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Research Article

On harmonic complex Narayana-Lucas sequences and harmonic hybrid Narayana-Lucas sequences

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Abstract: In this paper, we begin by introducing the concept of complex Narayana-Lucas sequence. Then we proceed to discuss the concept of harmonic number within the framework of complex Narayana-Lucas sequence. Furthermore, we introduce hybrid numbers in the context of harmonic Narayana-Lucas sequence, accompanied by a set of fundamental definitions and theorems pertaining to these sequence. Additionally, we present several mathematical properties, such as generating functions, Binet formulas, and other significant identities related to these newly introduced sequence. Finally, we also provide source Maple 13 code to verify the occurrence of these newly introduced sequence.

Keywords: Narayana-Lucas sequence, hybrid number, harmonic numbers.

AMS Subject classification: 11K31, 39B32, 15A15, 11B50, 11C20, 05A15

1. Introduction

Number theory is an area of mathematics concerned with the properties and interactions of numbers, particularly integers and finds applications in various disciplines, including cryptography, computer science, quantum physics, and others, which interms provides a theoretical foundation setup for better understanding the properties of integers, Number sequence provide a practical and visible embodiment of these mathematical concepts, frequently with applications in a variety of scientific and mathematical domains [16]. During the 14^{th} century, Narayana, an Indian origin mathematician, made significant contributions in this fascinating field by introducing the

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Narayana numbers while investigating issues related to a herd of cows and calves [1]. The problem posed was as follows: At the beginning of each year, a cow produces one calf. Starting from the fourth year, each calf then produces one calf at the beginning of each subsequent year. How many calves are there in total after 20 years? [1]. If m is the year, then the recurrence relation of Narayana sequence is defined as $N_{m+3} = N_{m+2} + N_m$ for all $m \geq 0$ with initial conditions $N_0 = 2, N_1 = 3, N_2 = 4$, (see [18]). The recurrence relation of Lucas numbers are defined as $L_n = L_{n-1} + L_{n-2}$ for all $n > 1$ with initial conditions $L_0 = 2$ and $L_1 = 1$. In [11] the author defined a new recurrence relation which is called k-Lucas numbers. In [20] author described the generalization of Narayana's numbers as well as two further special cases, the Narayana-Lucas sequence and Narayana-Perrin sequence, and also defined some of their identities. The recurrence relation for Narayana-Lucas sequence is defined as $U_{n+3} = U_{n+2} + U_n$ for all $n \geq 0$ with initial conditions $U_0 = 3, U_1 = 1, U_2 = 1$ [20]. Recently, In [17] Özedmir was the first to introduce the concepts of hybrid numbers. Hybrid numbers are an amalgamation of real, complex, hyperbolic, and dual numbers. The set of hybrid numbers H is defined as

$$H = \{z = a + b\iota + c\epsilon + dh; a, b, c, d \in \mathbb{R}\}$$

where ι, ϵ, h are operators such that $\iota^2 = -1, \epsilon^2 = 0, h^2 = 1, \iota h = -h\iota = \epsilon + \iota$. The conjugate of hybrid numbers z as:$\overline{z} = \overline{a + b\iota + c\epsilon + dh} = a - b\iota - c\epsilon - dh$. The character of hybrid number z is defined as the real number $C(z) = z\overline{z} = \overline{z}z = a^2 + b^2 - 2bc - d^2$ and the norm of hybrid number z is defined as $\sqrt{|C(z)|}$ and denoted by $\|z\|$ [17]. In the development of science and technology, the integer sequence have played a significant role. As the concept of hybrid numbers as well as hybrid sequence have widely utilization in science, engineering, designing, hypothetical physical science, and many other branches of mathematics like linear algebra, kinematics, geometry and number theory can also be benefited from the same, so the study of these hybrid numbers and sequence in extended forms due to these hybrid numbers becomes important. Hybrid numbers with various sequence, have earned a lot of interest in recent year [7, 15, 21–23].

The H_n is defined as the n^{th} harmonic number as follows:

$$H_n = \sum_{k=1}^{n} \frac{1}{k}, \text{ where } H_0 = 0. \tag{1.1}$$

Harmonic numbers are used extensively in a variety of scientific disciplines. Their application extends to high-energy physics, where the harmonic series is used in complex calculations. In computer science, harmonic numbers are important for the investigation of algorithm efficacy. Analytic number theory makes practical use of the evaluation of harmonic number sums, since it provides insights into the distribution of prime numbers and other mathematical phenomena. Sources like [4, 5, 8] and relevant literature offer comprehensive insights and perspectives, facilitating a thorough exploration of these subjects. H_n can be alternatively represented using

Stirling numbers, as elucidated in the following manner, as per [6]:

$$H_n = \frac{S(n+1,2)}{n!}$$

where the second-kind stirling numbers are denoted by $S(n, 2)$. Several intriguing properties characterize harmonic numbers, some of which include [2, 3, 24]:

$$\sum_{k=1}^{n-1} H_k = nH_n - n,$$
$$\sum_{k=1}^{n} H_k^2 = (n+1)H_{n+1}^2 - (2n+3)H_{n+1} + 2n + 2,$$
$$\sum_{k=1}^{n} kH_k^2 = \frac{(n-1)(n+2)}{2}H_{n+1}^2 - \frac{n^2-3n-7}{2}H_{n-1} + \frac{n^2-9n-10}{4}.$$

The difference operator for any function $h(x)$ and anti-difference operator $\sum$ respectively can be defined in [12] as :

$$\Delta h(x) = h(x+1) - h(x),$$
$$\sum_{a}^{b} p(x)\Delta q(x)\delta_x = p(x)q(x)|_a^{b+1} - \sum_{a}^{b} Eq(x)\Delta p(x)\delta_x, \ where\ Eq(x) = q(x+1). \tag{1.2}$$

