

## On maximum tolerant Radon partitions for all-paths convexity in graphs

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**Abstract:** In a connected graph  $G$ , the all-paths transit function  $A(u, v)$ , consists of the set of all vertices in the graph  $G$  which lies on some path connecting  $u$  and  $v$ . Convexity obtained by the all-paths transit function is called all-paths convexity. A Radon partition of a set  $P$  of vertices of a graph  $G$  is a partition of  $P$  into two disjoint non-empty subsets such that their convex hulls intersect. A Radon partition  $(P_t, Q_t)$  of  $P$  is called  $t$ -tolerant Radon partition, if for any set  $S \subseteq P$  with  $|S| \leq t$ , the intersection of the convex hulls  $\langle P_t \setminus S \rangle \cap \langle Q_t \setminus S \rangle \neq \emptyset$ . This paper is devoted to  $t$ -tolerant Radon partitions for the all-paths convexity of connected simple undirected graphs. It is proved that the minimum number of vertices needed for  $t$ -tolerant Radon partition is  $2t + 4$ . But, some selection of  $2t + 4$  vertices of  $G$  has a  $(t + 1)$ -tolerant Radon partition. In this paper, we discuss the necessary and sufficient condition to the existence of  $(t + 1)$ -tolerant Radon partition for  $2t + 4$  vertices of  $G$ . We also develop algorithms to construct the Radon partition,  $t$ -tolerant Radon partition, and  $(t + 1)$ -tolerant Radon partition of a set of  $2t + 4$  vertices, if it exists.

**Keywords:** all-paths convexity, Radon partition,  $(t + 1)$ -tolerant Radon partition.

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### 1. Introduction

In Discrete Geometry, Radon's theorem is one of the most useful theorems by its applications. In 1921, Radon [14] proved that every set  $P$  of  $d + 2$  points in  $\mathbb{R}^d$  possesses a partition of  $P$  into two non-empty sets  $(P, Q)$  such that their convex

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hulls intersect. Such a partition is called a Radon partition. The classical convexity invariant named as Radon number, which represents the smallest number  $k$  of points in the Euclidean space  $\mathbb{R}^d$  such that any set of  $k$  points has a Radon partition. The natural generalisation of Radon's theorem is the Tverberg's theorem. In 1966, Tverberg [21] partitioned  $(k - 1)(d + 1) + 1$  points in  $\mathbb{R}^d$  into  $k$  non-empty subsets such that their convex hulls intersect.

A Radon partition  $(P_t, Q_t)$  of a set  $P$  in  $\mathbb{R}^d$  has tolerance  $t$ , if by removing any  $t$  points from  $P$ , then  $(P_t, Q_t)$  still remains as a Radon partition. Generally, zero-tolerant Radon partition refers to the usual Radon partition. Tolerant Radon partitions of  $\mathbb{R}^d$  have applications in computational geometry. So many studies are done in the tolerance of Tverberg's partitions in the classical Euclidean convexity. Larman [12], proved that any set of  $2d + 3$  points in  $\mathbb{R}^d$  has one tolerant Radon partition and Garcia-Colin [7] gives the existence of  $t$ -tolerant Radon partition for any set of  $(t + 1)(d + 1) + 1$  points in  $\mathbb{R}^d$ . Further, Soberon et al. [19] proved that for any set of  $(t + 1)(k - 1)(d + 1) + 1$  points in  $\mathbb{R}^d$  has  $t$ -tolerant Tverberg partition. In 2022, Bereg et al. [2] developed different algorithms to compute the tolerance in Radon partitions in  $\mathbb{R}^d$ .

Tolerant Radon partitions in abstract convexity spaces, in particular in graph convexity spaces is an interesting area of the Radon partition problem. The theory of abstract convexity spaces was developed with the main intention of generalizing the classical convexity invariants such as Helly, Carathéodory, and Radon numbers in  $\mathbb{R}^d$ . Later these ideas are used in abstract convexity spaces by Sierksma [16–18], Duchet [10], van de Vel [22], to name a few important references.

In abstract convexity spaces, graph convexity spaces captured more attention and several authors have studied graph convexities in different settings. The most prominent types of graph convexities are defined in terms of paths in the graph. Important studies are done in terms of geodesic, induced path and all-paths convexity [1, 3–6, 9, 15].

The coarsest path convexity in a connected graph is the all-paths convexity [4, 6]. The convexity invariants of the all-paths convexity are determined in [6] for a graph  $G$ . It can be proved easily that the Carathéodory number,  $c(G) = 2$ ; Helly number,  $h(G) = 2$  and Radon number  $r(G)$  satisfies  $3 \leq r(G) \leq 4$ , for the all-paths convexity. In [20], it is proved that any set  $S$  of  $2t + 4$  vertices in a graph  $G$  guarantees that  $S$  has a  $t$ -tolerant Radon partition for the all-paths convexity. It may be noted that in some cases, a set of  $2t + 4$  vertices of  $G$  can have a Radon partition with maximum tolerance of  $t + 1$  for the all-paths convexity. In this paper, we give the necessary and sufficient condition for the existence of maximum tolerance of  $t + 1$  for a set of  $2t + 4$  vertices of  $G$  with respect to all-paths convexity and discuss algorithms to construct tolerant Radon partitions of a set of vertices and the maximum tolerant Radon partition of a set of vertices, if it exists.

We discuss all the possibilities to form  $(t + 1)$ -tolerant Radon partition to a set of  $2t + 4$  vertices of  $G$  for the all-paths convexity.

The paper is organised in the following way. In Section 2, we present the preliminary concepts, definitions, and results for the remaining sections. The necessary and sufficient condition to the existence of  $(t + 1)$ -tolerant Radon partition for  $2t + 4$  vertices of  $G$  is described in Section 3. In Section 4, we discuss the algorithms.

## 2. Preliminaries

In this paper, we consider a simple connected undirected graph  $G$  with vertex set  $V$ . Also, only the tolerant Radon partition with respect to the all-paths convexity in a graph  $G$  is taken into account. In this sense, the term convexity that we mention at times needs to be read as all-paths convexity.

A convexity space is a pair  $(V, \mathcal{C})$  where  $V$  is a non-empty set and  $\mathcal{C}$  is a collection of subsets of  $V$  such that  $\emptyset, V \in \mathcal{C}$ , arbitrary intersection of elements of  $\mathcal{C}$  is also in  $\mathcal{C}$ , and every nested union of elements of  $\mathcal{C}$  is also in  $\mathcal{C}$  [22]. If  $V$  is finite, then the nested union condition is redundant and not required.

The elements of  $\mathcal{C}$  are called convex sets. The smallest convex set containing a set  $S$  is called the convex hull of  $S$  and it is represented by  $\langle S \rangle_{\mathcal{C}}$ . A graph convexity space is a pair  $(G, \mathcal{C})$  formed with  $V$ , a vertex set and  $\mathcal{C}$ , a convexity on  $V$ .

According to Mulder in [13], a transit function on a finite set  $V$  is a function  $R : V \times V \rightarrow 2^V$  satisfying the following conditions

$u \in R(u, v)$  for any  $u, v \in V$ ,  $R(u, v) = R(v, u)$  for all  $u, v \in V$  and  $R(u, u) = \{u\}$  for all  $u \in V$ .

Let  $R$  be a transit function on  $V$ . A set  $W \subset V$  is an  $R$ -convex set if  $R(u, v) \subseteq W$  for all  $u, v \in W$ . The collection  $\mathcal{C}_R$  of all  $R$ -convex sets in  $V$  is a convexity in the sense that  $\mathcal{C}_R$  contains the empty set  $\emptyset$  and  $V$  itself and  $\mathcal{C}_R$  is closed under intersections and nested unions.

If  $R$  is a transit function on the vertex set  $V$  of a graph  $G$  then we say that  $R$  is a transit function on  $G$ . Transit function gives a rule to move around from an element  $u \in V$  to an element  $v \in V$  via the set  $R(u, v)$ . The geodesic interval function  $I$ , the induced path transit function  $J$ , and the all-paths transit function  $A$  [4, 6, 9, 15] are some examples of transit functions, where

$$I(u, v) = \{w \in V : w \text{ lies on some shortest } u - v \text{ path in } G\},$$

$$J(u, v) = \{w \in V : w \text{ lies on some induced } u - v \text{ path in } G\},$$

and

$$A(u, v) = \{w \in V : w \text{ lies on some } u - v \text{ path in } G\}.$$

The family of  $R$ -convex sets in a graph  $G$  is called  $R$ -convexity on  $G$ . Thus the convexities induced by the geodesic, induced path, and all-paths transit functions are

called geodesic convexity, induced path convexity and all-paths convexity, respectively. All these graph convexities are extensively studied, for e.g., in [4, 6, 9, 15].

For convenience, we denote the all-paths convex hull of a subset  $S$  of  $V$  by  $\langle S \rangle$ . It may be noted that the convex hull  $\langle S \rangle$  of subset  $S$  always induces a connected subgraph of  $G$ .

