfrak{r}i} \geq 0$, $\beta_{\mathfrak{r}i} \in \partial_c \mathcal{T}_{\mathfrak{r}}(\mathfrak{g},\xi_{\mathfrak{r}i})$, $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\mathfrak{g}), i \in I_{\mathfrak{r}} :=$

$\{1,\ldots,n_{\mathfrak{r}}\}, n_{\mathfrak{r}} \in \mathbb{N}$, $\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} = 1$ and $\mathfrak{f}_{\mathfrak{s}} \geq 0$, $\mathfrak{s} \in \mathcal{S}$, $\bar{w}_{\mathfrak{s}j} \geq 0$, $\lambda_{\mathfrak{s}j} \in \partial_c\zeta_{\mathfrak{s}}\big((\mathfrak{g},\mathfrak{l}),\rho_{\mathfrak{s}j}\big)$, $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\mathfrak{g},\mathfrak{l}), j \in J_{\mathfrak{s}} := \{1,\ldots,n_{\mathfrak{s}}\}, n_{\mathfrak{s}} \in \mathbb{N}$, $\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} = 1$, such that

$$0 = \vartheta + \mathfrak{x}a^*\eta + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\ \beta_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\ \lambda_{\mathfrak{s}j} \right). \tag{4.2}$$

Since $\Gamma(\cdot,\cdot)$, $\varphi(\cdot,\cdot)$, $\zeta_{\mathfrak{s}}\big((\cdot,\cdot),\rho_{\mathfrak{s}}\big)$, $\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}$ and $\mathcal{T}_{\mathfrak{r}}(\cdot,\xi_{\mathfrak{r}})$, $\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}$ are ∂_c-robust convex functions at $(\mathfrak{g},\mathfrak{l})$, by Definition 3, for any $\vartheta \in \partial_c\Gamma(\mathfrak{g},\mathfrak{l})$, $a^* > 0$, $\eta \in \partial_c\varphi(\mathfrak{g},\mathfrak{l})$, $\mathfrak{y}_{\mathfrak{r}} \geq 0$, $\mathfrak{r} \in \mathcal{R}$, $\bar{u}_{\mathfrak{r}i} \geq 0$, $\beta_{\mathfrak{r}i} \in \partial_c\mathcal{T}_{\mathfrak{r}}(\mathfrak{g},\xi_{\mathfrak{r}i})$, $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\mathfrak{g}), i \in I_{\mathfrak{r}} := \{1,\ldots,n_{\mathfrak{r}}\}, n_{\mathfrak{r}} \in \mathbb{N}$, and $\mathfrak{f}_{\mathfrak{s}} \geq 0$, $\mathfrak{s} \in \mathcal{S}$, $\bar{w}_{\mathfrak{s}j} \geq 0$, $\lambda_{\mathfrak{s}j} \in \partial_c\zeta_{\mathfrak{s}}\big((\mathfrak{g},\mathfrak{l}),\rho_{\mathfrak{s}j}\big)$, $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\mathfrak{g},\mathfrak{l}), j \in J_{\mathfrak{s}} := \{1,\ldots,n_{\mathfrak{s}}\}, n_{\mathfrak{s}} \in \mathbb{N}$, we deduce from (4.2) that

$$\begin{aligned}
0 =& \Big\langle \vartheta, (\mathfrak{z}_0,\mathfrak{k}_0) - (\mathfrak{g},\mathfrak{l}) \Big\rangle + \mathfrak{x}a^* \Big\langle \eta, (\mathfrak{z}_0,\mathfrak{k}_0) - (\mathfrak{g},\mathfrak{l}) \Big\rangle \\
&+ \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \Big\langle \beta_{\mathfrak{r}i}, (\mathfrak{z}_0,0) - (\mathfrak{g},0) \Big\rangle \right) \\
&+ \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \Big\langle \lambda_{\mathfrak{s}j}, (\mathfrak{z}_0,\mathfrak{k}_0) - (\mathfrak{g},\mathfrak{l}) \Big\rangle \right) \\
\leq& \Big[\Gamma(\mathfrak{z}_0,\mathfrak{k}_0) - \Gamma(\mathfrak{g},\mathfrak{l}) \Big] + \mathfrak{x}a^* \Big[\varphi(\mathfrak{z}_0,\mathfrak{k}_0) - \varphi(\mathfrak{g},\mathfrak{l}) \Big] \\
&+ \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \Big[\mathcal{T}_{\mathfrak{r}}(\mathfrak{z}_0,\xi_{\mathfrak{r}i}) - \mathcal{T}_{\mathfrak{r}}(\mathfrak{g},\xi_{\mathfrak{r}i}) \Big] \right) \\
&+ \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \Big[\zeta_{\mathfrak{s}}\big((\mathfrak{z}_0,\mathfrak{k}_0),\rho_{\mathfrak{s}j}\big) - \zeta_{\mathfrak{s}}\big((\mathfrak{g},\mathfrak{l}),\rho_{\mathfrak{s}j}\big) \Big] \right).
\end{aligned} \tag{4.3}$$