In the work by [12], the author employed a property of the finite difference operator to illustrate the validity of the identity. Numerous scientific disciplines depend on harmonic numbers, including high-energy physics computations, analyses of the effectiveness of algorithms used in computer science, and analytic number theory, where they are used to analyses harmonic number sums. In recent years, interest in the geometric and practical applications of hybrid numbers has significantly increased. Due to their applicability in different geometric and physical situations, these two-dimensional number systems have attracted the interest of several researchers. More details are given at the references [8, 19, 25]. In contrast, if u and v are real numbers, then any complex number can be expressed as $u + \iota v$. The denominator of a fraction that contains a complex number can be rewritten as follows:

$$\frac{1}{u+\iota v} = \frac{u-\iota v}{u^2+v^2} = \frac{u}{u^2+v^2} - \frac{\iota v}{u^2+v^2}.$$

Additionally, we explore generating functions, Binet formulas, and various other essential properties associated with these sequence. We also provide source Maple 13 code to verify the occurrence of these newly introduced sequence.

2. Complex Narayana-Lucas Sequences

Recently, many researcher work has been done in the field of complex and co-complex Fibonacci sequence [9, 10, 13]. In this section, we introduce the concept of complex Narayana-Lucas sequence by following a procedure mentioned in [9, 10, 13]. We substantiate our discussion with theorems and provide valuable Cassini's identity pertaining to these newly formed complex Narayana-Lucas sequence.

Definition 1. Let CU_n represent the n^{th} complex Narayana-Lucas sequence which can be defined as:

$$CU_n = U_n + \iota U_{n+1},$$

where U_n is the n^{th} Narayana-Lucas sequence. By using the recurrence relation and definition of Narayana-Lucas sequence [20], we obtain,

$$CU_n = (U_{n-1} + U_{n-3}) + \iota(U_n + U_{n-2}) = CU_{n-1} + CU_{n-3}.$$

Now, next we prove the Binet's formula, generating function and Cassini's identity for complex Narayana-Lucas sequence.

Theorem 1. *Let CU_n represent the n^{th} complex Narayana-Lucas sequence, then the Binet's formula can be defined as:*

$$CU_n = \mu^n \mu^{'} + \nu^n \nu^{'} + \lambda^n \lambda^{'}.$$

where μ, ν, λ are the roots of cubic equation $x^3 - x^2 - 1 = 0$ and $\mu^{'} = 1 + \iota\mu, \nu^{'} = 1 + \iota\nu, \lambda^{'} = 1 + \iota\lambda$.

Proof. Using the definition of complex Narayana-Lucas sequence and the Binet's formula for Narayana-Lucas sequence from [20], we have

$$\begin{aligned}
CU_n &= (\mu^n + \nu^n + \lambda^n) + \iota\left(\mu^{n+1} + \nu^{n+1} + \lambda^{n+1}\right), \\
&= (\mu^n + \iota\mu^{n+1}) + (\nu^n + \iota\nu^{n+1}) + (\lambda^n + \iota\lambda^{n+1}), \\
&= \mu^n(1 + \iota\mu) + \nu^n(1 + \iota\nu) + \lambda^n(1 + \iota\lambda), \\
CU_n &= \mu^n \mu^{'} + \nu^n \nu^{'} + \lambda^n \lambda^{'}.
\end{aligned}$$

where $\mu^{'} = 1 + \iota\mu, \nu^{'} = 1 + \iota\nu, \lambda^{'} = 1 + \iota\lambda$.
Thus the proof is completed for the Binet's formula of complex Narayana-Lucas sequence. □

Theorem 2. *Let CU_n represent the n^{th} complex Narayana-Lucas sequence, then the generating function can be defined as:*

$$g(t) = \frac{CU_0 + (CU_1 - CU_0)t + (CU_2 - CU_1)t^2}{1 - t - t^3}.$$

Proof. Let us define the generating function for the complex Narayana-Lucas sequence with the following formal power series as:

$$\begin{aligned} g(t) &= \sum_{k=1}^{n} CU_k t^k, \\ g(t) &= (CU_0 t^0 + CU_1 t^1 + CU_2 t^2 + \dots), \\ -tg(t) &= -t(CU_0 t^0 + CU_1 t^1 + CU_2 t^2 + \dots), \\ -t^3 g(t) &= -t^3(CU_0 t^0 + CU_1 t^1 + CU_2 t^2 + \dots). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} g(t) - tg(t) - t^3 g(t) =& (CU_0 t^0 + CU_1 t^1 + CU_2 t^2 + \dots) - t(CU_0 t^0 + CU_1 t^1 \\ &+ CU_2 t^2 + \dots) - t^3(CU_0 t^0 + CU_1 t^1 + CU_2 t^2 + \dots). \end{aligned}$$

We deduce that

$$g(t)(1 - t - t^3) = CU_0 + (CU_1 - CU_0)t + (CU_2 - CU_1)t^2.$$

Therefore, we have

$$g(t) = \frac{CU_0 + (CU_1 - CU_0)t + (CU_2 - CU_1)t^2}{(1 - t - t^3)}.$$

Thus the proof is completed for the generating function of complex Narayana-Lucas sequence. □

Theorem 3. *(Cassini's Identity) Let CU_n represent the n^{th} complex Narayana-Lucas sequence, then the following identity hold:*

$$\begin{aligned} CU_n^2 - CU_{n+1}CU_{n-1} =& 2U_n - \mu^{n-1}(\nu^{n+1} + \lambda^{n+1}) - \mu^{n-1}(\nu^{n+1} + \lambda^{n+1}) - \lambda^{n-1}(\mu^{n+1} \\ &+ \nu^{n+1}) - U_{n+1}^2 + \iota(U_n U_{n+1} - U_{n-1}U_{n+2}) \end{aligned}$$