For the basic graph theoretical terms, we refer to West [23]. A *cut vertex* of a graph  $G$  is a vertex whose removal increases the number of components of  $G$ . A subgraph  $H$  having vertex set  $S, S \subseteq V$ , is called an *induced subgraph* if for any two vertices  $u, v \in S$ ,  $u$  and  $v$  are adjacent in  $H$  if and only if  $u$  and  $v$  are adjacent in  $G$ . *Pendant vertex (leaf vertex)* of  $G$  is a vertex of  $G$  having degree one. The *diameter* of  $G$  is the maximum distance between any two vertices of  $G$ . The graph  $G$  is *two-connected* if the removal of any single vertex in  $G$  is not sufficient to disconnect the graph. A *block* of a graph  $G$  is a maximal two-connected subgraph of  $G$ . From the definition of  $A(u, v)$ , it is clear that  $A(u, u) = \{u\}$  for any vertex  $u$  of  $V$ , and if  $u, v$  are two vertices of a block, then  $A(u, v)$  is the set of all vertices of that block. For  $u, v$  in  $G$ ,  $A(u, v)$  contains all vertices of the blocks in  $G$ , in which there exists a  $u - v$  path traversing through that block. Thus if a  $u - v$  path contains an edge of a particular block, then the all-paths convex hull of  $\{u, v\}$  contains all vertices of that particular block. From the definition of  $A$ , it follows that, for any  $x, y \in V$ ,  $\langle \{x, y\} \rangle = A(x, y)$  where  $\langle \{x, y\} \rangle$  is the all-paths convex hull of  $\{x, y\}$ . The *block cut-vertex tree* denoted as  $B(G)$  of  $G$  has the blocks and cut vertices of  $G$  as its vertices and two vertices of  $B(G)$  are adjacent whenever one of them is a cut vertex of a block and the other is a block containing that cut vertex [3]. It is clear that the all-paths transit function has the structure reflecting the block-cut vertex tree structure of the graph. Blocks of a graph  $G$  having only one cut vertex are called *end blocks*.

The *Radon number* of a convexity space is the smallest integer  $r$  such that every  $r$  element subset  $P$  of  $V$  has a Radon partition. We denote the Radon partition of  $P$  concerning the all-paths convexity into sets  $P_0$  and  $Q_0$  as  $(P_0, Q_0)$ .

A Radon partition of  $P$  into two nonempty subsets  $P_t$  and  $Q_t$  is called  *$t$ -tolerant Radon partition* for some non-negative integer  $t$ , if for any set  $S \subseteq P$  with  $|S| \leq t$ , we have  $\langle P_t \setminus S \rangle \cap \langle Q_t \setminus S \rangle \neq \emptyset$ . For  $t$ -tolerant Radon partition, each set of a partition must have at least  $t + 1$  vertices. Clearly,  $t$ -tolerant Radon partition of a set is  $(t - 1)$ -tolerant Radon partition. But, the converse need not be true.

### 3. $(t + 1)$ -tolerant Radon partition

In [20], we proved that for any set of  $2t + 4$  vertices of  $G$  has a  $t$ -tolerant Radon partition for the all-paths convexity. Here  $t$  is the greatest lower bound of the tolerance of the Radon partition for any set of  $2t + 4$  vertices of  $G$ . But for some collection of  $2t + 4$  vertices of  $G$ , there exists a  $(t + 1)$ -tolerant Radon partition. In this section we find the different cases for the existence of  $(t + 1)$ -tolerant Radon partition for a set of  $2t + 4$  vertices of the graph. In [20], we proved that if  $G$  has no cut vertices, then

any collection of  $2t + 2$  vertices of  $P$  has a  $t$ -tolerant Radon partition.

**Proposition 1.** [20] *If a graph  $G$  has no cut vertices, then any set of  $2t + 2$  vertices of  $G$  has a  $t$ -tolerant Radon partition,  $t \geq 1$ .*

**Proposition 2.** *Let  $P$  be a set of  $2t + 4$  vertices of a graph  $G$  and let  $H$  be the subgraph induced by  $\langle P \rangle$ . If  $P$  contains only one vertex of some end block of  $H$  other than a cut vertex, then  $P$  has no  $(t + 1)$ -tolerant Radon partition.*

*Proof.* To form a  $(t + 1)$ -tolerant Radon partition, each set of a partition must have at least  $t + 2$  vertices of  $P$ . Since  $P$  contains only one vertex  $x$  of some end block  $B_i$  of  $H$  other than a cut vertex, it is clear that  $H$  contains more than one block. Consider any partition  $(P_1, Q_1)$ , both having  $t + 2$  vertices of  $P$ . Let  $x \in P_1$ . Let  $S$  be the set of  $t + 1$  vertices of  $P_1$  such that  $x \notin S$ . Then  $P_1 \setminus S = \{x\}$  and so  $\langle P_1 \setminus S \rangle = \{x\}$ . Since  $x$  is the only vertex from the end block  $B_i$  with possible exception of its cut vertex,  $Q_1$  contains no vertex of  $B_i$ . Thus  $\langle Q_1 \setminus S \rangle$  does not contain  $x$ . Hence  $\langle P_1 \setminus S \rangle \cap \langle Q_1 \setminus S \rangle = \emptyset$ . Thus  $P$  has no  $(t + 1)$ -tolerant Radon partition.  $\square$

**Proposition 3.** *Let  $P$  be a set of  $2t + 4$  vertices  $G$  other than cut vertices. If each block of  $G$  contains an even number (greater than 2) of vertices of  $P$ , then  $P$  has a  $(t + 1)$ -tolerant Radon partition;  $t \geq 0$ .*

*Proof.* Here we form partitions  $P^*$  and  $Q^*$  such that each partition contains half of the total number of vertices from each block. Then  $|P^*| = t + 2$  and  $|Q^*| = t + 2$ . We have to show that  $P^*$  and  $Q^*$  forms a  $(t + 1)$ -tolerant Radon partition.

**Case 1.** If we remove any  $t + 1$  vertices from  $P^*$  then one vertex of some block  $B_i$  remains in  $P^*$ . Since no vertices are removed from  $Q^*$ ,  $Q^*$  contains at least one vertex of that block  $B_i$ . So the all-paths convex hull of the remaining vertices of the partitions intersects at the remaining vertex of  $P^*$  from  $B_i$ . Similarly if we remove any  $t + 1$  vertices from  $Q^*$ , then also the convex hull of the remaining vertices of partitions intersects.

**Case 2.** If we remove any  $k, 1 \leq k < t + 1$ , vertices from  $P^*$  and  $t + 1 - k$  vertices from  $Q^*$ . After removing  $k$  vertices from  $P^*$ ,  $t + 2 - k$  vertices remain in  $P^*$ . Since  $P^*$  and  $Q^*$  contain the same number of vertices from each block, corresponding to these  $t + 2 - k$  vertices of  $P^*$  there exists  $t + 2 - k$  vertices in  $Q^*$ . Since  $(t + 2 - k) - (t + 1 - k) = 1$ , after removing any  $t + 1 - k$  vertices from  $Q^*$ , at least one vertex of the same block  $B_i$  remains in both  $P^*$  and  $Q^*$ . So the all-paths convex hull of the remaining vertices intersects at block  $B_i$ , because there are no cut vertices in  $P$  and so  $(P^*, Q^*)$  is a  $(t + 1)$ -tolerant Radon partition.  $\square$

Thus to form a  $(t + 1)$ -tolerant Radon partition from a set of  $2t + 4$  vertices of  $G$ ,  $P$  must contain at least two vertices from all the end blocks of the subgraph induced by  $\langle P \rangle$  other than cut vertices.

**Theorem 1.** *Let  $P$  be a set of  $2t + 4$  vertices of a graph  $G$  and let  $H$  be the subgraph induced by  $\langle P \rangle$ . Suppose that  $P$  contains at least one vertex from every interior block of  $H$  and at least two vertices from every end block of  $H$  other than cut vertices. Then  $P$  has a  $(t + 1)$ -tolerant Radon partition.*

*Proof.* Given a partition  $P = (P_1, Q_1)$ , we call  $(P_1, Q_1)$  is of the same cardinality, if  $|P_1| = |Q_1|$  and of different cardinality if  $|P_1| = |Q_1| + 1$ . We use this terminology in the proof.

Since  $H$  is a convex subgraph of  $G$ ,  $H$  is connected. Now, consider the block cut vertex tree,  $B(H)$  of the subgraph  $H$ . Blocks and cut vertices of  $H$  are the vertices of  $B(H)$ . We represent the vertices of  $P$  in  $B(H)$  in such a way that, all cut vertices of  $H$  in  $P$  can be represented in their corresponding places of  $B(H)$  and all other vertices of  $P$  from the same block can be represented in the corresponding vertex of  $B(H)$ . Thus, a vertex of  $B(H)$  may represent more vertices of  $P$ .

