

Some results on the strongly annihilator ideal graph of a lattice

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Abstract: For a lattice L , the strongly annihilator ideal graph of L is denoted by $SAnnIG(L)$. It is a graph with the vertex set, which consists of all ideals in L that have nontrivial annihilators such that any two distinct vertices I and J are adjacent in $SAnnIG(L)$ if and only if the annihilator of I contains a nonzero element of J and the annihilator of J contains a nonzero element of I . In this paper, we determine the radius, circumference, and domination number of $SAnnIG(L)$. We obtain necessary and sufficient conditions for $SAnnIG(L)$ to be in the class of paths, cycles, unicyclic, triangle-free, trees, complete multipartite, split or claw-free graphs. Among other results, we study the affinity between the strongly annihilator ideal and the annihilator ideal graph of a lattice.

Keywords: lattice, ideal, domination number and strongly annihilator ideal graph.

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1. Introduction

The concept of zero divisor graphs from algebraic structures is initiated by Beck in [7]. For a commutative ring R , he let all the elements of R as the vertex set such that any two distinct vertices x and y are adjacent if and only if $xy = 0$. In the said work, his complete focus was on coloring of the graph. Later, Anderson and Livingston studied the zero divisor graph of a commutative ring R in [5]. They considered the set of all non-zero zero-divisors in R as the vertex set and studied various graph-theoretic properties. For a lattice L with bottom element 0, the zero divisor graph of L is introduced by Estaji *et al.* in [9], denoted by $ZG(L)$. The $ZG(L)$ is an undirected

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graph with vertex set as $Z^*(L) = \{a \in L \mid a \neq 0, \exists b \neq 0 \text{ such that } a \wedge b = 0\}$ and for any $u, v \in Z^*(L)$, $u - v$ is an edge in $ZG(L)$ if and only if $u \wedge v = 0$. In [14], Kulal *et al.* studied the annihilator ideal graph of a lattice L , denoted by $AnnIG(L)$. It is a graph with the vertex set $N(L) = \{I \mid I \text{ is a non trivial ideal in } L \mid Ann(I) \neq \{0\}\}$ and two distinct vertices I and J are adjacent if and only if $I \cap Ann(J) \neq \{0\}$ or $J \cap Ann(I) \neq \{0\}$. To get more work related to the zero divisor graphs from algebraic structures, see [1–4, 6, 8, 11, 13, 15].

In [16], Tohidi *et al.* studied the strongly annihilating-ideal graph of a commutative ring R with the vertex set $A(R)^* = A(R) \setminus \{0\}$, where $A(R)$ is the set consists of all ideals with nonzero annihilator and any two distinct vertices I and J are adjacent if and only if $I \cap Ann(J) \neq 0$ and $J \cap Ann(I) \neq 0$. Motivated by this work, in [12], we have defined the strongly annihilator ideal graph of a lattice L , denoted by $SAnnIG(L)$. The vertex set of $SAnnIG(L)$ is a set consists of all ideals in L whose annihilator is nontrivial and any two distinct vertices I and J are adjacent if and only if $I \cap Ann(J) \neq 0$ and $J \cap Ann(I) \neq 0$.

Throughout this paper, $L = \langle L, \wedge, \vee \rangle$ is an atomic lattice with the least element 0 and greatest element 1. A sublattice I of L is said to be an ideal of L if $u \wedge i \in I$ for all $u \in L$ and $i \in I$. For two elements $u, v \in L$ we have $v < u$, if $v \wedge u = v$ and $v \vee u = u$. An element $u \in L$ is said to be least element if $u < v$ for every element $v \in L \setminus \{u\}$. An element $a \in L$ is said to be an atom if $0 < a$ and there is no u such that $0 < u < a$, where 0 is the least element in L . Dually, we have the concept of coatom. For any $u \in L$, the set $(u) = \{v \in L \mid v \leq u\}$ is the principal ideal in L generated by u and $Ann(u) = \{v \in L \mid u \wedge v = 0\}$ is the annihilator of u . Denote the set $[u]_z = \{v \in L \mid v \geq u \text{ and } v \in Z^*(L)\}$. We denote the set of all atoms in L by $A(L)$. Any two elements u and v in L are said to be incomparable if $u \not\leq v$ and $v \not\leq u$, we denote the same by $u \parallel v$. A lattice L is called atomic if, for each element $v \in L \setminus \{0\}$, there exists an atom $a_v \in A(L)$ such that $a_v \leq v$. If $p \in L$, then $\mathcal{B}(p) = \{a_p \in A(L) \mid a_p \leq p\}$ is defined as the base of p in L . For more concepts in lattice theory, we refer [10].

If V is a nonempty set (known as vertices) and E is a set of two-subsets of V (known as edges), then $G = (V, E)$ is known as a graph on the vertex set V and the edge set E . The two-subset $e = \{u, v\} \in E$ is called an edge between u and v , and in this case, we say that u and v are adjacent in G and this edge is denoted by $u - v$. If there is a path joining any two distinct vertices of G , then the graph G is said to be connected; otherwise, we say that G is disconnected. For two vertices, a and $b \in V(G)$, the length of a shortest path from a to b is denoted by $d(a, b)$, called the distance between a and b , and $diam(G) = \sup\{d(a, b) \mid a, b \in V(G)\}$ is called the diameter of G . The length of a shortest cycle in G is called the girth $gr(G)$ of G . If there are no cycles in G , we use $gr(G) = \infty$. If the set of vertices in a graph is empty, it is said to be empty. A connected acyclic graph is called a tree. A graph is planar if it can be drawn in a plane so that edges intersect at only vertices. For any vertex u , $nbd(u) = \{v \in V(G) \mid u \text{ and } v \text{ are adjacent in } G\}$ is called the neighbourhood of u . The degree of a vertex u in a graph, denoted by $deg(u)$, is the number of edges that are incident to vertex u . In other words, it is the count of vertices that are adjacent

to u . If the degree of all vertices in a graph G are the same, then G is called a regular graph. The circumference of a graph G , denoted by $\mathcal{C}(G)$, is defined as the length of longest cycle in G . The eccentricity $e(v)$ of a vertex v in a connected graph G is defined as the maximum distance from vertex v to any other vertex u in G . The radius of a graph G , denoted by $rad(G)$, is defined as the minimum eccentricity of G ; that is, $rad(G) = \min\{e(v) \mid v \in V(G)\}$. If every pair of distinct vertices in G is connected by an edge, then G is called a complete graph, and the complete graph with n vertices is denoted by K_n . If a graph G is the union of two disjoint sets of vertices, one of size m and the other of size n such that every vertex in the first set is connected to every vertex in the other set, and there are no edges within each set, then G is called a complete bipartite graph, denoted by $K_{m,n}$. The star graph is a complete bipartite graph $K_{1,n}$. For more concepts in graph theory, refer [17]. We recall the following results from [12].

Lemma 1 ([12], Lemma 2.1). *Let L be a lattice and $I, J \in N(L)$. Then I and J are adjacent in $SAnnIG(L)$ if and only if $Ann(I) \not\subseteq Ann(J)$ and $Ann(J) \not\subseteq Ann(I)$.*

Lemma 2 ([12], Lemma 2.3). *Let L be a lattice and $I, J \in N(L)$. Then I and J are adjacent in $SAnnIG(L)$ if and only if $A(I) \not\subseteq A(J)$ and $A(J) \not\subseteq A(I)$.*

Theorem 1 ([12], Theorem 3.2). *Let L be a lattice. Then $SAnnIG(L)$ is a complete bipartite or a star graph if and only if $|A(L)| = 2$.*

Theorem 2 ([12], Theorem 3.1 (iii)). *Let L be a lattice. If $S = \{\langle a \rangle \mid a \in A(L)\}$, then S forms an induced complete subgraph in $SAnnIG(L)$.*

Corollary 1 ([12], Corollary 3.3). *Let L be a lattice. Then $SAnnIG(L)$ is connected and $diam(SAnnIG(L)) \leq 2$.*

Theorem 3 ([12], Theorem 4.1(i)). *Let L be a lattice. Then $SAnnIG(L)$ is complete if and only if $N(L) = \{\langle a \rangle \mid a \in A(L)\}$.*

In the section 2 of this paper, we determine the radius, circumference, and domination number of $SAnnIG(L)$. We obtain necessary and sufficient conditions for $SAnnIG(L)$ to be in the class of paths, cycles, unicyclic, triangle-free, trees, complete multipartite, split or claw-free graphs. In the section 3, we study the relationship between strongly annihilator ideal and the annihilator ideal graph of a lattice.