Proof. In order to prove the identity, we use the definition of the complex Narayana-Lucas sequence (i.e. $CU_n = U_n + \iota U_{n+1}$), in the relation $CU_n^2 - CU_{n+1}CU_{n-1}$ and after on simplification, we get

$$\begin{aligned} CU_n^2 - CU_{n+1}CU_{n-1} =& (U_n^2 - U_{n-1}U_{n+1} - U_{n+1}^2 - U_n U_{n+2}) \\ &+ \iota(U_n U_{n+1} - U_{n-1}U_{n+2}) \end{aligned} \tag{2.1}$$

By using the Binet's formula for Narayana-Lucas sequence [20] and substituting in aforementioned equation, we obtain the relation in the complex form whose real and imaginary part are given below:
For the real part of the equation (2.1), we have

$$
\begin{aligned}
(U_n^2 - U_{n-1}U_{n+1} - U_{n+1}^2 - U_nU_{n+2}) &= (\mu^n + \nu^n + \lambda^n)^2 \\
&- (\mu^{n-1} + \nu^{n-1} + \lambda^{n-1})(\mu^{n+1} + \nu^{n+1} + \lambda^{n+1}) \\
&- (\mu^{n+1} + \nu^{n+1} + \lambda^{n+1})^2 \\
&- (\mu^n + \nu^n + \lambda^n)(\mu^{n+2} + \nu^{n+2} + \lambda^{n+2}) \\
&= 2(\mu^n\nu^n + \nu^n\lambda^n + \lambda^n\nu^n) - \mu^{n-1}(\nu^{n+1} + \lambda^{n+1}) \\
&- \nu^{n-1}(\mu^{n+1} + \lambda^{n+1}) - \lambda^{n-1}(\mu^{n+1} + \nu^{n+1}) \\
&- \mu^{(n+1)^2} - \nu^{(n+1)^2} - \lambda^{(n+1)^2} \\
&- 2(\mu^{n+1}\nu^{n+1} + \nu^{n+1}\lambda^{n+1} + \mu^{n+1}\lambda^{n+1})
\end{aligned}
$$

For the imaginary part of the equation (2.1), we have

$$
\begin{aligned}
U_nU_{n+1} - U_{n-1}U_{n+2} = U_nU_{n+1} - U_{n-1}U_{n+2} - \mu^n\mu^{n+1} - \nu^n\nu^{n+1} - \lambda^n\lambda^{n+1} \\
+ \mu^{n-1}\mu^{n+2} + \nu^{n-1}\nu^{n+2} + \lambda^{n-1}\lambda^{n+2}
\end{aligned}
$$

Make some necessary calculations, we get

$$
\begin{aligned}
CU_n^2 - CU_{n+1}CU_{n-1} =& 2U_n - \mu^{n-1}(\nu^{n+1} + \lambda^{n+1}) - \mu^{n-1}(\nu^{n+1} + \lambda^{n+1}) \\
&- \lambda^{n-1}(\mu^{n+1} + \nu^{n+1}) - U_{n+1}^2 + \iota(U_nU_{n+1} - U_{n-1}U_{n+2})
\end{aligned}
$$

Thus the proof is completed for the Cassini's identity of complex Narayana-Lucas sequence. □

3. Harmonic Complex Narayana-Lucas Sequences

In the preceding section 2, we have provided the definition of complex Narayana-Lucas sequence. In this section, we approach to combined the definition of harmonic number with complex Narayana-Lucas sequence which can be defined as:

$$H^cU_n = \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1}}.$$

The set of harmonic complex Narayana-Lucas sequence is as follows:

$$K = \left\{ \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1}} : \iota^2 = -1 \right\}$$

where H^cU_n represent the n^{th} harmonic complex Narayana-Lucas number and U_k is the k^{th} Narayana-Lucas numbers and $H^cU_0 = 0$.

Now, the scaler and vector parts of harmonic complex Narayana-Lucas sequence are:

$$H^cU_n = \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1}} = \sum_{k=1}^{n} \frac{U_k - \iota U_{k+1}}{U_k^2 + U_{k+1}^2}$$

Example 1. Let H^cU_n represent the n^{th} harmonic complex Narayana-Lucas number, for $n = 3$ we have

$$\begin{aligned} H^cU_3 &= \sum_{k=1}^{3} \frac{1}{U_k + \iota U_{k+1}} \\ &= \frac{1}{U_1 + \iota U_2} + \frac{1}{U_2 + \iota U_3} + \frac{1}{U_3 + \iota U_4} \\ &= \frac{1}{1+\iota} + \frac{1}{1+4\iota} + \frac{1}{4+5\iota} \\ &= \frac{915}{1394} - \frac{1195}{1394}\iota \end{aligned}$$

Algorithm 1 Maple code for the harmonic complex Narayana-Lucas sequence

```
return
with (Linear Algebra): with (Linalg):
n := −:
U:= proc(n)
if n ≥ 3 then
    return U(n − 1) + U(n − 3);
else
    return
end if;
SUM := (0);
for N from 1 to n by 1; do
    SUM := SUM + (1)/(U(N)^2) + ι ∗ U(N + 1)^2;
end for
end proc;
```

Now, we define the generating functions and Binet's formula for harmonic complex Narayana-Lucas sequence:

Theorem 4. *Let H^cU_n represent the n^{th} harmonic complex Narayana-Lucas sequence, then the generating functionx can be defined as:*

$$G(t) = \frac{H^cU_0 + (H^cU_1 - H^cU_0)t + (H^cU_2 - H^cU_1)t^2}{1 - t - t^3}.$$