**Case 1.  $B(H)$  is a path.**

Since  $2t + 4$  is even, the number of vertices of  $B(H)$  that corresponds to an odd number of vertices of  $P$  is always even. We construct  $P_1$  and  $Q_1$  by partitioning the vertices of  $P$  using  $B(H)$ , starting from a pendant vertex of  $B(H)$ . Suppose that the pendent vertex of  $B(H)$  represents  $x_1$  vertices of  $P$ . If  $x_1$  is even, then allocate  $\frac{x_1}{2}$  vertices to  $P_1$  and remaining  $\frac{x_1}{2}$  vertices to  $Q_1$ . If  $x_1$  is odd, then  $x_1 \geq 3$  and allocate  $\frac{x_1+1}{2}$  vertices to  $P_1$  and the remaining  $\frac{x_1-1}{2}$  vertices to  $Q_1$ . Continuing the traversal through  $B(H)$ , we come across the next  $x_2$  vertices of  $P$ . If  $x_2$  is even, then as in the previous case, allocate  $\frac{x_2}{2}$  vertices to  $P_1$  and remaining  $\frac{x_2}{2}$  vertices to  $Q_1$ . If  $x_2$  is odd then the allocation of  $\frac{x_2+1}{2}$  vertices depends on  $|P_1|$  and  $|Q_1|$ . If  $|P_1| = |Q_1|$ , then allocate  $\frac{x_2+1}{2}$  vertices to  $P_1$  and remaining  $\frac{x_2-1}{2}$  vertices to  $Q_1$ . If  $|P_1| > |Q_1|$ , then allocate  $\frac{x_2+1}{2}$  vertices to  $Q_1$  and remaining  $\frac{x_2-1}{2}$  to  $P_1$ . Thus, after the allocation of two sets of odd vertices, we have  $|P_1| = |Q_1|$ . Continuing this way, we allocate all vertices of  $P$  and construct  $(P_1, Q_1)$  such that  $|P_1| = |Q_1| = t + 2$ .

Since  $P_1$  and  $Q_1$  contains at least one vertex from all the end blocks,  $\langle P_1 \rangle = \langle Q_1 \rangle = V(H)$ . Let  $S$  be any set of  $t + 1$  vertices of  $P_1$ . Then it follows that  $\langle P_1 \setminus S \rangle \cap \langle Q_1 \rangle \neq \emptyset$ . Similar is the case, when  $S$  is any  $t + 1$  vertices of  $Q_1$ . If we remove  $k, 1 \leq k \leq t$  vertices from  $P_1$  and  $t + 1 - k$  vertices from  $Q_1$ , then  $t + 2 - k$  vertices remains in  $P_1$ . These  $t + 2 - k$  vertices may be from the same block or from different blocks. If the all-paths convex hull of the remaining  $t + 2 - k$  vertices of  $P_1$  contains  $t + 2 - k$  vertices of  $Q_1$ , after removing any  $t + 1 - k$  vertices from  $Q_1$ , at least one vertex of  $Q_1$  remains and so the convex hull of the remaining vertices of  $P_1$  and  $Q_1$  intersects.

If the all-paths convex hull of the remaining  $t + 2 - k$  vertices of  $P_1$  contains  $t + 1 - k$  vertices of  $Q_1$ , then by the way of construction of partition,  $Q_1$  contains at least one vertex from the block  $B$ , which is adjacent to the all-paths convex hull of the remaining vertices of  $P_1$ . So after removing any  $t + 1 - k$  vertices from  $Q_1$  the convex hull of the remaining vertices of the partitions intersects at a cut vertex of  $B$  or at a block. Thus the partition  $(P_1, Q_1)$  is a  $(t + 1)$ -tolerant Radon partition.

**Case 2.  $B(H)$  is a tree, which is not a path.**

We perform a traversal in  $B(H)$  to construct the Radon partition. We define a branch  $T$  of  $B(H)$  as a *sparse branch* if  $T$  starts with a leaf  $u$  and ends in a vertex  $w$ , which is the first vertex such that the neighbor of  $w$  as we traverse has a degree at least three in  $B(H)$  so that the path connecting  $u$  and  $w$  contains no vertex having degree  $\geq 3$ . We construct the partition as follows.

We construct  $P_1$  and  $Q_1$  by partitioning the vertices of  $P$  using  $B(H)$ , by following a traversal through the sparse branches of  $B(H)$ . Let the leaf vertex of a sparse branch, say  $T$  of  $B(H)$  represent  $x_1 \geq 2$  vertices of  $P$ . If  $x_1$  is even,  $\frac{x_1}{2}$  vertices are allotted to  $P_1$  and remaining  $\frac{x_1}{2}$  vertices are allotted to  $Q_1$ . If  $x_1$  is odd then  $x_1 \geq 3$ ,  $\frac{x_1+1}{2}$  vertices are allotted to  $P_1$  and remaining  $\frac{x_1-1}{2}$  vertices are allotted to  $Q_1$ . We have that  $|P_1| = |Q_1|$  or  $|P_1| = |Q_1| + 1$ .

We traverse along the vertices of  $T$  to find the next vertex, say  $w$  of  $T$ , which contains vertices of  $P$ . Let  $w$  contain  $x_2$  vertices of  $P$ . If  $x_2$  is even, then as in the previous case,  $\frac{x_2}{2}$  vertices are allotted to  $P_1$  and the remaining  $\frac{x_2}{2}$  vertices are allotted to  $Q_1$ . If  $x_2$  is odd and  $|P_1| = |Q_1|$ , then allocate  $\frac{x_2+1}{2}$  vertices to  $P_1$  and the remaining vertices of  $x_2$  to  $Q_1$ , otherwise, if  $|P_1| > |Q_1|$ , then allocate  $\frac{x_2-1}{2}$  vertices to  $P_1$  and the remaining vertices to  $Q_1$ . Continue the traversal and the partitioning of the vertices of  $P$  through the entire sparse branch  $T$ , thus obtaining the partition  $(P_1, Q_1)$  of vertices of  $P$  lying in  $T$ . From this procedure, we obtain that if the number of vertices of  $P$  lying in  $T$  is even (odd), then  $|P_1| = |Q_1|$  ( $|P_1| = |Q_1| + 1$ ). Then delete the branch  $T$  and continue the procedure on another sparse branch  $T_1$  to obtain the partition  $(P_2, Q_2)$  of vertices of  $P$  lying in  $T_1$ . Delete the branch and continue until all sparse branches are deleted and obtaining a sequence of partitions  $(P_1, Q_1), (P_2, Q_2), \dots, (P_m, Q_m)$  of  $P$ . Let there be  $k$  partitions having different cardinality  $(P_i, Q_i)$  ( $1 \leq i \leq k$ ) and  $m-k$  partitions  $(P_j, Q_j)$  ( $k+1 \leq j \leq m$ ) having same cardinality. Since the cardinality of  $P$  is even,  $k$  is even. Now, consider the partition  $(P_t, Q_t)$  defined as follows.

$$P_t = P_1 \cup P_2 \cup \dots \cup P_{\frac{k}{2}} \cup Q_{\frac{k}{2}+1} \cup \dots \cup Q_k \cup P_{k+1} \cup \dots \cup P_m$$

and

$$Q_t = Q_1 \cup Q_2 \cup \dots \cup Q_{\frac{k}{2}} \cup P_{\frac{k}{2}+1} \cup \dots \cup P_k \cup Q_{k+1} \cup \dots \cup Q_m$$

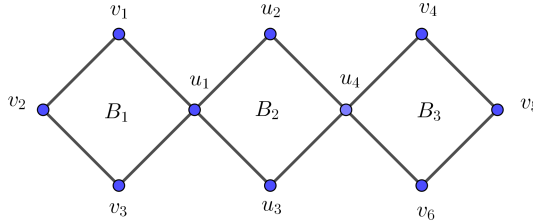
.

From construction, it follows that  $|P_t| = |Q_t| = t + 2$ .

We prove that  $(P_t, Q_t)$  is a  $(t+1)$ -tolerant Radon partition of  $P$ . From construction, it follows that  $P_t$  and  $Q_t$  contain at least one vertex from all the end blocks of  $H$  different from cut vertices and so  $\langle P_t \rangle = \langle Q_t \rangle = V(H)$ . Thus,  $(P_t, Q_t)$  is a Radon partition of  $P$ . Let  $S$  be a set of  $t+1$  vertices of  $P_t$ . Now  $P_t \setminus S$  consists of a single vertex, say  $z$ . Clearly  $Q_t \setminus S = Q_t$  and so  $\langle P_t \setminus S \rangle \cap \langle Q_t \setminus S \rangle = \{z\}$ . Similarly if  $S$  consists of  $t+1$  vertices of  $Q_t$ , then, also  $\langle P_t \setminus S \rangle \cap \langle Q_t \setminus S \rangle \neq \emptyset$ .