2. Characterizations of $SAnnIG(L)$

In the following theorem, we determine a necessary and sufficient condition such that $rad(SAnnIG(L)) = 1$.

Theorem 4. *Let L be a lattice. Then $\text{rad}(S\text{AnnIG}(L)) = 1$ if and only if $|[a]_z| = 1$ for some atom $a \in A(L)$.*

Proof. Let $\text{rad}(S\text{AnnIG}(L)) = 1$. On the contrary, suppose that $|[a]_z| > 1$ for all atoms $a \in A(L)$. If $(x] \in N(L)$ with $x \notin A(L)$, then there exists $a \in A(L)$ such that $a < x$. Therefore, $A([a]) \subseteq A((x])$ and hence by Lemma 2, vertex $(a]$ is not adjacent to $(x]$. By Corollary 1, we have $d((x], (a]) = 2$. Similarly, for every $(a] \in N(L)$ with $a \in A(L)$, there exists $(y] \in N(L)$ with $y > a$ such that $d((a], (y]) = 2$. Therefore, eccentricity of every vertex in $S\text{AnnIG}(L)$ is two and hence $\text{rad}(S\text{AnnIG}(L)) = 2$, a contradiction. Conversely, suppose $|[a]_z| = 1$ for some atom $a \in A(L)$. Then, for every $(x] \in N(L) \setminus \{(a]\}$, we have, $A([a]) \not\subseteq A((x])$ and $A((x]) \not\subseteq A([a])$. Hence, by Lemma 2, vertices $(a]$ and $(x]$ are adjacent. That is, $d((a], (x]) = 1$. Therefore, for every $(x] \in N(L) \setminus \{(a]\}$, we have, $e((x]) = 1$. Hence, $\min\{e(I) \mid I \in N(L)\} = 1$. Thus, $\text{rad}(S\text{AnnIG}(L)) = 1$. $\square$

By Corollary 1 and Theorem 4, we have the following immediate consequence.

Corollary 2. *Let L be a lattice. Then $\text{rad}(S\text{AnnIG}(L)) = 2$ if and only if $|[a]_z| > 1$ for all atoms $a \in A(L)$.*

Example 1. Let L be a lattice depicted in Figure 1. Observe that, $A(L) = \{a_1, a_2, a_3\}$ and $|[a_i]_z| > 1$ for every $i = 1, 2, 3$. Therefore, by Corollary 2, $\text{rad}(S\text{AnnIG}(L)) = 2$.

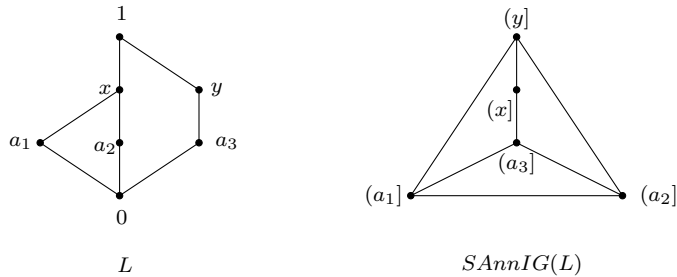


Figure 1. Some lattice L for which $\text{rad}(S\text{AnnIG}(L)) = 2$.

In the following result, we show that the circumference of $S\text{AnnIG}(L)$ does not exceed four.

Theorem 5. *Let L be a lattice. Then $\mathcal{C}(S\text{AnnIG}(L)) \leq 4$.*

Proof. Assume that there is an induced cycle $I_1 - I_2 - \dots - I_n - I_1$ in $S\text{AnnIG}(L)$ with length $n \geq 5$. Since there is no edge between I_1 and I_3 , by Lemma 2, suppose that $A(I_1) \subseteq A(I_3)$. If $A(I_4) \subseteq A(I_1)$, then $A(I_4) \subseteq A(I_3)$ and hence I_3 is not adjacent to

I_4 , which is not true. Therefore, $A(I_1) \subseteq A(I_4)$. Now, we show that $A(I_2) \subseteq A(I_n)$. As I_2 is not adjacent to I_4 , by Lemma 2, we have $A(I_2) \subseteq A(I_4)$ or $A(I_4) \subseteq A(I_2)$. If $A(I_4) \subseteq A(I_2)$, then as $A(I_1) \subseteq A(I_4)$, we have $A(I_1) \subseteq A(I_2)$ and hence I_1 is not adjacent to I_2 , which is incorrect. Therefore $A(I_2) \subseteq A(I_4)$. This process is therefore carried out for $I_5, \dots, I_n$. Then, we obtain $A(I_2) \subseteq A(I_5), \dots, A(I_2) \subseteq A(I_n)$.

Since $A(I_2) \subseteq A(I_5)$, hence from I_3 , we do the same argument on $I_5, \dots, I_n, I_1$. Then, we get $A(I_3) \subseteq A(I_5), \dots, A(I_3) \subseteq A(I_n), A(I_3) \subseteq A(I_1)$. Therefore $A(I_1) \subseteq A(I_3)$ and $A(I_3) \subseteq A(I_1)$ implies that $A(I_1) = A(I_3)$. But, since $A(I_3) \not\subseteq A(I_4)$ and $A(I_4) \not\subseteq A(I_3)$, we have $A(I_1) \not\subseteq A(I_4)$ and $A(I_4) \not\subseteq A(I_1)$, and hence I_1 is adjacent to I_4 , a contradiction. Similarly, if $A(I_3) \subseteq A(I_1)$, we have a contradiction. Thus $n \leq 4$. That is $\mathcal{C}(SAnnIG(L)) \leq 4$. $\square$

A subset D of the vertex set V of a graph G is called a dominating set if every vertex in the graph that is not in D is adjacent to at least one vertex in D . The domination number of a graph G , denoted by $\gamma(G)$, is defined as the size of the smallest dominating set of G . In the following result, we study the domination number of $SAnnIG(L)$.

Theorem 6. *Let L be a lattice. Then, the following statements hold.*

- (1) $\gamma(SAnnIG(L)) = 1$ if and only if $|[a]_z| = 1$ for some atom $a \in A(L)$.
- (2) $\gamma(SAnnIG(L)) \leq 2$.

Proof. (1) For a graph $G = (V, E)$ with $|V| \geq 2$, it is well known that $\gamma(G) = 1$ if and only if $rad(G) = 1$. Therefore, by Theorem 4, $\gamma(SAnnIG(L)) = 1$ if and only if $|[a]_z| = 1$ for some atom $a \in A(L)$.

(2) Let $b \in Z^*(L)$ be the element such that $|\mathcal{B}(a)| \leq |\mathcal{B}(b)|$, for all $a \in Z^*(L)$ and $\mathcal{B}(b) = S$. Consider $B = \{I \in N(L) \mid A(I) \subseteq S\}$. Then $B^c = \{J \in N(L) \mid A(J) \not\subseteq S\}$. Hence $N(L) = B \cup B^c$. Then there exists an ideal $(a_i]$ with $a_i \in A(L) \setminus S$. We show that $D = \{(b], (a_i)]\}$ is a dominating set. Note that, $A((a_i]) \not\subseteq A(I)$ and $A(I) \not\subseteq A((a_i])$ for every $I \in B$. Therefore, $(a_i] - I$ is an edge for every $I \in B$. Also, observe that $A(J) \not\subseteq A((b])$ and $A((b]) \not\subseteq A(J)$ for every $J \in B^c$. Therefore, $(b] - J$ is an edge for every $J \in B^c$. Therefore, D is a dominating set. Hence, we have, $\gamma(SAnnIG(L)) \leq 2$. $\square$

Example 2. Let L be a lattice depicted in Figure 2. Observe that, $A(L) = \{a_1, a_2, a_3\}$ and $|(a_3)_z| = 1$. Therefore, by Theorem 6, $\gamma(SAnnIG(L)) = 1$. Note that, $D = \{(a_3)\}$ is a dominating set.