Proof. Let us define the generating function for the harmonic complex Narayana-Lucas sequence with the following formal power series as:

$$G(t) = \sum_{k=1}^{n} H^c U_k t^k,$$

We deduce that

$$\begin{aligned} G(t) &= H^c U_0 t^0 + H^c U_0 t^1 + H^c U_0 t^2 + \dots + H^c U_n t^n + \dots, \\ -tG(t) &= -t(H^c U_0 t^0 + H^c U_1 t^1 + H^c U_2 t^2 + \dots + H^c U_n t^n + \dots), \\ -t^3 G(t) &= -t^3(H^c U_0 t^0 + H^c U_1 t^1 + H^c U_2 t^2 + \dots + H^c U_n t^n + \dots), \end{aligned}$$

Thus, we obtain

$$G(t)(1 - t - t^3) = H^c U_0 + (H^c U_1 - H^c U_0)t + (H^c U_2 - H^c U_1)t^2,$$

Therefore, we have

$$G(t) = \frac{H^c U_0 + (H^c U_1 - H^c U_0)t + (H^c U_2 - H^c U_1)t^2}{1 - t - t^3}.$$

Thus the proof is completed for the generating function of harmonic complex Narayana-Lucas sequence. □

Theorem 5. *Let $H^c U_n$ represent the n^{th} harmonic complex Narayana-Lucas sequence, then the Binet's formula can be defined as:*

$$H^c U_n = \sum_{k=1}^{n} \frac{1}{\mu^k \mu' + \nu^k \nu' + \lambda^k \lambda'}$$

where $\mu^{'} = 1 + \iota\mu, \nu^{'} = 1 + \iota\nu, \lambda^{'} = 1 + \iota\lambda.$

Proof. By using the definition of the harmonic complex Narayana-Lucas sequence and by using the Binet's formula for Narayana-Lucas sequence, we have:

$$\begin{aligned} H^c U_n &= \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1}}, \\ &= \sum_{k=1}^{n} \frac{1}{(\mu^k + \nu^k + \lambda^k) + \iota\,(\mu^{k+1} + \nu^{k+1} + \lambda^{k+1})}, \\ &= \sum_{k=1}^{n} \frac{1}{\mu^k(1 + \iota\mu) + \nu^k(1 + \iota\nu) + \lambda^k(1 + \iota\lambda)}, \\ H^c U_n &= \sum_{k=1}^{n} \frac{1}{\mu^k \mu' + \nu^k \nu' + \lambda^k \lambda'}. \end{aligned}$$

Thus the proof is completed for the Binet's formula of harmonic complex Narayana-Lucas sequence. □

Now, from [14], for all $H^cU_n = \sum_{k=1}^{n} \frac{1}{U_k+\iota U_{k+1}}, H^cU_m = \sum_{k=1}^{m} \frac{1}{U_k+\iota U_{k+1}} \in K$, then the fundamental operators can be defined as below:

(a) Addition:

$$(i)\quad \textit{If}\ \ m = n, H^cU_n + H^cU_m = 2\sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1}}.$$

$$(ii)\quad \textit{If}\ \ m < n, H^cU_n + H^cU_m = 2\sum_{k=1}^{m} \frac{1}{U_k + \iota U_{k+1}} + \sum_{k=m+1}^{n} \frac{1}{U_k + \iota U_{k+1}}.$$

$$(iii)\quad \textit{If}\ \ m > n, H^cU_n + H^cU_m = 2\sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1}} + \sum_{k=n+1}^{m} \frac{1}{U_k + \iota U_{k+1}}.$$

(b) Multiplication:

$$H^cU_n.H^cU_m = \left(\sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1}}\right) \cdot \left(\sum_{k=1}^{m} \frac{1}{U_k + \iota U_{k+1}}\right)$$

(c) Complex conjugate:

$$\overline{H^cU_n} = \sum_{k=1}^{n} \frac{U_k + \iota U_{k+1}}{U_k^2 + U_{k+1}^2}$$

Algorithm 2 Maple code for the harmonic complex conjugate operations on Narayana-Lucas sequence

```
return
with (Linear Algebra): with (Linalg):
n := −:
U:= proc(n)
if n ≥ 3 then
    return  U(n − 1) + U(n − 3);
else
    return
end if;
SUM := (0);
for N from 1 to n by 1; do
    SUM:=SUM+(U(N) + ιU(N + 1))/(U(N)^2 + U(N + 1)^2);
end for
end proc;
```

Theorem 6. *Let H^cU_n represent the n^{th} harmonic complex Narayana-Lucas sequence. Then we have*

$$\sum_{k=0}^{n} H^cU_n = nH^cU_n - \sum_{k=0}^{n-1} \frac{k+1}{C_{k+1}}$$

Proof. On substituting $p(k) = H^cU_k$ and $\Delta p(k) = 1$ in equation (1.2), we get $q(k) = k, Eq(k) = (k+1)$ and $\Delta p(k) = \frac{1}{C_{k+1}}$.
Thus, we have

$$\sum_{k=0}^{n-1} H^cU_n = nH^cU_n - \sum_{k=0}^{n-1} \frac{k+1}{C_{k+1}}.$$

□

Example 2. Let H^cU_n represent the n^{th} harmonic complex Narayana-Lucas sequence, for $n = 2$, we have

$$\begin{aligned}\sum_{k=0}^{1} H^cU_k &= 2H^cU_2 - \sum_{k=0}^{1} \frac{k+1}{C_{k+1}}\\ &= 2H^cU_2 - \left(\frac{1}{C_1} + \frac{2}{C_2}\right)\\ &= 2\left(\frac{19}{34} - \iota\frac{25}{34}\right) - \left(\frac{1}{1+\iota} + \frac{2}{1+\iota 4}\right)\\ &= H^cU_0 + H^cU_1.\end{aligned}$$

Theorem 7. *Let H^cU_n represent the n^{th} harmonic complex Narayana-Lucas sequence. Then we have*

$$\sum_{k=0}^{n-1}(H^cU_k)^2 = n(H^cU_n)^2 - \sum_{k=0}^{n-1} \frac{k+1}{C_{k+1}}\left(2H^cU_k + \frac{1}{C_{k+1}}\right).$$