Therefore

$$\begin{aligned}
\Gamma(\mathfrak{z}_0,\mathfrak{k}_0) \geq& \Gamma(\mathfrak{g},\mathfrak{l}) - \mathfrak{x}a^* \Big[\varphi(\mathfrak{z}_0,\mathfrak{k}_0) - \varphi(\mathfrak{g},\mathfrak{l}) \Big] \\
&- \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \Big[\mathcal{T}_{\mathfrak{r}}(\mathfrak{z}_0,\xi_{\mathfrak{r}i}) - \mathcal{T}_{\mathfrak{r}}(\mathfrak{g},\xi_{\mathfrak{r}i}) \Big] \right) \\
&- \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \Big[\zeta_{\mathfrak{s}}\big((\mathfrak{z}_0,\mathfrak{k}_0),\rho_{\mathfrak{s}j}\big) - \zeta_{\mathfrak{s}}\big((\mathfrak{g},\mathfrak{l}),\rho_{\mathfrak{s}j}\big) \Big] \right).
\end{aligned} \tag{4.4}$$

Since $(\mathfrak{z}_0,\mathfrak{k}_0) \in \mathcal{G}$, we have

$$\mathfrak{x}a^*\varphi(\mathfrak{z}_0,\mathfrak{k}_0) \leq 0, \qquad \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\ \mathcal{T}_{\mathfrak{r}}(\mathfrak{z}_0,\xi_{\mathfrak{r}i}) \right) \leq 0$$

and

$$\sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \ \zeta_{\mathfrak{s}}\big((\mathfrak{z}_0, \mathfrak{k}_0), \rho_{\mathfrak{s}j}\big) \right) \leq 0.$$

Hence

$$\begin{aligned}\Gamma(\mathfrak{z}_0, \mathfrak{k}_0) \geq \Gamma(\mathfrak{g}, \mathfrak{l}) + \mathfrak{x} a^* \varphi(\mathfrak{g}, \mathfrak{l}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \ \mathcal{T}_{\mathfrak{r}}(\mathfrak{g}, \xi_{\mathfrak{r}i}) \right) \\ + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \ \zeta_{\mathfrak{s}}\big((\mathfrak{g}, \mathfrak{l}), \rho_{\mathfrak{s}j}\big) \right).\end{aligned}$$

□

The next result addresses strong robust duality.

Theorem 5. ***(Strong Robust Duality):*** *Let $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ be a robust optimal solution for $(\mathcal{H})$. Suppose that the assumptions of Theorem 2 are satisfied. Then there exists $\overline{\Pi^*} = \big\{\big(\mathfrak{x},\ \mathfrak{y}_{\mathfrak{r}},\ \mathfrak{f}_{\mathfrak{s}}\big) :\ \mathfrak{x} > 0,\ \mathfrak{y}_{\mathfrak{r}} \geq 0,\ \mathfrak{f}_{\mathfrak{s}} \geq 0\big\}$, such that $((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \overline{\Pi^*})$ is a feasible point of $(\mathcal{WH})$ and respective objective values are equal. Moreover, if the conditions of Theorem 4 hold. Then $((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \overline{\Pi^*})$ is a robust optimal solution of $(\mathcal{WH})$.*