Corollary 3. *Let L be a Boolean lattice such that $|L| = 2^n, n \in \mathbb{N}$ with $n \geq 3$. Then $\gamma(SAnnIG(L)) = 2$.*

Proof. Since $|[a]_z| > 1$ for every atom $a \in A(L)$, by Theorem 6(1), $\gamma(SAnnIG(L)) \neq 1$. By Theorem 6(2), we have, $\gamma(SAnnIG(L)) = 2$. $\square$

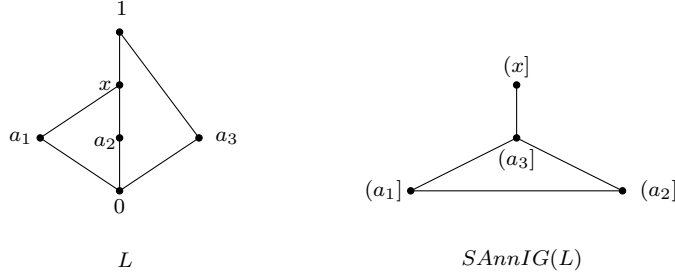


Figure 2. Some lattice L for which $\gamma(SAnnIG(L)) = 1$.

In the following result, we give the necessary and sufficient condition for $SAnnIG(L)$ to be a path or cycle.

Theorem 7. Let L be a lattice. Then, the following statements hold.

- (1) $SAnnIG(L)$ is a path if and only if $A(L) = \{a, b\}$ with $|[a]_z| \leq 2$ and $|[b]_z| = 1$.
- (2) $SAnnIG(L)$ is a cycle if and only if any of the following statement holds.
 - (a) $A(L) = \{a, b\}$ with $|[a]_z| = 2$ and $|[b]_z| = 2$.
 - (b) $A(L) = Z^*(L)$ with $|A(L)| = 3$.

Proof. (1) Suppose $SAnnIG(L)$ is a path. If $|A(L)| \geq 3$, then the ideals generated by three distinct atoms form a cycle of length three, a contradiction. Therefore $|A(L)| \leq 2$. If $|A(L)| = 1$, then $SAnnIG(L)$ is empty. Thus $|A(L)| = 2$. By Theorem 1, we have, $SAnnIG(L) = K_{|[a]_z|, |[b]_z|}$, where $A(L) = \{a, b\}$. Thus $|[a]_z| \leq 2$ and $|[b]_z| = 1$.

Conversely, suppose $A(L) = \{a, b\}$ such that $|[a]_z| \leq 2$ and $|[b]_z| = 1$. Then by Theorem 1, we have $SAnnIG(L) = K_{1,1}$ or $K_{2,1}$. Thus $SAnnIG(L)$ is a path.

(2) Suppose $SAnnIG(L)$ is a cycle. If $|A(L)| \geq 4$, then by Theorem 3, we have $SAnnIG(L)$ has a subgraph isomorphic to K_4 , which is not true. Therefore $|A(L)| \leq 3$. If $|A(L)| = 1$, then $SAnnIG(L)$ is empty, a contradiction. Thus $|A(L)| = 2$ or 3. Suppose $A(L) = \{a, b\}$. Then by Theorem 1, we have, $SAnnIG(L) = K_{|[a]_z|, |[b]_z|}$. Thus $|[a]_z| = 2$ and $|[b]_z| = 2$. Now, suppose $A(L) = \{a_1, a_2, a_3\}$. By Theorem 2, for every distinct i, j and k , we have $(a_j], (a_k] \in nbd((a_i])$ and hence $deg((a_i]) \geq 2$. If there exists $x \in Z^*(L)$ such that $x > a_j$, then there exists $a_i \in A(L)$ such that $a_i \notin (x]$. Then $A((a_i]) \not\subseteq A((x])$ and $A((x]) \not\subseteq A((a_i])$. Hence $(a_i] - (x]$ is an edge. Therefore $\{(a_j], (a_k], (x)\} \subseteq nbd((a_i])$ and hence $deg((a_i]) \geq 3$, a contradiction. Thus $Z^*(L) = A(L) = \{a_1, a_2, a_3\}$.

Conversely, suppose statement (a) holds. Then by Theorem 1, we have, $SAnnIG(L) = K_{2,2}$ and hence the result follows. Now, suppose statement (b) holds. Then by Theorem 2, we have, $SAnnIG(L) = K_3$ and hence $SAnnIG(L)$ is a cycle. $\square$

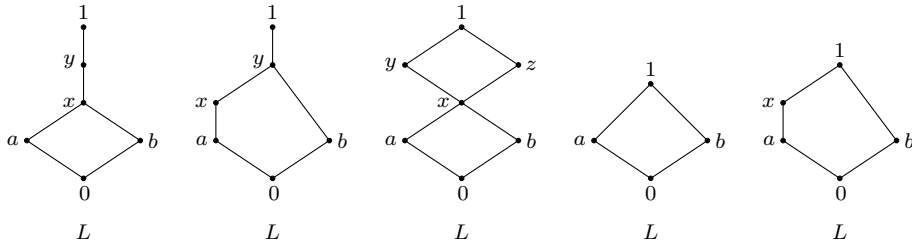


Figure 3. Some lattices L for which $SAnnIG(L)$ is a path.

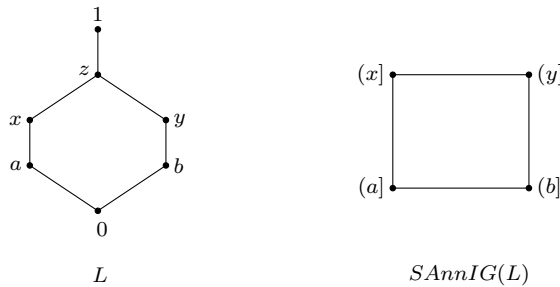


Figure 4. Some lattice L for which $SAnnIG(L)$ is a cycle.

A graph G is said to be unicyclic if there is only one cycle in G . Let $A_k = \{I \in N(L) \mid |A(I)| = k\}$. In the following result, we give the necessary and sufficient conditions for $SAnnIG(L)$ to be unicyclic.

Theorem 8. *Let L be a lattice. Then $SAnnIG(L)$ is unicyclic if and only if any of the following statement holds.*

- (1) $A(L) = \{a, b\}$ with $|[a]_z| = 2$ and $|[b]_z| = 2$.
- (2) $A(L) = \{a, b, c\}$ with $|A_1| = 3$ and no ideals $I, J \in A_2$ exist such that $A(I) \neq A(J)$.

Proof. Suppose $SAnnIG(L)$ is unicyclic. By Theorem 2, we have, $|A(L)| \leq 3$. If $|A(L)| = 1$, then $SAnnIG(L)$ is empty and hence it has no cycle, a contradiction. Suppose $A(L) = \{a, b\}$. Then $SAnnIG(L) = K_{|[a]_z|, |[b]_z|}$ and hence $|[a]_z| = 2$ and $|[b]_z| = 2$. Suppose $A(L) = \{a, b, c\}$. If $|A_1| \geq 4$, then there exists $(d) \in A_1$ with $d > a$. Since $A((d)) = \{a\}$, we have, $A((d)) \not\subseteq A((b))$ and $A((b)) \not\subseteq A((d))$. Also, $A((d)) \not\subseteq A((c))$ and $A((c)) \not\subseteq A((d))$. Therefore $(d) - (b)$ and $(d) - (c)$ are the edges. Hence $(d) - (b) - (c) - (d)$ is a cycle different from $(a) - (b) - (c) - (a)$. This contradicts the assumption that $SAnnIG(L)$ is unicyclic. Thus $|A_1| = 3$. Now, suppose there exist ideals I and J such that $A(I) = \{a, b\}$ and $A(J) = \{a, c\}$. Then $(a) - (b) - (c) - (a)$ and $I - (c) - (b) - J - I$ are the two distinct cycles in $SAnnIG(L)$ and hence $SAnnIG(L)$

is not unicyclic, a contradiction. Thus, there does not exist $I, J \in A_2$ such that $A(I) \neq A(J)$.