Proof. On Substituting $p(k) = H^cU_k^2$ and $\Delta q(k) = k$ in equation (1.2), we get $q(k) = k, Eq(k) = (k+1)$ and $\Delta p(k) = \frac{1}{C_{k+1}}\left(2H^cU_k + \frac{1}{C_{k+1}}\right)$. Thus, we have

$$\sum_{k=0}^{n-1}(H^cU_k)^2 = n(H^cU_n)^2 - \sum_{k=0}^{n-1} \frac{k+1}{C_{k+1}}\left(2H^cU_k + \frac{1}{C_{k+1}}\right).$$

□

Example 3. Let H^cU_n represent the n^{th} harmonic complex Narayana-Lucas sequence, for $n = 2$, we have

$$\begin{aligned}\sum_{k=0}^{1}(H^cU_k)^2 &= 2(H^cU_2)^2 - \sum_{k=0}^{1} \frac{k+1}{C_{k+1}}\left(2H^cU_k + \frac{1}{C_{k+1}}\right)\\ &= 2(H^cU_2)^2 - \left(\frac{1}{C_1}\left(2H^cU_0 + \frac{1}{C_1}\right) + \frac{2}{C_2}\left(2H^cU_1 + \frac{1}{C_2}\right)\right)\\ &= 2\left(\frac{19}{34} - \iota\frac{25}{34}\right)^2 - \left(\frac{1}{1+\iota}\left(2(0) + \frac{1}{1+\iota}\right) + \frac{2}{1+\iota 4}\left(2\left(\frac{1}{1+\iota}\right) + \frac{1}{1+\iota 4}\right)\right)\\ &= (H^cU_0)^2 + (H^cU_1)^2.\end{aligned}$$

Theorem 8. *Let H^cU_n represent the n^{th} harmonic complex Narayana-Lucas sequence and m be a non negative integer. Then we have*

$$\sum_{k=0}^{n-1}\binom{k}{m}(H^cU_k)=\binom{n}{m+1}H^cU_n-\sum_{k=0}^{n-1}\binom{k+1}{m+1}\left(\frac{1}{C_{k+1}}\right).$$

Proof. On substituting $p(k)=H^cU_k$ and $\Delta q_k=\binom{k}{m}$ in (1.2), we get $q(k)=\binom{k}{m+1}, Eq(k)=\binom{k+1}{m+1}$ and $\Delta p(k)=\frac{1}{C_{k+1}}$.
Thus, we have

$$\sum_{k=0}^{n-1}\binom{k}{m}(H^cU_k)=\binom{n}{m+1}H^cU_n-\sum_{k=0}^{n-1}\binom{k+1}{m+1}\left(\frac{1}{C_{k+1}}\right).$$

□

Example 4. Let H^cU_n represent the n^{th} harmonic complex Narayana-Lucas sequence, for $n=3$ and $m=0$, we have

$$\begin{aligned}\sum_{k=0}^{2}\binom{k}{m}(H^cU_k)&=\binom{n}{m+1}H^cU_n-\sum_{k=0}^{n-1}\binom{k+1}{m+1}\left(\frac{1}{C_{k+1}}\right)\\&=\binom{3}{1}H^cU_3-\sum_{k=0}^{2}\binom{k+1}{1}\left(\frac{1}{C_{k+1}}\right)\\&=3H^cU_3-\left(\frac{1}{C_1}+\frac{2}{C_2}+\frac{3}{C_3}\right)\\&=3\left(\frac{915}{1394}-\iota\frac{1195}{1394}\right)-\left(\frac{1}{1+\iota}+\frac{2}{1+\iota 4}+\frac{3}{4+\iota 5}\right)\\&=\binom{1}{0}H^cU_1+\binom{2}{0}H^cU_2.\end{aligned}$$

4. Harmonic Hybrid Narayana-Lucas Sequences

In this section, we define the harmonic hybrid Narayana-Lucas sequence. In addition to this, we define some basic identities and algebraic properties related to this newly introduced sequence. The real, complex, dual, and hyperbolic units can be represented respectively as mentioned below [17] :

$$1\leftrightarrow\{1,0,0,0\},\iota\leftrightarrow\{0,1,0,0\},\epsilon\leftrightarrow\{0,0,1,0\},h\leftrightarrow\{0,0,0,1\}.$$

In this context, these units are known as hybrid units. Additionally, the basis of a 2×2 matrix set linked to a hybrid units is represented by the four matrices [17] as :

$$1\leftrightarrow\begin{bmatrix}1&0\\0&1\end{bmatrix},\iota\leftrightarrow\begin{bmatrix}0&1\\-1&0\end{bmatrix},\epsilon\leftrightarrow\begin{bmatrix}1&-1\\1&-1\end{bmatrix},h\leftrightarrow\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

Now, the harmonic hybrid Narayana-Lucas sequence can be defined as :

$$HU_n = \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}},$$

The set of harmonic hybrid Narayana-Lucas sequence is as follows:

$$K = \left\{ \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}} : \iota^2 = -1, \epsilon^2 = 0, h^2 = 1, \iota h = -h\iota = \epsilon + \iota. \right\}$$

where U_k is the k^{th} Narayana-Lucas number.
As the harmonic hybrid Narayana-Lucas sequence can also be written as the combination of scaler and vector parts. So, the scaler and vector part of harmonic hybrid Narayana-Lucas sequence respectively can be defined as :

$$S(HU_n) = \sum_{k=1}^{n} \frac{U_k}{U_k^2 + (U_{k+1} - U_{k+2})^2 - U_{k+2}^2 - U_{k+3}^2},$$

and

$$V(HU_n) = \sum_{k=1}^{n} \frac{-\iota U_{k+1} - \epsilon U_{k+2} - hU_{k+3}}{U_k^2 + (U_{k+1} - U_{k+2})^2 - U_{k+2}^2 - U_{k+3}^2}.$$

If all of their components are equal, then the harmonic hybrid Narayana-Lucas numbers are also equal. Zero is defined as null elements. Additionally, the summation of each component determines the sum of harmonic hybrid Narayana-Lucas number. Harmonic hybrid Narayana-Lucas has associative and commutative addition operations. $-HU_n$ is the representation of the inverse of HU_n. Hence $(HU_n, +)$ is an abelian group.