Now, let  $K$  be any set of  $k$  ( $1 \leq k \leq t$ ) vertices of  $P_t$  and  $L$  be any set of  $t+1-k$  vertices of  $Q_t$  so that  $S = K \cup L$ . Then  $P_t \setminus K$  contains  $t+2-k$  vertices of  $P_t$ . Since  $B(H)$  is not a path,  $\langle P_t \setminus K \rangle$  contains some or no vertices of  $Q_t$ . If  $Q_t$  contains at

least  $t + 2 - k$  vertices from  $\langle P_t \setminus K \rangle$ , then at least one vertex remains in  $Q_t \setminus L$ , and so  $\langle P_t \setminus K \rangle \cap \langle Q_t \setminus L \rangle \neq \emptyset$ . If  $\langle P_t \setminus K \rangle$  contains  $t + 1 - k$  vertices of  $Q_t$ , then  $\langle Q_t \setminus L \rangle$  contains a vertex from a block, which is adjacent to  $\langle P_t \setminus K \rangle$  by our construction. Thus  $\langle P_t \setminus K \rangle$  and  $\langle Q_t \setminus L \rangle$  intersect at a cut vertex. If  $\langle P_t \setminus K \rangle$  contains less than  $t + 1 - k$  vertices of  $Q_t$ , then there exists a set  $T \subset Q_t$  with  $|T| \geq t + 2 - k$  such that  $\langle T \rangle$  contains the vertices of  $P_t \setminus K$ , consisting of vertices from different sparse branches of  $B(H)$ . Then  $T \setminus L \neq \emptyset$  and  $T \setminus L \subset Q_t \setminus L$ . Then  $\langle Q_t \setminus L \rangle$  contains at least one vertex of  $P_t \setminus K$ . Hence  $\langle Q_t \setminus L \rangle$  and  $\langle P_t \setminus K \rangle$  intersect, since the all-paths convex hull contains the entire block containing the vertices. Thus in all cases,  $\langle P_t \setminus S \rangle \cap \langle Q_t \setminus S \rangle \neq \emptyset$ .  $\square$



**Figure 1.** Graph  $H$

**Example 1.** Consider the graph  $H$  having three blocks  $B_1, B_2, B_3$  as in Figure 1. Let  $P = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ . Then  $|P| = 2t + 4$ ,  $t = 1$  and  $P$  has no vertex from an inner block  $B_2$ . We claim that  $P$  has no 2-tolerant Radon partition. Any possible 2-tolerant Radon partition  $P^*$  and  $Q^*$  should contain at least 3 vertices of  $P$ . Since  $|P| = 6$ ,  $P^*$  and  $Q^*$  should contain exactly 3 vertices of  $P$ . The possibilities of  $P^*$  and  $Q^*$  are as follows.  $P^*$  and  $Q^*$  consists of one vertex from  $B_1$  or  $B_3$  and two vertices from  $B_3$  or  $B_1$ . Let  $P^* = \{v_1, v_3, v_5\}$  and  $Q^* = \{v_2, v_4, v_6\}$  be one such partition. Let  $S = \{v_2, v_5\}$  with  $|S| = 2$ . Then  $P^* \setminus S = \{v_1, v_3\}$  and  $\langle P^* \setminus S \rangle = \{v_1, v_2, v_3, u_1\}$ . Also,  $Q^* \setminus S = \{v_4, v_6\}$  and  $\langle Q^* \setminus S \rangle = \{v_4, v_5, v_6, u_3\}$ . Thus  $\langle P^* \setminus S \rangle \cap \langle Q^* \setminus S \rangle = \emptyset$ . For any Radon partition  $(P^*, Q^*)$ , we can find a set  $S$  with  $|S| = 2$ , such that  $\langle P^* \setminus S \rangle \cap \langle Q^* \setminus S \rangle = \emptyset$ . Thus  $P$  has no 2-tolerant Radon partition.

Now, we see the existence of a  $(t + 1)$ -tolerant Radon partition even if there exists some blocks that contain no interior vertex of  $P$ .

For any set  $P$  of vertices, a block  $B$  of a graph  $G$  is called a *cut block with respect to  $P$*  if  $P$  contains no vertices of  $B$  other than cut vertices. If  $P$  changes, then the cut blocks of  $H$  with respect to  $P$  also change. Such a cut block  $B$  separates  $G$  into two or more components if  $B$  contains two or more cut vertices of  $G$ . Here, each component contains only one cut vertex of  $B$ . The components are formed by removing all vertices of  $B$  other than cut vertices from  $G$ . In Figure 1,  $B_2$  is a cut block with respect to  $P$ .  $B_2$  separates  $H$  into two components  $B_1$  and  $B_3$ .

**Theorem 2.** *Let  $P$  be a set of  $2t + 4$  vertices of a graph  $G$ , and let  $H$  be the subgraph induced by  $\langle P \rangle$ . Suppose that  $H$  has at least one cut block and  $P$  contains at least two vertices from every end block of  $H$  other than cut vertices. Then  $P$  has a  $(t + 1)$ -tolerant Radon partition if and only if every cut block of  $H$  that separates  $H$  into exactly two components, must contain an even number of vertices of  $P$ .*

*Proof.* Let  $P$  be a set of  $2t + 4$  vertices of a graph  $G$  and  $H$  be the subgraph induced by  $\langle P \rangle$  such that  $P$  contains at least two vertices from every end block of  $H$ , other than cut vertices. Let  $G$  contain a cut block  $C$  with respect to  $P$  such that the removal of  $C$  from  $H$  results in exactly two components with each component containing an even number of vertices of  $P$ .

Consider the block cut vertex tree  $B(H)$  of  $H$ . Identify all cut blocks of  $H$  that separate  $H$  into exactly two components and remove the vertices from  $B(H)$  that corresponds to these cut blocks in  $H$ . After the removal of these vertices, let  $B(H)$  be split into sub trees  $T_1, T_2, \dots, T_n$ . From the assumption, we observe that each  $T_i$  contains an even number of vertices of  $P$ . The sub trees  $T_i$  may contain vertices corresponding to some cut blocks of  $H$  that separate  $H$  into more than two components. Identify these cut blocks of  $H$  that separate each  $T_i$  into more than two components. Remove the vertices of  $T_i$  that correspond to these cut blocks from  $H$  so that each  $T_i$  can be split into  $m$  subtrees  $T_{i,1}, T_{i,2}, \dots, T_{i,m}$  such that each of them contains no cut blocks. Using the method of construction of the Radon partition described in the proof of Theorem 1, we form a partition of vertices of  $P$  that lie in each of the subtrees  $T_{i,1}, T_{i,2}, \dots, T_{i,m}$ . It may be noted that all these partitions may not be Radon. Let  $(P_{i,j}, Q_{i,j})$  be the partition of vertices of  $P$  that lies in  $T_{i,j}$ , for  $1 \leq j \leq m$ . Among these  $m$  partitions, let  $k$  partitions have different cardinality and the remaining  $m - k$  partitions have same cardinality. From the construction of the partitions as described in the proof of Theorem 1, it follows that for partitions with different cardinality,  $|P_{i,r}| = |Q_{i,r}| + 1$  for  $1 \leq r \leq k$  and for partitions with same cardinality,  $|P_{i,r}| = |Q_{i,r}|$  for  $k + 1 \leq r \leq m$ . Since each  $T_i$  contains an even number of vertices of  $P$ ,  $k$  must be even. Now we form a partition  $(P_i, Q_i)$  of  $T_i$  such as

$$P_i = P_{i,1} \cup P_{i,2} \cup \dots \cup P_{i,\frac{k}{2}} \cup Q_{i,\frac{k}{2}+1} \cup \dots \cup Q_{i,k} \cup P_{i,k+1} \cup \dots \cup P_{i,m}$$

and

$$Q_i = Q_{i,1} \cup Q_{i,2} \cup \dots \cup Q_{i,\frac{k}{2}} \cup P_{i,\frac{k}{2}+1} \cup \dots \cup P_{i,k} \cup Q_{i,k+1} \cup \dots \cup Q_{i,m}$$

for each  $i$ . Also,  $P_i$  and  $Q_i$  contain the same number of vertices of  $P$ .

Now we construct the partition,  $(P^*, Q^*)$  as

$$P^* = P_1 \cup P_2 \cup \dots \cup P_n$$

and

$$Q^* = Q_1 \cup Q_2 \cup \dots \cup Q_n.$$

It is clear that  $(P^*, Q^*)$  forms a Radon partition. Since  $|P_i| = |Q_i|$  for each  $i$ , we have  $|P^*| = |Q^*| = t + 2$ . We have to show that  $(P^*, Q^*)$  forms a  $(t + 1)$ -tolerant Radon partition.

