

The r -dynamic chromatic number of the corona product of graphs

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Abstract: Let G be a graph. An r -dynamic k -coloring of G is a proper k -coloring of G such that every vertex v in $V(G)$ has neighbors in at least $\min\{r, d_G(v)\}$ different color classes. The r -dynamic chromatic number of G , denoted by $\chi_r(G)$, is the least k such that G has an r -dynamic k -coloring. We determine the r -dynamic chromatic number of the corona product $G \odot H$ of graphs G and H , in terms of the dynamic chromatic numbers of G and H .

Keywords: dynamic coloring, corona product.

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1. Introduction

We refer the book [7] for graph theoretical notation and terminology. Let G be a finite, simple, connected, undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, $d_G(v)$ and $N_G(v)$ denote, respectively, the degree of v , and the neighborhood of v . The minimum degree and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A k -coloring of G is a map $c : V(G) \rightarrow S$, where $|S| = k$; it is *proper* if adjacent vertices receive different colors. An r -dynamic k -coloring is a proper k -coloring c of G such that on each vertex neighborhood $N_G(v)$ at least $\min\{r, d_G(v)\}$ colors are used, i.e., $|c(N_G(v))| \geq \min\{r, d_G(v)\}$. The r -dynamic chromatic number, introduced by Montgomery [23] and written as $\chi_r(G)$, is the minimum k such that G has an r -dynamic k -coloring.

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The 1-dynamic chromatic number of G is its chromatic number $\chi(G)$, i.e., $\chi_1(G) = \chi(G)$. The 2-dynamic chromatic number was introduced as *dynamic chromatic number* by Montgomery [23]. Given a graph G , form G^2 by adding edges joining nonadjacent vertices having a common neighbor in G . From the definition, we have the following (see [13]):

$$\chi_r(G) = \chi_{\Delta(G)}(G), \text{ when } r \geq \Delta(G);$$

$$\chi_{\Delta(G)}(G) = \chi(G^2);$$

$$\chi_1(G) \leq \chi_2(G) \leq \chi_3(G) \leq \cdots \leq \chi_{\Delta(G)}(G); \text{ and}$$

$$\chi_r(G) \geq \min\{r, \Delta(G)\} + 1.$$

Hence, $\chi_r(G) = \chi_{\Delta(G)}(G) \geq \Delta(G) + 1$, when $r \geq \Delta(G)$; and $\chi_r(G) \geq r + 1$, when $r \leq \Delta(G)$.

The r -dynamic chromatic number has been studied by several authors (see [2–6, 8–10, 12–15, 20–23, 25]). In particular, for the corona product, it has been studied in [1, 16–19, 24].

Given two simple graphs G and H with $V(G) = \{v_1, v_2, \dots, v_n\}$ and n disjoint copies $H_1, H_2, \dots, H_n$ of H , the *corona product of G and H* , denoted by $G \odot H$, is the simple graph obtained from the disjoint union $G \cup (H_1 \cup H_2 \cup \cdots \cup H_n)$ by making the vertex v_i of G adjacent to every vertex of H_i , $i \in \{1, 2, \dots, n\}$ (defined by Frucht and Harary [11]). In considering the corona product $G \odot H$, assume throughout that G is a connected graph with at least two vertices.

Observe that, for $i \in \{1, 2, \dots, n\}$, $d_{G \odot H}(v_i) = d_G(v_i) + |V(H)|$ and for $u \in V(H_i)$, $d_{G \odot H}(u) = d_H(u) + 1$. Hence, $\delta(G \odot H) = \delta(H) + 1$ and $\Delta(G \odot H) = \Delta(G) + |V(H)|$. For disjoint graphs G_1 and G_2 , the graph $G_1 \vee G_2$ is the simple graph obtained from the disjoint union $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 .

In Section 2, for $1 \leq r \leq \Delta(G) + |V(H)|$, we compute the r -dynamic chromatic number of the corona product $G \odot H$.

2. Result

Theorem 1.

1. $\chi_1(G \odot H) = \chi(G \odot H) = \max\{\chi(G), 1 + \chi(H)\}$.
2. $\chi_2(G \odot H) = \begin{cases} 3 & \text{if } \chi(G) = 2 \text{ and } E(H) = \emptyset, \\ \chi_1(G \odot H) & \text{otherwise.} \end{cases}$
3. For $3 \leq r \leq \Delta(H) + 1$,
 $\chi_r(G \odot H) = \max\{\chi(G), 1 + \chi_{r-1}(H)\}$.
4. For $\Delta(H) + 2 \leq r \leq \chi_{\Delta(H)}(H)$,
 $\chi_r(G \odot H) = \max\{\chi(G), 1 + \chi_{\Delta(H)}(H)\}$.
5. For $\chi_{\Delta(H)}(H) + 1 \leq r \leq |V(H)|$,
 $\chi_r(G \odot H) = \max\{\chi(G), r + 1\}$.
6. For $|V(H)| + 1 \leq r \leq |V(H)| + \Delta(G)$,
 $\chi_r(G \odot H) = \max\{\chi_{r-|V(H)|}(G), r + 1\}$.

Proof. *Proof of 1.* Let $k = \max\{\chi(G), 1 + \chi(H)\}$.

As G and $K_1 \vee H$ are subgraphs of $G \odot H$, we have, $\chi(G \odot H) \geq \chi(G)$ and $\chi(G \odot H)$

$\geq \chi(K_1 \vee H) = 1 + \chi(H)$. Hence, $\chi(G \odot H) \geq k$.