Example 5. Let $H^c U_n$ represent the n^{th} harmonic hybrid Narayana-Lucas sequence, for $n = 2$ we have

$$\begin{aligned} HU_2 &= \sum_{k=1}^{2} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}} \\ &= \frac{1}{U_1 + \iota U_2 + \epsilon U_3 + hU_4} + \frac{1}{U_2 + \iota U_3 + \epsilon U_4 + hU_5} \\ &= \frac{1}{1 + \iota + 4\epsilon + 5h} + \frac{1}{1 + 4\iota + 5\epsilon + 6h} \\ &= \frac{-90}{1829} + \frac{183}{1829}\iota + \frac{391}{1829}\epsilon + \frac{481}{1829}h \end{aligned}$$

Algorithm 3 Maple code for the harmonic hybrid Narayana-Lucas sequence

```
return
with (Linear Algebra): with (Linalg):
n := −:
U:= proc(n)
if n ≥ 3 then
   return U(n − 1) + U(n − 3);
else
   return
end if;
SUM := (0); ϵ^2 = 0 : h^2 = 1 : I.h = −h.I := ϵ + I;
for N from 1 to n by 1; do
   SUM:=SUM+(U(N) + ι.U(N + 1) − ϵ.U(N + 2) − h.U(N + 3))/(U(N)^2 + (U(N +
   1) − U(N + 2))^2 − U(N + 2)^2 − U(N + 3)^2);
end for
end proc;
```

For any two harmonic hybrid Narayana-Lucas sequence (where m is different from n)

$$HU_n = \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}, HU_m = \sum_{k=1}^{m} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}$$

the inner product

$$\left(\sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}\right) \cdot \left(\sum_{k=1}^{m} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}\right)$$

is arrived at by applying each unit product in accordance with the multiplication table and distributing the terms to the right.

.	1	ι	ϵ	h
1	1	ι	ϵ	h
ι	ι	-1	1-h	ϵ+ι
ϵ	ϵ	h+1	0	-ϵ
h	h	-ϵ - ι	ϵ	1

The table demonstrates that the multiplication operation in the harmonic hybrid Narayana-Lucas sequence is not commutative. However, it does satisfy the associative property.

Definition 2. The Harmonic hybrid Narayana-Lucas conjugate sequence, denoted by $\overline{HU_n}$ and can be defined as

$$\overline{(HU_n)} = \sum_{k=1}^{n} \frac{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}{U_k^2 + (U_{k+1} - U_{k+2})^2 - U_{k+2}^2 - U_{k+3}^2},$$

Example 6. Let $\overline{HU_n}$ be the n^{th} harmonic hybrid Narayana-Lucas conjugate number, for $n = 2$ we have

$$\overline{HU_2} = \frac{-90}{1829} - \frac{183}{1829}\iota - \frac{391}{1829}\epsilon - \frac{481}{1829}h.$$

Additionally, based on inner product, we obtain $HU_n\overline{HU_n} = \overline{HU_n}HU_n$. The real number

$$\begin{aligned} C(HU_n) &= HU_n\overline{HU_n} = \overline{HU_n}HU_n \\ &= -\left\langle \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}, \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}} \right\rangle \\ &= - < A_n, A_n > \end{aligned}$$

where $\quad A_n = \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}.$

Moreover, the real number $\sqrt{C(HU_n)}$ represents the norm of the harmonic hybrid Narayana-Lucas number and is also denoted as $||HU_n||$.

Definition 3. The inverse of harmonic hybrid Narayana-Lucas number, $||HU_n|| \neq 0$ is defined as:

$$\begin{aligned} HU_n^{-1} &= \frac{\overline{HU_n}}{C(HU_n)} \\ &= \frac{\sum_{k=1}^{n} \frac{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}{U_k^2 + (U_{k+1} - U_{k+2})^2 - U_{k+2}^2 - U_{k+3}^2}}{(\sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}})^2} \end{aligned}$$

Consequently, it is evident that the set of harmonic hybrid Narayana-Lucas numbers is non-commutative in terms of addition and multiplication operations.

Example 7. For $n = 2$, the inverse of $HU_2 = \sum_{k=1}^{2} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}$, $||HU_2|| \neq 0$ is:

$$\begin{aligned} HU_2^{-1} &= \frac{\overline{HU_2}}{CHU_2} = \frac{\frac{-90 - 183\iota - 391\epsilon - 481h}{1829}}{(\frac{-90 - 183\iota - 391\epsilon - 481h}{1829})(\frac{-90 + 183\iota + 391\epsilon + 481h}{1829})} \\ HU_2^{-1} &= \frac{1829}{-90 + 183\iota + 391\epsilon + 481h} \end{aligned}$$

The scaler product of HU_n and HU_m can be defined as follows:

$$h : K \times K \to \mathbb{R},$$

$$(HU_n, HU_m) \to h(HU_n, HU_m) = \frac{HU_n.\overline{HU_m} + HU_m.\overline{HU_n}}{2},$$

where $HU_n = \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}$ and $HU_m = \sum_{k=1}^{m} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}$.