Proof. Since $(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$ be a robust optimal solution for $(\mathcal{H})$ and all conditions of Theorem 2 are met, it follows that there exist $\mathfrak{x} > 0,\ \vartheta \in \partial_c \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}),\ a^* > 0,\ \eta \in \partial_c \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}),\ \mathfrak{y}_{\mathfrak{r}} \geq 0,\ \mathfrak{r} \in \mathcal{R},\ \bar{u}_{\mathfrak{r}i} \geq 0,\ \beta_{\mathfrak{r}i} \in \partial_c \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}),\ \xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}}), i \in I_{\mathfrak{r}} := \{1, \ldots, n_{\mathfrak{r}}\}, n_{\mathfrak{r}} \in \mathbb{N}$, and $\mathfrak{f}_{\mathfrak{s}} \geq 0,\ \mathfrak{s} \in \mathcal{S},\ \bar{w}_{\mathfrak{s}j} \geq 0,\ \lambda_{\mathfrak{s}j} \in \partial_c \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big),\ \rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), j \in J_{\mathfrak{s}} := \{1, \ldots, n_{\mathfrak{s}}\}, n_{\mathfrak{s}} \in \mathbb{N}$, such that $\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} = 1,\quad \sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} = 1$,

$$0 = \vartheta + \mathfrak{x} a^* \eta + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \ \beta_{\mathfrak{r}i} \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \ \lambda_{\mathfrak{s}j} \right) \tag{4.5}$$

and

$$\mathfrak{y}_{\mathfrak{r}} \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \ \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}) = 0, \qquad \mathfrak{r} \in \mathcal{R}, \tag{4.6a}$$

$$\mathfrak{f}_{\mathfrak{s}} \max_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \ \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}}\big) = 0, \qquad \mathfrak{s} \in \mathcal{S}. \tag{4.6b}$$

This implies that $((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \overline{\Pi^*}) \in \mathcal{G}_{\mathcal{WH}}$ and the values of the two objective functions are equal. By Theorem 4, for any $((\mathfrak{g}, \mathfrak{l}),\ \Pi^*) \in \mathcal{G}_{\mathcal{WH}}$ we have

$$\begin{aligned}\Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) \geq \Gamma(\mathfrak{g}, \mathfrak{l}) + \mathfrak{x} a^* \varphi(\mathfrak{g}, \mathfrak{l}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \ \mathcal{T}_{\mathfrak{r}}(\mathfrak{g}, \xi_{\mathfrak{r}i}) \right) \\ + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \ \zeta_{\mathfrak{s}}\big((\mathfrak{g}, \mathfrak{l}), \rho_{\mathfrak{s}j}\big) \right).\end{aligned} \tag{4.7}$$

Since $\xi_{\mathfrak{r}i} \in \Omega_{\mathfrak{r}}(\bar{\mathfrak{z}})$,

$$\mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) = \max_{\xi_{\mathfrak{r}} \in \Omega_{\mathfrak{r}}} \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}}), \qquad \forall\, i \in I_{\mathfrak{r}}, \quad \forall\, \mathfrak{r} \in \mathcal{R}.$$

Thus, it follows by (4.6a) that

$$\mathfrak{y}_{\mathfrak{r}} \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) = 0, \qquad \forall\, i \in I_{\mathfrak{r}}, \quad \forall\, \mathfrak{r} \in \mathcal{R}.$$

And since $\rho_{\mathfrak{s}j} \in \Lambda_{\mathfrak{s}}(\bar{\mathfrak{z}}, \bar{\mathfrak{k}})$,

$$\zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) = \max_{\rho_{\mathfrak{s}} \in \Lambda_{\mathfrak{s}}} \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}}\big) \qquad \forall\, j \in J_{\mathfrak{s}}, \quad \mathfrak{s} \in \mathcal{S}.$$

Thus, it follows by (4.6b) that

$$\mathfrak{f}_{\mathfrak{s}}\; \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) = 0 \qquad \forall\, j \in J_{\mathfrak{s}}, \quad \mathfrak{s} \in \mathcal{S}.$$

As $\varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = 0$, This implies that

$$\mathfrak{x}a^* \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = 0, \quad \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\; \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) \right) = \sum_{\mathfrak{r}=1}^{p} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i} \mathfrak{y}_{\mathfrak{r}}\; \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) \right) = 0$$

and

$$\sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\; \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) \right) = \sum_{\mathfrak{s}=1}^{q} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j} \mathfrak{f}_{\mathfrak{s}} \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) \right) = 0.$$

Therefore, we have

$$\Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) = \Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) + \mathfrak{x}a^* \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\; \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\; \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) \right). \tag{4.8}$$