Conversely, suppose the statement (1) holds. Then $SAnnIG(L) = K_{2,2}$ and hence $SAnnIG(L)$ is unicyclic. Now, suppose the statement (2) holds. Then $(a] - (b] - (c] - (a]$ is the only cycle formed by elements in A_1 . If $I \in A_3$ then $I \notin N(L)$ and hence $N(L) = A_1 \cup A_2$. Since no any two elements in A_2 are adjacent to each other, they never form a cycle of length 3 or 4. Also, since any element of A_1 is adjacent to at most one element of A_2 and vice versa, and no two elements of A_2 are adjacent, we have elements in A_1 and A_2 do not form a cycle of length 3 or 4. Thus, by Theorem 5, we have $SAnnIG(L)$ is unicyclic. $\square$

Example 3. Consider a lattice L depicted in Figure 5. We have, $A(L) = \{a, b, c\}$, $|A_1| = |\{(a], (b], (c)\}| = 3$ and $A_2 = \{(x], (y], (z)\}$. Observe that, $A(I) = \{a, b\}$ for every $I \in A_2$. Thus, by Theorem 8(2), $SAnnIG(L)$ is unicyclic.

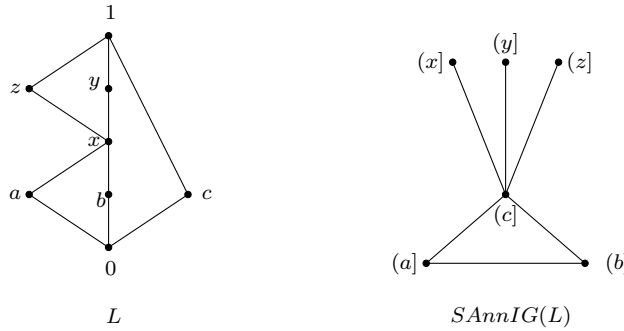


Figure 5. Some lattice L for which $SAnnIG(L)$ is unicyclic.

The following corollary is an immediate consequence of Theorem 7(2) and 8.

Corollary 4. Let L be a lattice such that $|A(L)| = 2$. Then $SAnnIG(L)$ is a cycle if and only if $SAnnIG(L)$ is unicyclic.

An undirected graph is a triangle-free graph if none of its three vertices combine to form a triangle with edges. An undirected graph is a tree if there is only one path connecting any two vertices. In the following result, we give the characterization for $SAnnIG(L)$ to be in the class of triangle-free or tree.

Proposition 1. Let L be a lattice such that $SAnnIG(L)$ is nonempty. Then, the following statements hold.

- (1) $SAnnIG(L)$ is a triangle-free graph if and only if $|A(L)| = 2$.
- (2) $SAnnIG(L)$ is a tree if and only if $A(L) = \{a, b\}$ with $|(a)_z| = 1$ or $|(b)_z| = 1$.

Proof. Follows by Theorem 1 and 2 □

In general, the triangle-free graph is not necessarily a bipartite graph. For example, a cycle C_n with n is odd, $n \geq 5$. But, in the following corollary, we show that it holds in $SAnnIG(L)$,

Corollary 5. *Let L be a lattice such that $SAnnIG(L)$ is nonempty. Then $SAnnIG(L)$ is a triangle-free graph if and only if it is a bipartite graph.*

Proof. By Theorem 1 and Proposition 1(1), it is trivial. □

In the following result, we give a characterization for $SAnnIG(L)$ to be a complete multipartite graph.

Theorem 9. *Let L be a lattice. Then the following statements are equivalent:*

1. $SAnnIG(L)$ is a complete multipartite graph.
2. $|A(I)| = 1$ for each $I \in N(L)$.
3. $AnnIG(L) = SAnnIG(L)$ is a complete multipartite graph.

Proof. (1) $\Rightarrow$ (2) Assume that $SAnnIG(L)$ is a complete multipartite graph. Hence, there exists a partition $N(L) = \bigcup_{\lambda \in \Lambda} V_\lambda$ such that no two vertices of V_λ are adjacent in $SAnnIG(L)$ for each $\lambda \in \Lambda$ and for any distinct $\lambda, \lambda' \in \Lambda$, any $I \in V_\lambda, J \in V_{\lambda'}$, I and J are adjacent in $SAnnIG(L)$. Let $I \in N(L)$. Suppose that $|A(I)| > 1$. Let $a, b \in A(I)$ with $a \neq b$. Observe that $A([a]) \subset A(I)$. Therefore, $[a]$ and I are not adjacent in $SAnnIG(L)$ by Lemma 2. Therefore, both $[a]$ and I must belong to V_λ for some $\lambda \in \Lambda$. Similarly, as $A([b]) \subset A(I)$, it follows that $[b]$ and I are not adjacent in $SAnnIG(L)$. Hence, $[b]$ must be in V_λ . This is impossible, since $[a]$ and $[b]$ are adjacent in $SAnnIG(L)$. Therefore, $|A(I)| = 1$ for each $I \in N(L)$.

(2) $\Rightarrow$ (3) Assume that $|A(I)| = 1$ for each $I \in N(L)$. Let $A(L) = \{a_\lambda \mid \lambda \in \Lambda\}$. The relation $\sim$ defined on $N(L)$ by for any $I, J \in N(L)$, $I \sim J$ if and only if $A(I) = A(J)$ is an equivalence relation. For any $\lambda \in \Lambda$, let us denote the equivalence class containing (a_λ) determined by $\sim$ by $[(a_\lambda)]$. Since $|A(I)| = 1$ for each $I \in N(L)$, it follows that $\{[(a_\lambda)] \mid \lambda \in \Lambda\}$ is the collection of all equivalence classes determined by $\sim$. Let $\lambda \in \Lambda$. If $I, J \in [(a_\lambda)]$, then $A(I) = A(J)$ and so, I and J are not adjacent in $SAnnIG(L)$. Let $\lambda' \in \Lambda$ be such that $\lambda \neq \lambda'$. Let $I \in [(a_\lambda)]$ and $J \in [(a_{\lambda'})]$. Then $A(I) = \{a_\lambda\} \not\subseteq A(J) = \{a_{\lambda'}\}$ and similarly $A(J) \not\subseteq A(I)$. Therefore, I and J are adjacent in $SAnnIG(L)$. This shows that $SAnnIG(L)$ is a complete multipartite graph whose parts are $\{[(a_\lambda)] \mid \lambda \in \Lambda\}$. Observe that $V(AnnIG(L)) = N(L) = V(SAnnIG(L))$. By assumption, $|A(I)| = 1$ for each $I \in N(L)$. Let $I, J \in N(L)$ be distinct. Notice that I and J are adjacent in $SAnnIG(L)$ if and only if $A(I) \not\subseteq A(J)$ and $A(J) \not\subseteq A(I)$ by Lemma 2, if and only if $A(I) \neq A(J)$, if and only if I and J are adjacent in $AnnIG(L)$ by [[14], Lemma 2.2]. Therefore, $AnnIG(L) = SAnnIG(L)$ is a complete multipartite graph.

(3) $\Rightarrow$ (1) This is clear. $\square$

Example 4. Let $L = \{\phi\} \cup \{\{2\}, \{4\}, \{6\}, \dots\} \cup \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots\} \cup \mathbb{N}$, where $\mathbb{N}$ = set of all natural numbers. Define a partial order relation on L such that for any two sets $A, B \in L$, we have, $A \leq B$ if and only if $A \subseteq B$. Then, $(L, \leq)$ is a lattice as shown in Figure 6 in which ϕ is a least element and $\mathbb{N}$ is a greatest element, and $A(L) = \{\{2\}, \{4\}, \{6\}, \dots\}$ and $C(L) = \{\{1, 2\}, \{3, 4\}, \{5, 6\}, \dots\}$, where $C(L)$ be the set containing all coatoms in L . Observe that, $L = \phi \cup A(L) \cup C(L) \cup \mathbb{N}$ and $|A(I)| = 1$ for every $I \in N(L)$. Therefore, by Theorem 9, $SAnnIG(L)$ is a complete multipartite graph.