In addition, the vector product of HU_n and HU_m is :

$$h : K \times K \to K, (HU_n \times HU_m) = \frac{HU_n.\overline{HU_m} - HU_m.\overline{HU_n}}{2}$$

The rules for the vector product satisfied in the following table :

$\times$	1	ι	ϵ	h
1	0	$-\iota$	$-\epsilon$	$-h$
ι	ι	0	h	$-\epsilon$-ι
ϵ	ϵ	$-h$	0	ϵ
h	h	-$\epsilon + \iota$	$-\epsilon$	0

A matrix representation of harmonic hybrid Narayana-Lucas number simplifies the multiplication of harmonic hybrid Narayana-Lucas number. By demonstrating an isomorphism between 2×2 matrices and harmonic hybrid Narayana-Lucas number, we can easily multiply harmonic hybrid Narayana-Lucas number and also define various properties of them.

Theorem 9. *An isomorphism exists between the harmonic hybrid Narayana-Lucas number of ring K and the ring of 2×2 matrices $M_{2\times2}$.*

Proof. Define the map $\psi : K \to M_{2\times2}$,

$$\psi\left(\sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}}\right)$$

is corresponds to the matrix representation:

$$\frac{1}{T}\begin{pmatrix} -\sum_{k=1}^{n} U_{k+1} & -2\sum_{k=1}^{n} U_{k+1} \\ -2\sum_{k=1}^{n} U_{k+2} & \sum_{k=1}^{n}(U_k + U_{k+2}) \end{pmatrix}$$

where $T = \sum_{k=1}^{n}(U_k + U_{k+2}) - 4\sum_{k=1}^{n}(U_{k+1}U_{k+2})$.
The ring isomorphism of this mapping is satisfied. Simple demonstration is sufficient to establish that these equality's are valid:

$$\psi(HU_nHU_m) = \psi(HU_n)\psi(HU_m)$$
$$\psi(HU_n + HU_m) = \psi(HU_n) + \psi(HU_m)$$

in which the harmonic hybrid Narayana-Lucas number HU_n and HU_m are defined. Additionally, this mapping possesses the desirable properties of being bijective. Alternatively, for any 2×2 real matrix, we have

$$A = \begin{pmatrix} \sum_{k=1}^{n} \frac{U_k}{U_k^2+(U_{k+1}-U_{k+2})^2-U_{k+2}^2-U_{k+3}^2} & \sum_{k=1}^{n} \frac{-U_{k+2}}{U_k^2+(U_{k+1}-U_{k+2})^2-U_{k+2}^2-U_{k+3}^2} \\ \sum_{k=1}^{n} \frac{-U_{k+1}}{U_k^2+(U_{k+1}-U_{k+2})^2-U_{k+2}^2-U_{k+3}^2} & \sum_{k=1}^{n} \frac{-U_{k+3}}{U_k^2+(U_{k+1}-U_{k+2})^2-U_{k+2}^2-U_{k+3}^2} \end{pmatrix}$$

there is a harmonic hybrid Narayana-Lucas number

$$HU_n = \left(\frac{\sum_{k=1}^{n}(\frac{2U_k}{U_k^2+(U_{k+1}-U_{k+2})^2-U_{k+2}^2-U_{k+3}^2})}{2}\right) + \left(\frac{\sum_{k=1}^{n}(\frac{-U_{k+1}+U_{k+2}-U_{k+3}}{U_k^2+(U_{k+1}-U_{k+2})^2-U_{k+2}^2-U_{k+3}^2})}{2}\right)\iota$$
$$+ \left(\frac{\sum_{k=1}^{n}(\frac{-U_k-U_{k-3}}{U_k^2+(U_{k+1}-U_{k+2})^2-U_{k+2}^2-U_{k+3}^2})}{2}\right)\epsilon + \left(\frac{\sum_{k=1}^{n}(\frac{-2U_{k+1}-2U_{k+2}}{U_k^2+(U_{k+1}-U_{k+2})^2-U_{k+2}^2-U_{k+3}^2})}{2}\right)h$$

where $\psi(H_n) = A$. Hence ψ is a ring isomorphism. □

Theorem 10. *(Binet Formulas) For $n \geq 1$, the Binet formulas for harmonic hybrid Narayana-Lucas sequence is given as follows:*

$$HU_n = \sum_{k=1}^{n} \frac{1}{\mu^k \mu_1 + \nu^k \nu_1 + \lambda^k \lambda_1}$$

where $\mu_1 = 1 + \iota\mu + \epsilon\mu^2 + h\mu^3, \nu_1 = 1 + \iota\nu + \epsilon\nu^2 + h\nu^3, \lambda_1 = 1 + \iota\lambda + \epsilon\lambda^2 + h\lambda^3$.

Proof. Now, by using the Binet formulas for Narayana-Lucas sequence and and using the definition of harmonic hybrid number for Narayana-Lucas sequence can be define as:

$$\begin{aligned} HU_n &= \sum_{k=1}^{n} \frac{1}{U_k + \iota U_{k+1} + \epsilon U_{k+2} + hU_{k+3}} \\ &= \sum_{k=1}^{n} \frac{1}{(\mu^k + \nu^k + \lambda^k) + \iota(\mu^{k+1} + \nu^{k+1} + \lambda^{k+1}) + \epsilon(\mu^{k+2} + \nu^{k+2} + \lambda^{k+2}) + h(\mu^{k+3} + \nu^{k+3} + \lambda^{k+3})} \\ &= \sum_{k=1}^{n} \frac{1}{(\mu^k(1 + \iota\mu + \epsilon\mu^2 + h\mu^3) + \nu^k(1 + \iota\nu + \epsilon\nu^2 + h\nu^3) + \lambda^k(1 + \iota\lambda + \epsilon\lambda^2 + h\lambda^3))} \end{aligned}$$

$$HU_n = \sum_{k=1}^{n} \frac{1}{\mu^k \mu_1 + \nu^k \nu_1 + \lambda^k \lambda_1}$$

Thus the proof is completed for the harmonic hybrid Narayana-Lucas sequences. □

5. Conclusion

In the present paper, we introduced the concept of harmonic complex Narayana-Lucas sequence. Firstly, we introduced the concept of complex Narayana-Lucas sequence and then introduced the harmonic number concept in the complex Narayana-Lucas sequence. We also introduced the hybrid number in the harmonic Narayana-Lucas sequence and also defined Binet formulas, generating functions, and some other algebraic identities for the harmonic complex Narayana-Lucas sequence. Moreover, we also provide a Maple source code to validate our findings.