From equations (4.7) and (4.8) we have

$$\begin{aligned} &\Gamma(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) + \mathfrak{x}a^* \varphi(\bar{\mathfrak{z}}, \bar{\mathfrak{k}}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\; \mathcal{T}_{\mathfrak{r}}(\bar{\mathfrak{z}}, \xi_{\mathfrak{r}i}) \right) + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\; \zeta_{\mathfrak{s}}\big((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \rho_{\mathfrak{s}j}\big) \right) \\ &\geq \Gamma(\mathfrak{g}, \mathfrak{l}) + \mathfrak{x}a^* \varphi(\mathfrak{g}, \mathfrak{l}) + \sum_{\mathfrak{r}=1}^{p} \mathfrak{y}_{\mathfrak{r}} \left(\sum_{i=1}^{n_{\mathfrak{r}}} \bar{u}_{\mathfrak{r}i}\; \mathcal{T}_{\mathfrak{r}}(\mathfrak{g}, \xi_{\mathfrak{r}i}) \right) \\ &\quad + \sum_{\mathfrak{s}=1}^{q} \mathfrak{f}_{\mathfrak{s}} \left(\sum_{j=1}^{n_{\mathfrak{s}}} \bar{w}_{\mathfrak{s}j}\; \zeta_{\mathfrak{s}}\big((\mathfrak{g}, \mathfrak{l}), \rho_{\mathfrak{s}j}\big) \right). \end{aligned}$$

Hence, $((\bar{\mathfrak{z}}, \bar{\mathfrak{k}}), \overline{\Pi^*})$ is a robust optimal solution of $(\mathcal{WH})$. □

The following example shows the applicability of our duality results.

Example 2. We revisit problem $(\mathcal{EH})$, previously discussed in Example 1, to analyze its Wolfe-type robust dual problem

$$(\mathcal{WEH})\begin{cases} \underset{\mathfrak{g},\mathfrak{l}}{max}\left\{\Gamma(\mathfrak{g},\mathfrak{l})+\mathfrak{x}a^*\varphi(\mathfrak{g},\mathfrak{l})+\mathfrak{y}_1\ \mathcal{T}_1(\mathfrak{g},\xi_1)+\mathfrak{f}_1\ \zeta_1\big((\mathfrak{g},\mathfrak{l}),\rho_1\big)\right\}\\ s.\ t.\\ \quad (0,0)\in\partial_c\Gamma(\mathfrak{g},\mathfrak{l})+\mathfrak{x}a^*\partial_c\varphi(\mathfrak{g},\mathfrak{l})+\mathfrak{y}_1\ \partial_c\mathcal{T}_1(\mathfrak{g},\xi_1)+\mathfrak{f}_1\ \partial_c\zeta_1\big((\mathfrak{g},\mathfrak{l}),\rho_1\big),\\ \quad \Pi^*=\Big\{\big(\mathfrak{x},\ \mathfrak{y},\ \mathfrak{f}\big):\ \mathfrak{x}>0,\ \mathfrak{y}\geq 0,\ \mathfrak{f}\geq 0\Big\},\\ \quad (\mathfrak{g},\mathfrak{l})\in\mathbb{R}^2. \end{cases}$$

Let us claim that $(\mathfrak{g},\mathfrak{l})=(-1,0)$ is a feasible point of $(\mathcal{WEH})$.
The sets

$$\partial_c\Gamma(-1,0)=\left\{\Big(\frac{1}{3},\frac{1}{3}\Big),\Big(\frac{1}{3},-\frac{1}{3}\Big)\right\},\ \partial_c\varphi(-1,0)=\{(0,1)\},$$

$$\partial_c\mathcal{T}_1(\cdot,0)(-1)=\Big\{-1\Big\},\ \text{and}\ \partial_c\zeta_1\big((\cdot,\cdot),0\big)(-1,0)=\{(0,-1)\},$$

are the Clarke subdifferentials of Γ, φ, $\mathcal{T}_1$ and ζ_1 at $(\mathfrak{g},\mathfrak{l})$. Note that, by Definition 3 that $\Gamma(\cdot,\cdot)$, $\varphi(\cdot,\cdot)$, $\mathcal{T}_1(\cdot,\xi_1)$ and $\zeta_1\big((\cdot,\cdot),\rho_1\big)$ are ∂_c-robust convex at $(\mathfrak{g},\mathfrak{l})$.
For $\Pi^*=\Big(1,\frac{1}{3},\frac{7}{6}\Big)$, we have

$$\Big(\frac{1}{3},-\frac{1}{3}\Big)+1\frac{3}{2}\Big(0,1\Big)+\frac{1}{3}\Big(-1,0\Big)+\frac{7}{6}\Big(0,-1\Big)=\Big(0,0\Big).$$