Consider a cut block that separates  $B(H)$  into two components  $C_1$  and  $C_2$ . By construction of the partition, vertices of  $C_1$  has a partition  $(P_1, Q_1)$  and vertices of  $C_2$  has a partition  $(P_2, Q_2)$ . Since each component contains an even number of vertices of  $P$ , clearly  $|P_1| = |Q_1|$  and  $|P_2| = |Q_2|$ . Form  $P^* = P_1 \cup P_2$  and  $Q^* = Q_1 \cup Q_2$ . Let  $S \subset P$ , has cardinality  $t + 1$ . Clearly  $\langle P^* \rangle = \langle Q^* \rangle = V(H)$ . If  $S \subset P^*$  (or  $S \subset Q^*$ ), then  $\langle P^* \setminus S \rangle \cap \langle Q^* \setminus S \rangle \neq \emptyset$ , since  $\langle P^* \setminus S \rangle$  (or  $\langle Q^* \setminus S \rangle$ ) contains one vertex of  $P^*$  (or  $Q^*$ ). Now, consider the case when  $S$  contains  $k$ ,  $(1 < k < t + 1)$  vertices of  $P^*$  and  $t + 1 - k$  vertices of  $Q^*$ . If  $P^* \setminus S$  and  $Q^* \setminus S$  contain vertices from both components  $C_1$  and  $C_2$ , then  $\langle P^* \setminus S \rangle$  and  $\langle Q^* \setminus S \rangle$  intersect at the cut block. If  $P^* \setminus S$  or  $Q^* \setminus S$  contains the vertices from only one component, then  $\langle P^* \setminus S \rangle$  and  $\langle Q^* \setminus S \rangle$  intersect in a cut vertex of the cut block or at a block other than the cut block. Thus for any set  $S$  having  $t + 1$  vertices,  $\langle P^* \setminus S \rangle \cap \langle Q^* \setminus S \rangle \neq \emptyset$ . This shows that  $(P^*, Q^*)$  forms a  $(t + 1)$ -tolerant Radon partition.

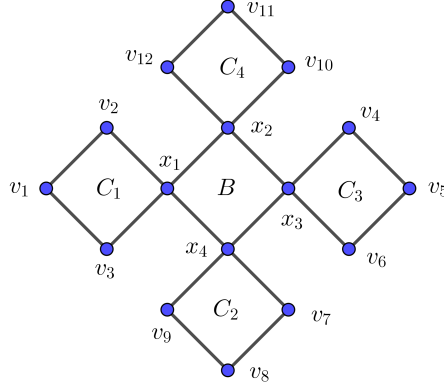
Conversely, let  $P$  be a set of  $2t + 4$  vertices of  $G$  such that  $P$  has a  $(t + 1)$ -tolerant Radon partition. We have to prove that for all cut blocks that separate  $H$  into two components, each of the components must contain an even number of vertices of  $P$ . Suppose there exists a cut block  $B$  that separates  $H$  into two components  $C_1$  and  $C_2$  such that  $C_1$  and  $C_2$  contain odd number of vertices of  $P$ . Let  $C_1$  contains  $p$  vertices of  $P$  and  $C_2$  contains  $q$  vertices of  $P$ . We have  $p + q = 2t + 4$ . Using  $B(H)$ , we can construct partitions  $P^*$  and  $Q^*$  such that  $|P^*| = |Q^*| = t + 2$ . Since  $p$  and  $q$  are odd,  $P^*$  contains  $\frac{p+1}{2}$  vertices of  $P$  from  $C_1$  and  $\frac{q-1}{2}$  vertices of  $P$  from  $C_2$ . Similarly  $Q^*$  contains  $\frac{p-1}{2}$  vertices of  $P$  from  $C_1$  and  $\frac{q+1}{2}$  vertices of  $P$  from  $C_2$ . We can form a set  $S$  of  $t + 1$  vertices of  $P$  such that  $S$  contains all the  $\frac{p-1}{2}$  vertices of  $Q^*$  from  $C_1$  and all the  $\frac{q-1}{2}$  vertices of  $P^*$  from  $C_2$ . Thus  $P^* \setminus S$  contains  $\frac{p+1}{2}$  vertices of  $C_1$  and  $Q^* \setminus S$  contains  $\frac{q+1}{2}$  vertices of  $C_2$ . Clearly  $|S| = \frac{p+q-2}{2} = t + 1$ .

Since there is a cut block  $B$  between  $C_1$  and  $C_2$ , we have  $\langle P^* \setminus S \rangle \cap \langle Q^* \setminus S \rangle = \emptyset$ . Thus  $(P^*, Q^*)$  is not a  $(t + 1)$ -tolerant Radon partition, which completes the converse part of the theorem.  $\square$

We observe the following remark, which can be established in the same procedure as we have described in Theorem 2.

**Remark 1.** For a set  $P$  of vertices of a graph  $G$ , the cut blocks that separate  $G$  into more than two components will not reduce the tolerance of the Radon partition even if the components contain an odd number of vertices of  $P$ .

An illustration of the Remark 1 is given in Figure 2, where  $B$  is a cut block with respect to  $P = \{v_1, v_2, \dots, v_{12}\}$  and  $C_1, C_2, C_3, C_4$  are the components. Here  $P_1 = \{v_1, v_3, v_5, v_7, v_9, v_{11}\}$  and  $Q_1 = \{v_2, v_4, v_6, v_8, v_{10}, v_{12}\}$  form a 5-tolerant Radon partition of  $P$ .



**Figure 2.** A 5-tolerant Radon partition as an illustration for Remark 1

#### 4. Algorithms for tolerant Radon partitions

In this section, we develop five algorithms related to tolerant Radon partitions for a given set of vertices of a graph. We assume that  $G$  is a connected graph having  $n$  vertices and  $m$  edges. The following Proposition (Corollary 3.1 from [20]) guarantees that a given set of  $2t + 3$  vertices of a path has a  $t$ -tolerant Radon partition.

**Proposition 4 (Corollary 3.1[20]).** *Any set of  $2t+3$  vertices of a path has a  $t$ -tolerant Radon partition.*

---

**Algorithm 1.** Algorithm for  $t$ -tolerant Radon partition on paths.

---

**Input :** A path  $X$  represented by a list of vertices  $(v_1, v_2, \dots, v_n)$  and a set  $P$  of  $2t + 3$  vertices of  $X$ .

**Output:** A  $t$ -tolerant Radon partition  $(P_t, Q_t)$  of  $P$ .

Initially, set  $P_t = \emptyset$  and  $Q_t = \emptyset$ .

Starting from the first vertex,  $v_1$  of  $X$  and make a traversal from  $v_1$  and add each vertex of  $X$  encountered in the traversal alternatively to  $P_t$  and  $Q_t$  respectively. The resulting partition  $(P_t, Q_t)$  is a  $t$ -tolerant Radon partition.

**Running time analysis:** Each vertex and edge is considered only once during the traversal and so the time complexity of the traversal is  $O(n+m)$  where  $m$  is the number of edges. For a path,  $m = n - 1$  and hence the Algorithm 1 has time complexity  $O(n)$ .

**Theorem 3.** *The partition  $(P_t, Q_t)$  formed in Algorithm 1 is a  $t$ -tolerant Radon partition of a set  $P$  of  $2t + 3$  vertices of a path.*

*Proof.* Suppose that during the traversal through  $X$ , we renamed the vertices of  $P$  as  $v_1, v_2, v_3, \dots, v_{2t+3}$  in the order in which they are encountered. From the algorithm, we formed a partition to the set  $P$  as  $P_t = \{v_1, v_3, \dots, v_{2t+3}\}$  and  $Q_t = \{v_2, v_4, \dots, v_{2t+2}\}$  with  $|P_t| = t + 2$  and  $|Q_t| = t + 1$ .

**Case 1.** Removing any  $t$  vertices from  $P_t$ .

Then two vertices remain in  $P_t$  having an odd label. In this case no vertices are removed from  $Q_t$ . Between the two odd labeled vertices of  $P_t$ , there exists at least one even labeled vertex and that vertex is in  $Q_t$ . Thus the convex hull of the remaining vertices of the partitions intersects.

**Case 2.** Removing any  $t$  vertices from  $Q_t$ .

Then only one vertex remains in  $Q_t$ . Clearly the remaining vertex of  $Q_t$  has even label. In this case no vertex is removed from  $P_t$  and so there exists at least two vertices in  $P_t$  having odd label, between them the vertex of  $Q_t$  lies. Thus the convex hull of the remaining vertices intersects.

**Case 3.** Removing any  $k$  vertices from  $P_t$  and  $t - k$  vertices from  $Q_t$  for  $1 \leq k \leq t - 1$ . Then  $t + 2 - k$  vertices remain in  $P_t$ . So, the convex hull of these  $t - k + 2$  vertices of  $P_t$  contains at least  $t - k + 1$  vertices of  $Q_t$ . After removing  $t - k$  vertices from  $Q_t$ , at least one vertex of  $Q_t$  remains and that vertex is in the convex hull of the remaining vertices of  $P_t$ . Thus, the convex hull of the remaining vertices intersects. Hence the partition  $(P_t, Q_t)$  is a  $t$ -tolerant Radon partition to the given set of  $2t + 3$  vertices of a path.  $\square$

Next, we discuss the algorithm for determining  $t$ -tolerant Radon partition for a given input of  $2t + 4$  vertices of a tree. Before discussing the algorithm on trees, we need an algorithm for constructing the Radon partition. It follows from Theorem 11 in [6] that every set of four points in a tree has a Radon partition. In this algorithm, at first, we construct a subtree  $W$  of the given tree  $T$  such that all pendant vertices of  $W$  are from the given input set  $P$  of four vertices. Then it follows that  $W = \langle P \rangle$ . The algorithm determines the Radon partition of  $P$  depending upon the number of pendant vertices of  $W$ .