Conflict of interest. The authors declare that they have no conflict of interest.

Data Availability. Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

References

[1] J.P. Allouche and T. Johnson, *Narayana's cows and delayed morphisms*, Journées d'Informatique Musicale, 1996.

[2] M. Bahşi, *On the norms of r-circulant matrices with the hyperharmonic numbers*, J. Math. Inequalities **10** (2016), no. 2, 445–458. https://doi.org/10.7153/jmi-10-35.

[3] M. Bahsi and S. Solak, *On the matrices with harmonic numbers*, Gazi Univ. J. Sci. **23** (2010), no. 4, 445–448.

[4] B.C. Berndt, *Ramanujan's Theory of Divergent Series*, Ramanujan's Notebooks: Part I, Springer, New York, 1985, pp. 133–149.

[5] ______, *Ramanujan's Theory of Prime Numbers*, Ramanujan's Notebooks: Part IV, Springer, New York, 1994, pp. 111–137.

[6] L. Carlitz, *Weighted Stirling numbers of the first and second kind—I*, Fibonacci Quart. **18** (1980), no. 2, 147–162. https://doi.org/10.1080/00150517.1980.12430168.

[7] P. Catarino, *On k-pell hybrid numbers*, J. Discrete Math. Sci. Cryptogr. **22** (2019), no. 1, 83–89. https://doi.org/10.1080/09720529.2019.1569822.

[8] M.W. Coffey and N. Lubbers, *On generalized harmonic number sums*, Appl. Math. Comput. **217** (2010), no. 2, 689–698. https://doi.org/10.1016/j.amc.2010.06.006.

[9] Ö. Deveci, S. Hulku, and A.G. Shannon, *On the co-complex-type k-Fibonacci numbers*, Chaos Solit. Fractals **153** (2021), Article ID: 111522. https://doi.org/10.1016/j.chaos.2021.111522.

[10] Ö. Deveci and A.G. Shannon, *The complex-type k-Fibonacci sequences and their applications*, Commun. Algebra **49** (2021), no. 3, 1352–1367. https://doi.org/10.1080/00927872.2020.1834573.

[11] S. Falcon, *On the k-Lucas numbers*, Int. J. Contemp. Math. Sci. **6** (2011), no. 21, 1039–1050.

[12] R.L. Graham, D.E. Knuth, and O. Patashnik, *Fibonacci and Lucas Numbers with Applications, Volume 2*, Addison-Wesley Professional, 1994.

[13] A.F. Horadam, *Complex Fibonacci numbers and Fibonacci quaternions*, Am. Math. Mon. **70** (1963), no. 3, 289–291.

[14] E. Karaca and F. Yilmaz, *Some characterizations for harmonic complex fibonacci sequences*, International Conference on Mathematics and its Applications in Science and Engineering (Cham) (F. Yilmaz, Araceli Queiruga-Dios, M.J. Santos Sánchez, D. Rasteiro, V. Gayoso Martínez, and J. Martín Vaquero, eds.), Springer International Publishing, 2021, pp. 159–165.

[15] C. Kızılateş, *A new generalization of Fibonacci hybrid and Lucas hybrid numbers*, Chaos Solit. Fractals **130** (2020), Article ID: 109449.
https://doi.org/10.1016/j.chaos.2019.109449.

[16] T. Koshy, *Fibonacci and Lucas Numbers with Applications, Volume 2*, John Wiley & Sons, 2019.

[17] M. Özdemir, *Introduction to hybrid numbers*, Adv. Clifford Algebras **28** (2018), no. 1, 1–32.
https://doi.org/10.1007/s00006-018-0833-3.

[18] S. Petroudi, M. Pirouz, and A. Özkoç, *The Narayana polynomial and Narayana hybrinomial sequences*, Konuralp J. Math. **9** (2021), no. 1, 90–99.

[19] D. Quadling, *Ramanujan's notebooks, by bruce c. berndt. part 1. pp 357. dm188. 1985. isbn 3-540-96110-0 (springer)*, Math. Gaz. **70** (1986), no. 451, 84–85.
https://doi.org/10.2307/3615877.

[20] Y. Soykan, *On generalized Narayana numbers*, Int. J. Adv. Appl. Math. Mech. **7** (2020), no. 3, 43–56.

[21] A. Szynal-Liana, *The Horadam hybrid numbers*, Discus. Math., Gen. Algebra Appl. **38** (2018), no. 1, 91–98.
http://dx.doi.org/10.7151/dmgaa.1287.

[22] A. Szynal-Liana and I. Włoch, *On Pell and Pell- Lucas hybrid numbers*, Comment. Math. **58** (2018), no. 1-2, 11–17.
http://dx.doi.org/10.14708/cm.v58i1-2.6364.

[23] ______, *On Jacobsthal and Jacobsthal-Lucas hybrid numbers*, Ann. Math. Sil. **33** (2019), 276–283.
https://doi:10.2478/amsil-2018-0009.

[24] N. Tuglu, C. Kızılateş, and S. Kesim, *On the harmonic and hyperharmonic Fibonacci numbers*, Adv. Differ. Equ. **2015** (2015), no. 1, Article ID: 297.
https://doi.org/10.1186/s13662-015-0635-z.

[25] I.J. Zucker, *Ramanujan's notebooks: Part 2, edited by bruce c. berndt. pp 359. dm 168. 1989. isbn 3-540-96794-x (springer)*, Math. Gaz. **74** (1990), no. 467, 95–97.
https://doi.org/10.2307/3618899.