This simplifies

$$(0,0)\in\mathfrak{x}\partial_c\Gamma(-1,0)+\mathfrak{x}a^*\partial_c\varphi(-1,0)+\mathfrak{y}_1\ \partial_c\mathcal{T}_1(-1,\xi_1)+\mathfrak{f}_1\ \partial_c\zeta_1\big((-1,0),\rho_1\big).$$

* Since $\mathcal{G}=\Big\{(x,0)\in\mathbb{R}\times\mathbb{R}\mid x\geq 0\Big\}$, for any feasible solution $(\mathfrak{z},\mathfrak{k})\in\mathcal{G}$ of $(\mathcal{REH})$ and any feasible solution $(\mathfrak{g},\mathfrak{l},\Pi^*)\in\mathcal{G}_{\mathcal{WH}}$ of $(\mathcal{WEH})$, we have

$$\Gamma(\mathfrak{z},\mathfrak{k})\geq\Gamma(\mathfrak{g},\mathfrak{l})+\mathfrak{x}a^*\varphi(\mathfrak{g},\mathfrak{l})+\mathfrak{y}_1\ \mathcal{T}_1(\mathfrak{g},\xi_1)+\mathfrak{f}_1\ \zeta_1\big((\mathfrak{g},\mathfrak{l}),\rho_1\big).$$

  Therefore, Theorem 4 is valid for both $(\mathcal{EH})$ and $(\mathcal{WEH})$.

* We knows that $(\bar{\mathfrak{z}},\bar{\mathfrak{k}})=(0,0)$ is a robust optimal solution of $(\mathcal{EH})$ where (ENMFCQ) holds and that (3.7) and (3.8) is satisfied at $\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\bar{\Pi^*}\big)=\Big((0,0),\frac{1}{2},\frac{1}{6},\frac{7}{6}\Big)$. Since $\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\bar{\Pi^*}\big)$ is a feasible point of $(\mathcal{WEH})$, then for any feasible point $\big((\mathfrak{g},\mathfrak{l}),\Pi^*\big)$ of $(\mathcal{WEH})$, we have

$$\begin{aligned}\Gamma(\bar{\mathfrak{z}},\bar{\mathfrak{k}})+\mathfrak{x}a^*\varphi(\bar{\mathfrak{z}},\bar{\mathfrak{k}})+\mathfrak{y}_1\ \mathcal{T}_1(\bar{\mathfrak{z}},\xi_1)+\mathfrak{f}_1\ \zeta_1\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\rho_1\big)&\\ \geq\Gamma(\mathfrak{g},\mathfrak{l})+\mathfrak{x}a^*\varphi(\mathfrak{g},\mathfrak{l})+\mathfrak{y}_1\ \mathcal{T}_1(\mathfrak{g},\xi_1)+\mathfrak{f}_1\ \zeta_1\big((\mathfrak{g},\mathfrak{l}),\rho_1\big).&\end{aligned}$$

  Therefore, $\big((\bar{\mathfrak{z}},\bar{\mathfrak{k}}),\bar{\Pi^*}\big)$ is the robust optimal solution of $(\mathcal{WEH})$, which proves that Theorem 5 is valid.

5. Conclusions

Robust bilevel programming is an emerging topic in optimization theory. This study explores sufficient optimality conditions and duality results for a uncertain bilevel model where uncertainty affects constraints at both levels. To tackle this problem, we employed an optimal value reformulation along with an exact penalization approach, transforming it into a single-level surrogate. Utilizing principles from robust counterpart optimization and ∂_c-robust convexity, we derived sufficient conditions for robust optimality and established both weak and strong robust duality results through Wolfe-type robust dual models. Several examples were provided to illustrate the relevance of these theoretical results in uncertain bilevel optimization. To our knowledge, this work is among the first to address these aspects in this setting.
Future research could explore extensions to uncertain multiobjective bilevel optimization problem, particularly in cases where uncertainty also affects the objective function.

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