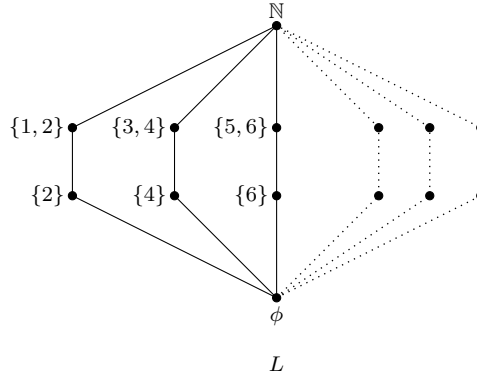


Figure 6. Atomic lattice L with infinite number of atoms satisfying the condition (2) of the statement in Theorem 9.

Example 5. Let L be a lattice as shown in Figure 7. Observe that the vertex (b) is not adjacent to (e) as well as to (d) . Hence, vertices (b) , (e) and (d) are the members of same partition. Since, vertex (e) is adjacent to (d) , we have, $SAnnIG(L)$ is not a complete multipartite graph. Since, $A(I) \neq A(J)$ for every $I, J \in N(L)$, we have, $AnnIG(L)$ is a complete multipartite graph. Note that, there exists ideal $(d) \in N(L)$ such that $|A((d))| > 1$.

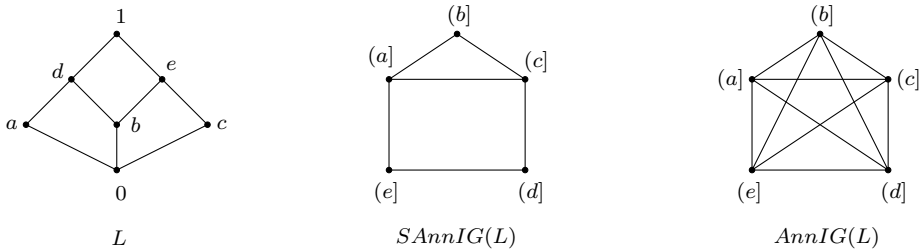


Figure 7. Some lattice L for which $SAnnIG(L)$ is not complete multipartite, while $AnnIG(L)$ is complete multipartite.

Corollary 6. Let L be a lattice such that $|A(L)| = n$. Then $SAnnIG(L)$ is a complete multipartite graph if and only if $SAnnIG(L)$ is a complete n -partite graph.

Example 6. Let L be a lattice as shown in Figure 8. Observe that, $|A(L)| = 3$, $|A(I)| = 1$ for every $I \in N(L)$ and $SAnnIG(L) = AnnIG(L)$ is a complete 3-partite graph.

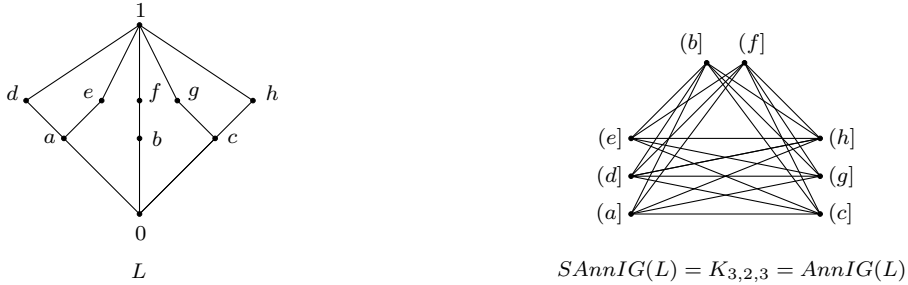


Figure 8. Some lattice L for which $SAnnIG(L) = AnnIG(L)$ is a complete multipartite graph.

A graph with its vertices divided into an independent set and a clique is called a split graph. If L is a lattice such that $A(L) = \{a, b\}$, then $SAnnIG(L) = K_{|[a]_z|, |[b]_z|}$. Hence, if $A(L) = \{a, b\}$, then $SAnnIG(L)$ is a split graph if and only if $|[a]_z| = 1$ or $|[b]_z| = 1$.

Let L be a lattice with $A(L) = \{a_1, a_2, \dots, a_n\}$. Also, let $S_i = \{I \in N(L) \mid A(I) = \{a_i\}\}$. For distinct i and j , let $S_{i,j} = \{I \in N(L) \mid A(I) = \{a_i, a_j\}\}$. For distinct i, j and k , let $S_{i,j,k} = \{I \in N(L) \mid A(I) = \{a_i, a_j, a_k\}\}$ and so on. In the following result, we give the necessary and sufficient condition for $SAnnIG(L)$ to be a split graph when L is a lattice with $|A(L)| \geq 3$.

Theorem 10. Let L be a lattice such that $A(L) = \{a_1, a_2, \dots, a_n\}$, $n \geq 3$. Then $SAnnIG(L)$ is a split graph if and only if $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \dots \cup S_{1,2,3,\dots,n-1} \cup \left(\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i) \right)$.

Proof. Let $SAnnIG(L)$ be a split graph with Q as a clique and U as an independent set. Let $B_k = \{I \in N(L) \setminus V(Q) \mid |A(I)| = k\}$. Observe that, $N(L) = V(Q) \cup V(U)$, where $V(U) = \bigcup_{i=1}^{n-1} B_i$. If there exist ideals $(a_i), (a_j) \in V(U)$ with $i \neq j$, then $A((a_i)) \not\subseteq A((a_j))$ and $A((a_j)) \not\subseteq A((a_i))$ and hence (a_i) and (a_j) are adjacent vertices, a contradiction. Thus, we have the following two cases.

Case (i) $(a_i) \notin V(U)$ for all $a_i \in A(L)$.

Then, by Theorem 2, we have $(a_i) \in V(Q)$ for all $a_i \in A(L)$. Now, for every $(x) \in N(L)$ with $x \notin A(L)$, if $(x) \in V(Q)$ then for $a_j < x$, we have $A((a_j)) \subseteq A((x))$ and hence (a_j) and (x) are non adjacent vertices, a contradiction. Thus, $V(Q) =$

$\{(a_1], (a_2], \dots, (a_n]\}$. If $V(U) = \phi$, then $S_{1,2} = \phi = S_{1,2,3} = \dots = S_{1,2,\dots,n-1}$. Hence, $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \dots \cup S_{1,2,3,\dots,n-1} \cup \left(\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i] \right)$, where $S_1 = \{(a_1]\}$.