We use the adjacency list for representing trees and graphs in all the algorithms discussed from here onwards.

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**Algorithm 2.** Algorithm for Radon partition of a set of four vertices of a tree.

---

**Input :** A tree  $T$  and a set  $P$  of four vertices  $\{a, b, c, d\}$  of  $T$ .

**Output:** A Radon partition  $(P_0, Q_0)$  of  $P$ .

Initially set  $P_0 = \emptyset$  and  $Q_0 = \emptyset$ .

**Step 1:** Start a Breadth First Search traversal from any one vertex of  $P$  to identify the remaining three vertices and find the path connecting the three pairs of vertices. Combination of these three paths represents  $\langle P \rangle$ . Let  $W$  be the newly obtained tree.

**Step 2:** Make a Breadth First Search traversal through  $W$  to identify the degree of all vertices of  $W$ .

**Step 3:** If  $W$  contains only two pendant vertices  $a$  and  $d$ , we form  $P_0 = \{a, d\}$  and  $Q_0 = \{b, c\}$ . Return  $(P_0, Q_0)$ .

**Step 4:** If  $W$  contains three pendant vertices  $a, b, c$ ; then make a Depth First Search traversal from the unique vertex  $x$  of degree 3 in  $W$  towards  $a$ . If we find  $d$  during this traversal then construct  $P_0 = \{a, b\}$  and  $Q_0 = \{c, d\}$ . Otherwise construct  $P_0 = \{b, c\}$  and  $Q_0 = \{a, d\}$ . Return  $(P_0, Q_0)$ .

**Step 5:** In this case all four vertices of  $P$  are pendant vertices of  $W$ . If  $W$  contains a unique vertex of degree 4, construct  $P_0 = \{a, b\}$  and  $Q_0 = \{c, d\}$ . Return  $(P_0, Q_0)$ . Otherwise go to Step 6.

**Step 6:** If  $W$  contains two vertices  $u_1$  and  $u_2$  having degree 3, then make a Depth First Search traversal from  $a$  to  $b$ . If  $u_1$  and  $u_2$  are encountered, then  $P_0 = \{a, b\}$  and  $Q_0 = \{c, d\}$ . Otherwise form  $P_0 = \{a, c\}$  and  $Q_0 = \{b, d\}$ . Return  $(P_0, Q_0)$ .

**Running time analysis:** Time complexity to do the BFS traversal is  $O(n + m)$ . Since  $m = n - 1$ , time taken to complete the first Step of the algorithm is  $O(n)$ . In Step 2, each vertex and each edge is listed at most once and so time complexity is  $O(n + m)$  and hence it is  $O(n)$ . Time complexity of Step 3 is  $O(1)$ . In the Step 4 and Step 6, one more traversal is needed in each Step and so the time complexity of each Step is  $O(n + m)$  and it is reduced to  $O(n)$ . Time complexity of Step 5 is  $O(1)$ . Thus the total time complexity of the Algorithm 2 is  $O(n)$ .

**Theorem 4.** *The partition  $P_0$  and  $Q_0$  formed in Algorithm 2 is a Radon partition of a set  $P$  of four vertices of a tree.*

*Proof.* If  $W$  contains only two pendant vertices  $a$  and  $d$ , then  $P_0 = \{a, d\}$  and  $Q_0 = \{b, c\}$ . Here  $\langle P_0 \rangle$  contains all vertices of  $\langle Q_0 \rangle$ . If  $W$  contains three pendant vertices  $a, b, c$ ; then there exists the unique vertex  $x$  in  $W$  having degree 3. If  $d$  lies in the path connecting  $a$  and  $x$ , then  $P_0 = \{a, b\}$  and  $Q_0 = \{c, d\}$ . Otherwise  $d$  lies in the path connecting  $b$  and  $c$ . In this case  $P_0 = \{b, c\}$  and  $Q_0 = \{a, d\}$ . So  $\langle P_0 \rangle$  and

$\langle Q_0 \rangle$  intersect at  $d$ . If all vertices of  $W$  are pendant vertices, then  $W$  contains either a unique vertex of degree 4 or two vertices of degree 3. If  $W$  contains the vertex of degree 4, then  $P_0 = \{a, b\}$  and  $Q_0 = \{c, d\}$ . If  $W$  contains two vertices of degree 3, then we can form  $\langle P_0 \rangle$  and  $\langle Q_0 \rangle$  are paths containing these two vertices. Hence in all cases,  $\langle P_0 \rangle$  and  $\langle Q_0 \rangle$  intersect and hence the partition is a Radon partition.  $\square$

Now, we find  $t$ -tolerant Radon partitions on trees by using the Radon partition obtained from Algorithm 2. The following Theorem from [20] guarantees that for a given input of  $2t + 4$  vertices of the tree  $T$  has a  $t$ -tolerant Radon partition. We continuously remove two special vertices from  $P$  in the tree  $T$  and form a partition  $(P_t, Q_t)$  by putting one of the vertices in  $P_t$  and the other in  $Q_t$ . After  $t$  steps, the Radon partition of the remaining four vertices is obtained and the  $t$ -tolerant Radon partition is developed by using it.

**Theorem 5.** [20] *Any set of  $2t + 4$  vertices of a tree has a  $t$ -tolerant Radon partition.*

---

**Algorithm 3.** Algorithm for  $t$ -tolerant Radon partitions on trees.

---

**Input :** A tree  $T$  and a set  $Y$  of  $2t + 4$  vertices of  $T$ .

**Output :** A  $t$ -tolerant Radon partition  $(P_t, Q_t)$  of  $Y$ .

Initially set  $P_t = \emptyset$  and  $Q_t = \emptyset$ .

**Step 1:** Compute the distance from each vertex of  $Y$  to all other vertices of the tree  $T$  and store the distances in a matrix  $M$ . Let  $Y_1 = Y$

**For each  $j, 1 \leq j \leq t$ , repeat Steps 2 to 6**

**Step 2:** Start a Breadth First Search traversal from any one vertex of  $P$  to identify the remaining  $2t + 3$  vertices and find the path connecting the  $2t + 3$  pairs of vertices. Combination of these paths represents  $\langle Y \rangle$ . Let  $T_j$  be the newly obtained tree.

**Step 3:** We start a Depth First Search traversal in  $T_j$  from one pendant vertex  $u_j$ , to find another vertex  $v_j$  of  $Y_j$  or to find a vertex  $x_j$  of  $T_j$  having degree  $\geq 3$ . If we met  $v_j$  first, then form the set  $S_j = \{u_j, v_j\}$ . Then  $P_t = P_t \cup \{u_j\}$  and  $Q_t = Q_t \cup \{v_j\}$ .

**Step 4:** If we met  $x_j \notin Y_j$  first, then from the distance matrix  $M$ , find the pendant vertex  $w_j$  of  $T_j$  having minimum distance with  $x_j$ , other than  $u_j$ . Repeat Step 4 starting from  $w_j$ .

**Step 5:** If all the intermediate vertices of the  $w_j - x_j$  path are not from  $Y_j$  and if there is no intermediate vertex having degree  $\geq 3$  in the  $w_j - x_j$  path, then we form  $S_j = \{u_j, w_j\}$ . Rename  $w_j$  as  $v_j$  and form  $S_j = \{u_j, v_j\}$ . Construct

$P_t = P_t \cup \{u_j\}$  and  $Q_t = Q_t \cup \{v_j\}$ .

**Step 6:** Form  $Y_{j+1} = Y_j \setminus S_j$ .

**Step 7:** Use Algorithm 2 to form a Radon partition  $(P_0, Q_0)$  to  $Y_{t+1}$ .

**Step 8:** Form  $P_t = P_t \cup P_0$  and  $Q_t = Q_t \cup Q_0$  and Stop.

**Running time analysis:** Time complexity to find the distance from a vertex of  $Y$  to all other vertices of the tree is  $O(n)$  where  $n$  is the number of vertices of  $T$ . This can be done for  $2t + 4$  vertices. So, the time complexity of the first Step is  $O(nt)$ . Time complexity to do the BFS traversal is  $O(n + m)$ . Since  $m = n - 1$ , it is reduced to  $O(n)$ . The process can be repeated for  $t$  iterations and so the time complexity of Step 2 of the algorithm is  $O(nt)$ . For each Step 3 to 5, the time complexity of the traversal to find the set  $S_j$  for  $1 \leq j \leq t$  is  $O(n)$ . The process is continued for a maximum of  $t$  iterations. So the total time complexity of the algorithm from Step 3 to Step 5 is  $O(nt)$ . The total time complexity of Step 6 is  $O(t)$ . For the Step 7, the time complexity of constructing a Radon partition is  $O(n)$ . Hence the total time complexity of the Algorithm 3 is  $O(nt)$ .

**Theorem 6.** *Algorithm 3 correctly computes the  $t$ -tolerant Radon partition for vertices in a tree  $T$ , for the convexity defined by the paths in  $T$ .*

*Proof.* We have constructed the sets  $S_j = \{u_j, v_j\}$ , for  $1 \leq j \leq t$ , either in Step 4 or in Step 6. It is to be shown that the path connecting every vertex of  $S_j$  to a vertex of  $Y_{j+1}$ , as defined in Step 7 of the Algorithm, must pass through a common vertex  $x_j$  of  $T_j$ . Clearly  $S_j$  contains one pendant vertex or two pendant vertices. Also  $\langle S_j \rangle$  does not contain other vertices of  $Y_{j+1}$ . If  $u_j$  is a pendant vertex and  $v_j$  is not a pendant vertex, then the path from a vertex of  $S_j$  to a vertex of  $Y_{j+1}$  must pass through the vertex  $v_j$ . Here, with out loss of generality, we take  $x_j = v_j$ . If  $u_j$  and  $v_j$  are pendant vertices, then there is no other vertex of  $Y_j$  in the interior of the  $u_j - v_j$  path. So the path connecting a vertex of  $S_j$  to a vertex of  $Y_{j+1}$  must pass through an interior vertex  $x_j$  that lies in the  $u_j - v_j$  path such that  $\text{degree}(x_j) \geq 3$ . Thus in any case, for each  $j, 1 \leq j \leq t$ , both paths connecting a vertex of  $S_j$  to any vertex of  $Y_{j+1}$  will intersect at a vertex  $x_j$  of the tree  $T_j$ .

From the algorithm described above we constructed partitions as  $P_t = P_0 \cup \{u_1, u_2, \dots, u_t\}$  and  $Q_t = Q_0 \cup \{v_1, v_2, \dots, v_t\}$ . Then  $|P_t| = t + 2$  and  $|Q_t| = t + 2$ . The following cases are considered for proving the theorem.

**Case 1.** Removing  $t$  vertices from  $P_t$ .

In this case, no vertex is removed from  $Q_t$  and hence two vertices will remain in  $P_t$ . If the remaining two vertices are from  $P_0$ , since  $(P_0, Q_0)$  is a Radon partition, the convex hull of the remaining vertices intersects.

Suppose the remaining vertices of  $P_t$  are  $u_i, u_j$  where  $i < j$ , where  $u_i \in S_i$  and  $u_j \in S_j$ . Then by the method of selection of  $S_i$ , the path connecting every vertex of  $S_i$  to every vertex of  $S_j$  must pass through a common vertex  $x_i$ . Since  $v_i$  and  $v_j$  are in  $Q_t$ , the convex hull of the remaining vertices of  $P_t$  and  $Q_t$  intersect at  $x_i$ .

Suppose that one vertex  $u_i$  and one vertex of  $P_0$  remains in  $P_t$ . Then also the path connecting  $u_i$  to a vertex of  $P_0$  intersects with the path connecting  $v_i$  to  $Q_0$  and so the convex hull of the remaining vertices intersects.

**Case 2.** Removing  $t$  vertices from  $Q_t$ .

By using the symmetric arguments used in the previous case, we conclude that the convex hull of the remaining vertices intersects.

**Case 3.** Removing any  $k$  vertices from  $P_t$  and  $t - k$  vertices from  $Q_t$  for  $1 \leq k \leq t - 1$ . Consider the partitions  $P_t = \{u_1, u_2, \dots, u_t\} \cup P_0$  and  $Q_t = \{v_1, v_2, \dots, v_t\} \cup Q_0$ . Suppose that  $(P_0, Q_0)$  intersects at  $x_0$  and suppose that two paths from vertices of  $S_j$  to a vertex of  $P_0 \cup Q_0 \cup S_1 \cup S_2 \cup \dots \cup S_{j-1}$  must pass through a common vertex  $x_j$  for  $1 \leq j \leq t$ . Thus  $\langle P_t \rangle$  and  $\langle Q_t \rangle$  intersect at least  $t + 1$  vertices of the tree (intersecting points), counted with repetition. Removing of one vertex from one partition may reduce the intersection points by one. If we remove any  $t - k$  vertices from  $Q_t$ , then at least  $k + 2$  vertices remain in  $Q_t$ . The convex hull of the remaining  $k + 2$  vertices of  $Q_t$  and  $\langle P_t \rangle$  intersect with at least  $k + 1$  intersecting points, counted with repetition. After removing any  $k$  vertices from  $P_t$ , then again the convex hull of the remaining vertices of  $P_t$  and the convex hull of the remaining vertices of  $Q_t$  intersect with at least one intersecting point. Thus the partition  $(P_t, Q_t)$  is a  $t$ -tolerant Radon partition to the set of  $2t + 4$  vertices of a tree.  $\square$

The all-paths convexity reflects the block-cut vertex tree structure of graphs. Now we design the algorithm to construct  $t$ -tolerant Radon partition on an arbitrary connected graph  $G$ . The idea is that first we construct a spanning tree of  $G$  using the simple standard Breadth First Search algorithm. Then form  $t$ -tolerant Radon partition to the vertices of the spanning tree using Algorithm 3. We prove that the Radon partition of the spanning tree obtained using Algorithm 3 is the same that of the given graph  $G$ .

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**Algorithm 4.** Algorithm for  $t$ -tolerant Radon partitions of an arbitrary connected graph.

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**Input:** A connected graph  $G = G(V, E)$  and a set  $P$  of  $2t + 4$  vertices of a graph  $G$

**Output:** A  $t$ -tolerant Radon partition  $(P_t, Q_t)$  of  $P$ .

Initially set  $P_t = \emptyset$  and  $Q_t = \emptyset$ .

**Step 1:** Construct a spanning tree  $T = G(V, E')$  of  $G$  by using Breadth First Search algorithm, where  $E' \subseteq E$ .

**Step 2:** Use Algorithm 3 to construct a  $t$ -tolerant Radon partition  $(P_t, Q_t)$  to the set of  $2t + 4$  vertices of the spanning tree  $T$ . Return  $(P_t, Q_t)$  and Stop.

**Running time analysis:** The construction of spanning tree has complexity  $O(m + n)$  by a Breadth First Search algorithm, where  $m$  is the number of edges and  $n$  is the number of vertices of the graph. The time complexity of constructing  $t$ -tolerant Radon partition of vertices of a tree is  $O(nt)$ . Hence the time complexity of the Algorithm 4 is  $O(m + nt)$ .

We proved the following Theorem in [20], which guarantees that any set of  $2t + 4$  vertices in a graph  $G$  has a  $t$ -tolerant Radon partition on all-paths convexity.

**Theorem 7.** [20] *Any set of  $2t + 4$  vertices of a graph  $G$  has  $t$ -tolerant Radon partition with respect to all-paths convexity if the block cut vertex tree of  $G$  is not a path.*

**Theorem 8.** *The Algorithm 4 correctly computes the  $t$ -tolerant Radon partition of vertices of a given connected graph  $G$ .*

*Proof.* In the Algorithm 4, a spanning tree  $T$  of the graph is constructed at first and a  $t$ -tolerant Radon partition  $(P_t, Q_t)$  to a set  $P$  of vertices of the tree  $T$  is formed. Thus for any set  $C \subset P$  with  $|C| \leq t$ ,  $\langle P_t \setminus C \rangle \cap \langle Q_t \setminus C \rangle \neq \emptyset$ . In the spanning tree  $T$  of the graph  $G$ , we are considering only one path between any two vertices of  $G$ . Since the all-paths transit function considers all paths connecting any two vertices of  $G$ , we have  $\langle P_t \setminus C \rangle \subseteq A\langle P_t \setminus C \rangle$  and  $\langle Q_t \setminus C \rangle \subseteq A\langle Q_t \setminus C \rangle$ , where  $\langle P_t \setminus C \rangle$  represents the convex hull of  $P_t \setminus C$  in the tree  $T$  and  $A\langle P_t \setminus C \rangle$  represents the all-paths convex hull of  $P_t \setminus C$  in the graph  $G$ . So for any set  $C \subset P$  with  $|C| \leq t$ ,  $A\langle P_t \setminus C \rangle \cap A\langle Q_t \setminus C \rangle \neq \emptyset$ . Thus the same partition  $(P_t, Q_t)$  is a  $t$ -tolerant Radon partition of  $P$  with respect to all-paths convexity.  $\square$