Suppose $I_1 \in V(U)$. Then, $I_1 \in B_k$ for some k . If there exists $I_2 \in B_k$ such that $A(I_1) \neq A(I_2)$, then by Lemma 2, vertices I_1 and I_2 are adjacent, which is impossible as $I_2 \in V(U)$. Therefore, for every $I \in B_k$, we have $A(I) = \{a_1, a_2, \dots, a_k\}$. That is, for every $I \in B_k$, we have $I \in S_{1,2,\dots,k}$. If there exists $I_3 \in B_{k-1}$, then since $I_3 \in V(U)$, by Lemma 2, $A(I_3) \subset A(I)$ for every $I \in B_k$, and for any $I_4 \in B_{k-1}$ if $A(I_3) \neq A(I_4)$, then by Lemma 2, vertices I_3 and I_4 are adjacent, which is impossible as $I_4 \in V(U)$. Thus, for every $J \in B_{k-1}$, we have $A(J) = \{a_1, a_2, \dots, a_{k-1}\}$. That is, for every $J \in B_{k-1}$, we have $J \in S_{1,2,\dots,k-1}$. Continuing in this way, we have, if $K \in B_1$, then $K \in S_1$. Now, if there exists $I_5 \in B_{k+1}$, then since $I_5 \in V(U)$, by Lemma 2, $A(I) \subset A(I_5)$ for every $I \in B_k$, and for any $I_6 \in B_{k+1}$ if $A(I_5) \neq A(I_6)$, then by Lemma 2, vertices I_5 and I_6 are adjacent, which is impossible as $I_6 \in V(U)$. Thus, for every $M \in B_{k+1}$, we have $A(M) = \{a_1, a_2, \dots, a_k, a_{k+1}\}$. That is, for every $M \in B_{k+1}$, we have $M \in S_{1,2,\dots,k+1}$. Continuing in this way, we have, if $N \in B_{n-1}$, then $N \in S_{1,2,\dots,n-1}$. This shows that $V(U) = S_1 \setminus \{(a_1]\} \cup S_{1,2} \cup S_{1,2,3} \cup \dots \cup S_{1,2,3,\dots,n-1}$. Therefore, $N(L) = V(Q) \cup V(U) = \{(a_1], (a_2], \dots, (a_n]\} \cup \{S_1 \setminus \{(a_1]\} \cup S_{1,2} \cup S_{1,2,3} \cup \dots \cup S_{1,2,3,\dots,n-1}\} = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \dots \cup S_{1,2,3,\dots,n-1} \cup \left(\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i] \right)$.

Case (ii) $(a_i] \in V(U)$ for some $a_i \in A(L)$.

Suppose $(a_1] \in V(U)$. By Theorem 2, for every $2 \leq i \leq n$, we have $(a_i] \notin V(U)$ and hence $(a_i] \in V(Q)$. Also, if there exists $J \in V(Q)$ such that $J \neq (a_i], \forall i$ and $a_i \in A(J)$ with $a_i \neq a_1$, then $A((a_i]) \subseteq A(J)$ and hence $(a_i] - J$ is not an edge, which is not true. Thus, $V(Q) = \{(a_1], (a_2], \dots, (a_n]\}$ with $A(I) = \{a_1\}$ or $V(Q) = \{(a_2], \dots, (a_n]\}$. Observe that, $(a_1] \in S_1 \cap B_1$. By Lemma 2, for every $J \in B_1$, we have $A(J) = \{a_1\}$. That is, for every $J \in B_1$, we have $J \in S_1$. From B_2 onward, we give the similar proof as in the Case (i) and conclude that $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \dots \cup S_{1,2,3,\dots,n-1} \cup \left(\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i] \right)$.

Conversely, let $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \dots \cup S_{1,2,3,\dots,n-1} \cup \left(\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i] \right)$. Then,

for any $I, J \in \bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i]$, we have, $A(I) \not\subseteq A(J)$ and $A(J) \not\subseteq A(I)$ and hence

$I - J$ is an edge. Therefore, elements in $\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i]$ forms a clique. Now, for

any $I, J \in N(L) \setminus \left(\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i] \right)$, we have, $A(I) \subseteq A(J)$ or $A(J) \subseteq A(I)$ and

hence I and J are non adjacent vertices. Therefore, $N(L) \setminus \left(\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i] \right)$ forms

an independent set. Thus, $SAnnIG(L)$ is split. $\square$

Example 7. Let L be a lattice as shown in Figure 9. We have, $A(L) = \{a_1, a_2, a_3, a_4\}$, $S_1 = \{(a_1], (a_1'], (a_1'')]\}$, $S_{1,2} = \{(a_{12}], (a_{12'}]\}$, $S_{1,2,3} = \{(a_{123}], (a_{123'}]\}$ and $N(L) = S_1 \cup$

$S_{1,2} \cup S_{1,2,3} \cup \left(\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i) \right)$. Observe that, $SAnnIG(L)$ is a split graph.

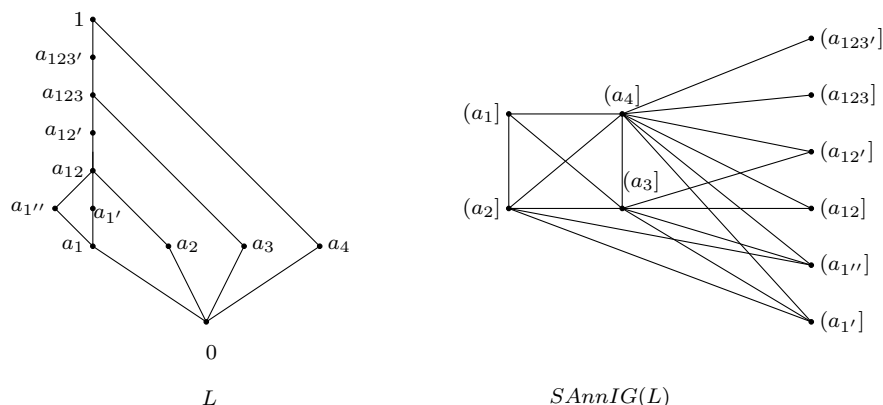


Figure 9. Some lattice L for which $SAnnIG(L)$ is a split graph.

Example 8. Let L be a lattice as shown in Figure 10. We have, $A(L) = \{a_1, a_2, a_3, a_4\}$, $S_1 = \{(a_1), (a_1'), (a_1''), (a_1''')\}$, $S_{1,2} = \emptyset$, $S_{1,2,3} = \{(a_{123}), (a_{123}'), (a_{123}'')\}$ and $N(L) = S_1 \cup S_{1,2} \cup S_{1,2,3} \cup \left(\bigcup_{a_i \in A(L) \setminus \{a_1\}} (a_i) \right)$. Observe that, $SAnnIG(L)$ is a split graph.

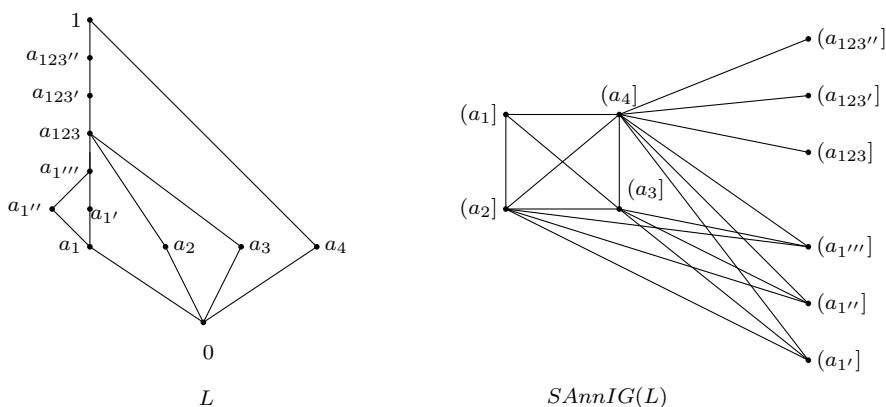


Figure 10. Some lattice L for which $SAnnIG(L)$ is a split graph.

The complete bipartite graph $K_{1,3}$ is a tree called claw. A claw-free graph is a graph that does not have a claw as an induced subgraph. In the following result, we give the characterization for $SAnnIG(L)$ to be a claw-free graph.