#### 4.1. Algorithm for $(t + 1)$ -tolerant Radon partition

Here we design a decision algorithm to check the existence of  $(t + 1)$ -tolerant Radon partition for any input of  $2t + 4$  vertices and the algorithm outputs a  $(t + 1)$ -tolerant Radon partition, if there is such a partition. If  $P$  has no  $(t + 1)$ -tolerant Radon partition, then the algorithm outputs "False". Such an algorithm is called a *decision algorithm*. The main idea of the algorithm can be described as follows. At first, we determine all the cut vertices of  $G$ , which is available in [8]. Then we determine all blocks of  $G$ . For this we make use of an algorithm by Hopcroft et al. in [11]. Then we construct the block cut vertex tree of  $G$ ,  $B(G)$ , with blocks and cut vertices of  $G$  as its vertices. From  $B(G)$ , we construct a subtree of  $B(G)$ , which is precisely the block cut vertex tree  $B(H)$  of the subgraph  $H$ , where  $H$  is the subgraph induced by  $\langle P \rangle$  in  $G$ . A vertex of  $B(H)$  represents either a cut vertex of  $H$  or a block of  $H$ . Vertex of  $B(H)$  corresponding to a block will represent all vertices of  $P$  which are from a particular block of  $H$  other than cut vertices. We count the number of vertices of  $P$

in each block. We can construct  $(t + 1)$ -tolerant Radon partition, if it exists using the results that we have presented in the previous Section (Section 3) with the help of  $B(H)$ . A block which has only one cut vertex is an end block. Now, we are ready to describe the algorithm.

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**Algorithm 5.** A decision algorithm to determine a  $(t + 1)$ -tolerant Radon partition on a set of  $2t + 4$  vertices of a graph  $G$ .

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**Input :** A graph  $G$  represented as adjacency list and a set  $P$  of  $2t + 4$  vertices of  $G$ .

**Output :** A  $(t + 1)$ -tolerant Radon partition of  $P = (X, Y)$  or False, that returns that  $P$  has no such partition.

Initially set  $X = \emptyset$ ,  $Y = \emptyset$ .

**Step 1:** Determine the set  $C$  of cut vertices of  $G$  by using an application of the standard DFS algorithm and the set  $\{B_1, B_2, \dots, B_k\}$  of blocks of  $G$  by the algorithm of Hopcroft and Tarjan from [11].

**Step 2:** Identify the sets  $P_C = P \cap C$ , the set of cut vertices of  $P$ ; and  $P_{B_i} = P \cap (B_i \setminus C)$ , the set of vertices of  $P$  which are not cut vertices from the blocks  $B_i$ . Let  $b_i$  be the cardinality of  $P_{B_i}$  for  $1 \leq i \leq k$ .

**Step 3:** Identify the sets  $C_{B_i} = B_i \cap C$  of cut vertices of a particular block  $B_i$  for each  $i$ .

**Step 4:** Construct block cut vertex tree  $B(G)$  of  $G$  by using the adjacency of blocks and cut vertices.

**Step 5:** If  $b_i = 0$  for any pendant vertex  $B_i$  of  $B(G)$ , remove it from  $B(G)$ . Repeat this process until  $b_i > 0$  for all pendant vertices. The resulting subtree is precisely the block-cut vertex tree  $B(H)$  of the subgraph  $H$  induced by  $\langle P \rangle$ .

**Step 6:** Check whether  $b_i = 1$  for any pendant vertex  $B_i$  of  $B(H)$ , then go to Step 17, otherwise continue.

**Step 7:** Identify the vertices of  $B(H)$ , which are  $B_i$ 's with degree two such that  $b_i = 0$ . Rename these vertices as  $u_1, u_2, \dots, u_q$ . If there is no such vertex then go to Step 10.

**For each  $r, 1 \leq r \leq q$ , repeat Step 8 to Step 9.**

**Step 8:** Find the components  $C_1, C_2$  of  $B(H) \setminus \{u_r\}$  and find the total number of vertices of  $P$  in  $C_1$  (say  $x$ ).

**Step 9:** If  $x$  is odd, then go to Step 17.

**Step 10:** If  $B(H)$  contains only one vertex, (say  $B_s$ ) then  $\frac{b_s}{2}$  vertices of  $P_{B_s}$  are added to  $X$  and remaining vertices of  $P_{B_s}$  are added to  $Y$  and go to Step 16. Otherwise start partitioning from one pendant vertex  $B_i$  of  $B(H)$  for some  $i$ .

**Step 11:** If  $b_i$  is even,  $\frac{b_i}{2}$  vertices of  $P_{B_i}$  are added to  $X$  and remaining vertices of  $P_{B_i}$  are added to  $Y$ . Go to Step 13.

**Step 12:** If  $b_i$  is odd and  $|X| = |Y|$ , then  $\frac{b_i+1}{2}$  vertices of  $P_{B_i}$  are added to  $X$  and remaining vertices of  $P_{B_i}$  are added to  $Y$ . If  $b_i$  is odd and  $|X| > |Y|$  then  $\frac{b_i+1}{2}$  vertices of  $P_{B_i}$  are added to  $Y$  and remaining vertices of  $P_{B_i}$  are added to  $X$ .

**Step 13:** If a new vertex adjacent to the already partitioned vertex has degree  $> 2$  during the DFS (completed one sparse branch), select another pendant vertex  $B_l$  of  $B(H)$ . Remove the already partitioned vertex from  $B(H)$  and continue partitioning from  $B_l$ . put  $i = l$  and go to Step 11.

**Step 14:** If a new vertex adjacent to the already partitioned vertex has degree  $\leq 2$  (still in the same sparse branch), select that new vertex for partitioning and remove the already partitioned vertex from  $B(H)$ . If the newly selected vertex is a block  $B_j$  then  $i = j$  and go to Step 11.

**Step 15:** If the vertex is a cut vertex of  $P_C$ , then that vertex is added to  $X$  if  $|X| = |Y|$  or that vertex is added to  $Y$  if  $|X| > |Y|$ .

**Step 16:** If  $B(H)$  contains only one vertex, then return  $(X, Y)$  and Stop.

**Step 17:** Return 'False' and Stop.

**Running time analysis:** At first, all the cut vertices and blocks of the given graph  $G$  are identified. Each part of Step 1 is a Depth First Search algorithm with time complexity  $O(n + m)$ . Thus the time complexity of this Step of the algorithm is  $O(n + m)$ . In Step 2, by visiting only the vertices of  $B_i, 1 \leq i \leq k$  we get the vertices of  $P$  which are cut vertices and which are non cut vertices in blocks. In Step 3, the cut vertices of a particular block are identified. The time complexity of Steps 2 and 3 is  $O(n)$ . To construct block cut vertex tree  $B(G)$  of  $G$ , we have to check whether which of the blocks contains a particular cut vertex and which of the cut vertices are connected to a particular block. Represent  $B(G)$  as adjacency list of blocks and cut vertices and the time complexity of Step 4 is  $O(n)$ . The time complexity of the construction of  $B(H)$  is also  $O(n)$ . Cut blocks of  $H$  which separates  $H$  into two components are identified in Step 7 with time complexity  $O(n + m)$ . In Step 8, all blocks in one component can be identified by using the adjacency list of vertices of  $B(H)$ . Here we find one cut vertex of the cut block. Then find all other blocks which contains this cut vertex. Then find the cut vertices of these selected blocks. Continuing this way we can select the blocks of one component. Since all blocks and cut vertices are already listed, time complexity of this Step is  $O(n)$ . The process is repeated at most  $r, r < n$  steps and so the total time complexity is  $O(n^2)$ . Time complexity of Steps 11 to 16 is  $O(n + m)$ . Thus the total time complexity of the algorithm is  $O(n^2)$ .

**Theorem 9.** *The Algorithm 5 is correct.*

*Proof.* In Step 1, we find all the cut vertices of the graph [8] and the blocks of the graph using the Hopcroft-Tarjan Algorithm described in [11]. The proof of the algo-

rithm completely depends on the theorems we proved in Section 3. In Proposition 2 it is proved that if an end block of  $G$  contains only one vertex of  $P$  other than cut vertex then the partition we obtained is not  $(t + 1)$ -tolerant Radon partition. From Theorem 1, it is clear that if  $G$  contains no cut blocks and end blocks contains at least two vertices of  $P$  other than cut vertices, then  $G$  has a  $(t + 1)$ -tolerant Radon partition. Theorem 2 proves that if  $G$  contains cut blocks that separate  $G$  into two components such that each component contains an even number of vertices of  $P$ , then  $P$  has a  $(t + 1)$ -tolerant Radon partition. Here, each end block must contain at least two vertices of  $P$  other than the cut vertex. Also, Theorem 2 states that if  $G$  contains a cut block that separates  $G$  into two components such that each component contains an odd number of vertices of  $P$ , then  $P$  has no  $(t + 1)$ -tolerant Radon partition. The proof of all these theorems completes the proof of the algorithm.  $\square$

**Conflict of Interest:** The authors declare that they have no conflict of interest.

**Data Availability:** Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

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