Theorem 11. *Let L be a lattice such that $|A(L)| \geq 2$. Then $SAnnIG(L)$ is a claw-free graph if and only if for every $a \in A(L)$ there does not exist distinct $b_i, b_j \in [a]_z \setminus \{a\}$ such that $A((b_i]) \subseteq A((b_j])$.*

Proof. Suppose $SAnnIG(L)$ is a claw-free graph. For some $a_i \in A(L)$, if there exists distinct $b_i, b_j \in [a_i]_z \setminus \{a_i\}$ such that $A((b_i]) \subseteq A((b_j])$, then $\{(a_i], (b_i], (b_j]\}$ forms an independent set. Since $(x] \in N(L)$ for every $x \in [a_i]_z$, there exists $a_j \in A(L) \setminus \{a_i\}$ such that $A((a_j]) \not\subseteq A((x])$ and $A((x]) \not\subseteq A((a_j])$. Therefore ideal $(a_j]$ is adjacent to $(x]$ for every $x \in [a_i]_z$. Thus, the set $\{(a_i], (b_i], (b_j], (a_j]\}$ forms a claw, a contradiction. Conversely, suppose for every $a \in A(L)$ there does not exist distinct $b_i, b_j \in [a]_z \setminus \{a\}$ such that $A((b_i]) \subseteq A((b_j])$. On the contrary, assume that $SAnnIG(L)$ is not a claw-free graph. Therefore, there exist an independent set, say $S = \{I_1, I_2, I_3\}$. By Lemma 2, suppose that $A(I_1) \subseteq A(I_2)$, $A(I_1) \subseteq A(I_3)$ and $A(I_2) \subseteq A(I_3)$. Assume that $a \in A(L) \cap A(I_1)$, $I_2 = (b_i]$ and $I_3 = (b_j]$, with $b_i \neq b_j$. Clearly, there does exist distinct $b_i, b_j \in [a]_z \setminus \{a\}$ such that $A((b_i]) \subseteq A((b_j])$, a contradiction. Similarly, in all the other possibilities, we get a contradiction. Thus $SAnnIG(L)$ is a claw-free graph. $\square$

Example 9. Consider a lattice as shown in Figure 8. Since, there exists $a \in A(L)$ and $d, e \in [a]_z \setminus \{a\}$ such that $A((d]) = \{a\} = A((e])$, we have, $SAnnIG(L)$ is not a claw-free graph. Note that, the elements in the set $\{(a], (d], (e], (b)\}$ forms a claw in $SAnnIG(L)$.

In the following result, we study $SAnnIG(L)$ when L is a lattice of positive divisors.

Theorem 12. *Let n be a natural number, and $L = D(n)$ be the lattice of all divisors of n , and $n = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$, $1 \leq m_1 \leq m_2 \leq \dots \leq m_k$ be the prime factorization of n , where $n > 1$, $k \geq 2$,*

- (1) *Then $\mathcal{C}(SAnnIG(L)) = 4$ provided $SAnnIG(L)$ admits a cycle.*
- (2) *$\gamma(SAnnIG(L)) = 1$ if and only if $n = p_1 p_2^{m_2}$.*
- (3) *$\gamma(SAnnIG(L)) = 2$ if and only if $n \neq p_1 p_2^{m_2}$.*
- (4) *$SAnnIG(L)$ is a path if and only if $k = 2$ with $m_1 = 1, m_2 \leq 2$.*
- (5) *$SAnnIG(L)$ is a cycle if and only if $k = 2$ with $m_1 = 2 = m_2$ if and only if $SAnnIG(L)$ is unicyclic.*
- (6) *$SAnnIG(L)$ is a tree if and only if $k = 2$ with $m_1 = 1$.*
- (7) *$SAnnIG(L)$ is a triangle-free graph if and only if $k = 2$ if and only if $SAnnIG(L)$ is a complete multipartite graph.*
- (8) *$SAnnIG(L)$ is a split graph if and only if $k = 2$ with $m_1 = 1$.*
- (9) *$SAnnIG(L)$ is a split graph if and only if $SAnnIG(L)$ is a tree.*
- (10) *$SAnnIG(L)$ is a claw-free graph if and only if $k = 2$ with $m_2 \leq 2$ or $k = 3$ with $m_1 = m_2 = m_3 = 1$.*

Proof. (1) If $k = 2$, then $SAnnIG(L) = K_{m_1, m_2}$ and hence no cycle of length 3 exists. If $k \geq 3$, then $SAnnIG(L)$ contains a cycle $(p_1] - (p_2] - (p_1p_3] - (p_2p_3] - (p_1]$ of length 4. Thus by Theorem 5, we have $\mathcal{C}(SAnnIG(L)) = 4$.

(2) Here $A(D(n)) = \{p_1, \dots, p_n\}$. Now, $|[p_i]_z| = 1$ for some i if and only if $n = p_1p_2^{m_2}$. Thus by Theorem 6, we have $\gamma(SAnnIG(L)) = 1$.

(3) By (2) and Theorem 6, it is obvious.

(4) Suppose $SAnnIG(L)$ is a path. By Theorem 2, $k = 2$ and hence $SAnnIG(L) = K_{m_1, m_2}$. Thus $m_1 = 1, m_2 \leq 2$. The converse of the statement is trivial.

(5) Suppose $SAnnIG(L)$ is a cycle. If $k \geq 3$, then $|Z^*(L)| \geq 6$ and by Theorem 7, we have, $k = 2$. Therefore $A(L) = \{p_1, p_2\}$. Hence by Theorem 7, we have $|[p_1]_z| = m_1 = 2 = m_2 = |[p_2]_z|$ and then by Theorem 8, $SAnnIG(L)$ is unicyclic. Conversely, suppose $SAnnIG(L)$ is unicyclic. By Theorem 8, we have $k \leq 3$. Suppose $k = 3$. Then $|A_1| = 3$ if and only if $m_i = 1, \forall i$. But, then there exists ideals $I = (p_i p_j]$ and $J = (p_j p_k]$ such that $A(I) \neq A(J)$ and hence by Theorem 8, we have, $SAnnIG(L)$ is not unicyclic, a contradiction. Thus $k = 2$. Therefore by Theorem 8, we have $|[p_1]_z| = m_1 = 2 = m_2 = |[p_2]_z|$. Thus by Theorem 7(2), we have, $SAnnIG(L)$ is a cycle.

(6) Suppose $SAnnIG(L)$ is a tree. Then $SAnnIG(L) = K_{|[p_1]_z|, |[p_2]_z|} = K_{m_1, m_2}$ if and only if $k = 2$. Thus by Proposition 1(2), we have $m_1 = 1$. The converse of the statement is trivial.

(7) Suppose $SAnnIG(L)$ is a triangle-free graph. By Theorem 2 and Corollary 5, we have $k = 2$. Therefore $SAnnIG(L) = K_{m_1, m_2}$. Conversely, suppose $SAnnIG(L)$ is a complete multipartite graph. If $k \geq 3$, then there exists ideal $I = (p_i p_j]$ such that $|A(I)| = 2 > 1$ and a contradiction to Theorem 9. Therefore $k = 2$ and hence $|A(L)| = 2$. Thus, by Proposition 1(1), we have, $SAnnIG(L)$ is a triangle-free graph.

(8) Suppose $SAnnIG(L)$ is a split graph. If $k \geq 3$, then $S_1 = \{I \in N(L) \mid A(I) = \{p_1\}\} \neq \emptyset$ and $S_2 = \{I \in N(L) \mid A(I) = \{p_2\}\} \neq \emptyset$. Therefore, by Theorem 10, we have, $SAnnIG(L)$ is not split, a contradiction. Thus $k = 2$. Hence $SAnnIG(L) = K_{m_1, m_2}$. Thus $m_1 = 1$. The converse of the statement is trivial.

(9) Result follows by statements (6) and (8).

(10) Suppose $SAnnIG(L)$ is a claw-free graph. If $k \geq 4$, then there exists $p_1p_2, p_1p_2p_3 \in [p_1]_z$ such that $A((p_1p_2]) \subset A((p_1p_2p_3])$ and hence by Theorem 11, we have, $SAnnIG(L)$ is not claw-free, a contradiction. Thus $k \leq 3$. Suppose $k = 2$. Then $SAnnIG(L) = K_{m_1, m_2}$. Thus $m_2 \leq 2$. Now, suppose $k = 3$. If $m_3 > 1$, then there exists $p_1p_3, p_1p_3^2 \in [p_3]_z$ such that $A((p_1p_3]) = A((p_1p_3^2])$ and hence by Theorem 11, we have, $SAnnIG(L)$ is not claw-free, a contradiction. Therefore $m_3 = 1$. Thus $m_1 = m_2 = m_3 = 1$. The converse of the statement is trivial. $\square$

3. Comparison between $SAnnIG(L)$ and $AnnIG(L)$

In this section, we determine the condition on lattice L so that $SAnnIG(L)$ is identical to $AnnIG(L)$. Also, we study the relationship between the diameters of $SAnnIG(L)$ and $AnnIG(L)$ and the relationship between their girths.

Recall the following result from [12].

Theorem 13 ([12], Theorem 2.1). *Let L be a lattice. Then $SAnnIG(L)$ is a subgraph of $AnnIG(L)$.*

Recall the following result from [14].

Theorem 14 ([14], Theorem 2.3). *Let L be a lattice. Then $AnnIG(L)$ is a complete multipartite graph.*

The following result gives the characterization for $SAnnIG(L)$ and $AnnIG(L)$ to be identical.

Theorem 15. *Let L be a lattice. Then $SAnnIG(L)$ and $AnnIG(L)$ are identical if and only if $|A(I)| = 1$, for all $I \in N(L)$.*

Proof. Let $SAnnIG(L)$ and $AnnIG(L)$ be the identical graphs. Hence, by Theorem 14, $SAnnIG(L)$ is a complete multipartite graph. Therefore, according to Theorem 9, we have $|A(I)| = 1$, for all $I \in N(L)$. The converse of the statement follows from Theorem 9. $\square$

The following corollary is an immediate consequence of Theorem 15.

Corollary 7. *Let L be a lattice with $|A(L)| \leq 2$. Then $SAnnIG(L)$ and $AnnIG(L)$ are identical.*

Theorem 16. *Let L be a lattice such that $SAnnIG(L)$ is nonempty. Then the following statements are equivalent:*

- (1) $gr(SAnnIG(L)) = 4$
- (2) $SAnnIG(L) = AnnIG(L)$ and $gr(AnnIG(L)) = 4$.
- (3) $gr(AnnIG(L)) = 4$.
- (4) $|A(L)| = 2$ and there exist elements x_a and x_b such that $a \prec x_a$, $b || x_a$ and $b \prec x_b$, $a || x_b$, where $A(L) = \{a, b\}$.
- (5) $AnnIG(L) = K_{m,n}$, $m, n \geq 2$;
- (6) $SAnnIG(L) = K_{m,n}$, $m, n \geq 2$;

Proof. (1) $\Rightarrow$ (2) By [[12], Corollary 3.2(2)], we have, $|A(L)| = 2$. Then by Corollary 7, we have, $SAnnIG(L) = AnnIG(L)$. Thus, the statement is clear.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (4) Let $gr(AnnIG(L)) = 4$. By [[14], Corollary 2.5(ii)], the statement is clear.

(4) $\Rightarrow$ (5) Let $|A(L)| = 2$. By Corollary 7, $AnnIG(L) = K_{|[a]_z|, |[b]_z|}$. Then by [[14], Corollary 2.5(ii)], $AnnIG(L) = K_{m,n}$, $m, n \geq 2$;

(5) $\Rightarrow$ (6) This is clear by Corollary 7.

(6) $\Rightarrow$ (1) The graph $SAnnIG(L)$ is complete bipartite that is not a star. Thus, the statement is clear. $\square$

Theorem 17. *Let L be a lattice such that $SAnnIG(L)$ is nonempty. Then the following statements are equivalent:*

(1) $gr(SAnnIG(L)) = \infty$

(2) $SAnnIG(L) = AnnIG(L)$, and $gr(AnnIG(L)) = \infty$.

(3) $gr(AnnIG(L)) = \infty$.

(4) $|A(L)| = 2$ and there doesn't exist elements x_a and x_b such that $a \prec x_a$, $b || x_a$ and $b \prec x_b$, $a || x_b$, where $A(L) = \{a, b\}$.

(5) $AnnIG(L) = K_{1,n}$, $n \geq 1$;

(6) $SAnnIG(L) = K_{1,n}$, $n \geq 1$;

Proof. (1) $\Rightarrow$ (2) By [[12], Corollary 3.2(2)], we have $|A(L)| = 2$. Then by Corollary 7, we have $SAnnIG(L) = AnnIG(L)$. Thus, the statement is obvious.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (4) Let $gr(AnnIG(L)) = \infty$. By [[14], Corollary 2.5(iii)], the statement is clear.

(4) $\Rightarrow$ (5) Let $|A(L)| = 2$. By Corollary 7, $AnnIG(L) = K_{|[a]_z|, |[b]_z|}$. Then by [[14], Corollary 2.5(iii)], $AnnIG(L) = K_{1,n}$, $n \geq 1$;

(5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1). Similar to the proof of (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (1) of Theorem 16. $\square$

In the following result, for the graphs $SAnnIG(L)$ and $AnnIG(L)$, we give the relationship between their diameters.

Theorem 18. *Let L be a lattice such that $|A(L)| \geq 2$. Then the following statements hold:*

(1) If $diam(SAnnIG(L)) = 1$, then $diam(AnnIG(L)) = 1$.

(2) If $diam(SAnnIG(L)) = 2$, then $diam(AnnIG(L)) = 1$ or 2.

(3) If $diam(AnnIG(L)) = 1$, then $diam(SAnnIG(L)) = 1$ or 2.

(4) If $diam(AnnIG(L)) = 2$, then $diam(SAnnIG(L)) = 2$.

Proof. (1) By Theorem 3, we have, $N(L) = \{[a] \mid a \in A(L)\}$. Thus, $A(I) \neq A(J)$ for every distinct $I, J \in N(L)$. By [[14], Lemma 2.2], we have, $diam(AnnIG(L)) = 1$.

- (2) Let $\text{diam}(S\text{AnnIG}(L)) = 2$. If $A(I) \neq A(J)$ for every distinct $I, J \in N(L)$, then by [[14], Lemma 2.2], we have $\text{diam}(\text{AnnIG}(L)) = 1$, otherwise $\text{diam}(\text{AnnIG}(L)) = 2$.
- (3) Suppose $\text{diam}(\text{AnnIG}(L)) = 1$. If $N(L) = \{(a) \mid a \in A(L)\}$, then we have, $\text{diam}(S\text{AnnIG}(L)) = 1$; otherwise $\text{diam}(S\text{AnnIG}(L)) = 2$ by Corollary 1.
- (4) It is obvious by Theorem 13. $\square$

By [[12], Corollary 3.2] and [[14], Corollary 2.5], we have the following immediate result.

Corollary 8. *Let L be a lattice. Then $\text{gr}(S\text{AnnIG}(L)) = \text{gr}(\text{AnnIG}(L))$.*

Corollary 9. *Let L be a lattice. If $S\text{AnnIG}(L) \neq \text{AnnIG}(L)$, then $\text{gr}(S\text{AnnIG}(L)) = \text{gr}(\text{AnnIG}(L)) = 3$.*

The converse of Corollary 9 is not true. To observe this, refer to the following example.

Example 10. For a diamond lattice $L = D_3$, we have, $\text{AnnIG}(L) = K_3 = S\text{AnnIG}(L)$. Therefore, we have $\text{diam}(S\text{AnnIG}(L)) = 1 = \text{diam}(\text{AnnIG}(L))$ and $\text{gr}(S\text{AnnIG}(L)) = 3 = \text{gr}(\text{AnnIG}(L))$.

Example 11. Let $L = (D(30), |)$. Then $\text{AnnIG}(L) = K_6$ and $S\text{AnnIG}(L)$ is a 3-regular graph. Therefore $\text{diam}(\text{AnnIG}(L)) = 1$ and $\text{diam}(S\text{AnnIG}(L)) = 2$.

Example 12. Let L be a lattice as shown in Figure 11. Observe that, $\text{diam}(S\text{AnnIG}(L)) = 2 = \text{diam}(\text{AnnIG}(L))$.

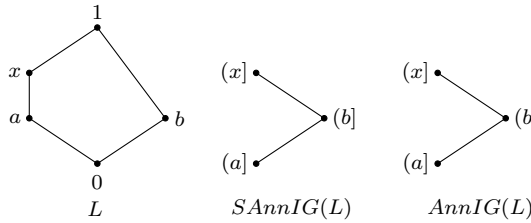


Figure 11. Some lattice L for which $\text{diam}(S\text{AnnIG}(L)) = 2 = \text{diam}(\text{AnnIG}(L))$.

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