Let c' be a proper $\chi(G)$ -coloring of G with colors $\{1, 2, \dots, \chi(G)\}$ and let c'' be a proper $\chi(H)$ -coloring of H with colors $\{1, 2, \dots, \chi(H)\}$. Define $c'_1 : V(G \odot H) \rightarrow \{1, 2, \dots, k\}$ as follows: for $i \in \{1, 2, \dots, n\}$, $c'_1(v_i) = c'(v_i)$ and c'_1 restricted to $V(H_i)$ is c'' . We now recolor $G \odot H$ as follows. For $i \in \{1, 2, \dots, n\}$, if $c'(v_i) \in \{1, 2, \dots, \chi(H)\}$, recolor the vertices $u \in V(H_i)$ satisfying $c''(u) = c'(v_i)$ with the color $1 + \chi(H)$. This yields a proper k -coloring $c_1 : V(G \odot H) \rightarrow \{1, 2, \dots, k\}$ of $G \odot H$. Hence, $\chi(G \odot H) \leq k$.

Thus, $\chi_1(G \odot H) = \chi(G \odot H) = k$.

Proof of 2. Let $k = \max\{\chi(G), 1 + \chi(H)\}$. Since $\chi_1(G \odot H) \leq \chi_2(G \odot H)$, we have $k \leq \chi_2(G \odot H)$. We consider two cases.

Case 1. $E(H) \neq \emptyset$.

Then, consider the k -coloring c_1 defined in the proof of 1. Take $c_2 = c_1$. Let $i \in \{1, 2, \dots, n\}$. Since $E(H) \neq \emptyset$, we have $|c_2(N_{G \odot H}(v_i))| \geq |c_2(V(H_i))| \geq 2$. As $\min\{2, d_{G \odot H}(v_i)\} = \min\{2, d_G(v_i) + |V(H)|\} = 2$, we have $|c_2(N_{G \odot H}(v_i))| \geq \min\{2, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. For $d_H(u) \geq 1$, as $c_2(v_i) \notin c_2(N_{H_i}(u))$, we have $|c_2(N_{G \odot H}(u))| \geq |c_2(v_i)| + |c_2(N_{H_i}(u))| \geq 1 + 1 = 2 = \min\{2, 1 + d_H(u)\} = \min\{2, d_{G \odot H}(u)\}$; and for $d_H(u) = 0$, we have $|c_2(N_{G \odot H}(u))| = 1 = \min\{2, 1\} = \min\{2, 1 + d_H(u)\} = \min\{2, d_{G \odot H}(u)\}$. Hence, c_2 is a 2-dynamic k -coloring of $G \odot H$, and so $\chi_2(G \odot H) \leq k$. Thus, $\chi_2(G \odot H) = k$.

Case 2. $E(H) = \emptyset$.

Then, $H = K_m^c$, the complement of K_m , and $k = \max\{\chi(G), 1 + \chi(H)\} = \chi(G)$. We consider two subcases.

Subcase 2.1. $\chi(G) \geq 3$.

Let c' be a proper $\chi(G)$ -coloring of G with colors $\{1, 2, \dots, \chi(G)\}$.

For $m \geq 2$, extend c' to obtain $c_2 : V(G \odot H) \rightarrow \{1, 2, \dots, \chi(G)\}$ so that, for each $i \in \{1, 2, \dots, n\}$, $c'(v_i) \notin c_2(V(H_i))$ and $|c_2(V(H_i))| \geq 2$. Such an extension is possible. Let $i \in \{1, 2, \dots, n\}$. Then $|c_2(N_{G \odot H}(v_i))| \geq |c_2(V(H_i))| \geq 2$. Since $\min\{2, d_{G \odot H}(v_i)\} = \min\{2, d_G(v_i) + |V(H)|\} = 2$, we have $|c_2(N_{G \odot H}(v_i))| \geq \min\{2, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $d_H(u) = 0$, and so $|c_2(N_{G \odot H}(u))| = 1 = \min\{2, 1\} = \min\{2, 1 + d_H(u)\} = \min\{2, d_{G \odot H}(u)\}$.

Now consider $m = 1$. Let $i \in \{1, 2, \dots, n\}$, and let $v_j^{(i)}$ be an arbitrary vertex of G which is adjacent to v_i in G . Extend c' to obtain $c_2 : V(G \odot H) \rightarrow \{1, 2, \dots, \chi(G)\}$ so that, for each $i \in \{1, 2, \dots, n\}$, $c_2(V(H_i)) \in \{1, 2, \dots, \chi(G)\} \setminus \{c'(v_i), c'(v_j^{(i)})\}$.

Such an extension is possible. Then $|c_2(N_{G \odot H}(v_i))| \geq |\{c'(v_j^{(i)}), c_2(V(H_i))\}| \geq 2$. Since $\min\{2, d_{G \odot H}(v_i)\} = \min\{2, d_G(v_i) + |V(H)|\} = 2$, we have $|c_2(N_{G \odot H}(v_i))| \geq \min\{2, d_{G \odot H}(v_i)\}$. Also, let $V(H_i) = \{u\}$. Then $d_H(u) = 0$, and so $|c_2(N_{G \odot H}(u))| = 1 = \min\{2, 1\} = \min\{2, 1 + d_H(u)\} = \min\{2, d_{G \odot H}(u)\}$.

Hence, for $m \geq 1$, c_2 is a 2-dynamic $\chi(G)$ -coloring of $G \odot H$, and so $\chi_2(G \odot H) \leq \chi(G)$. Thus $\chi_2(G \odot H) = \chi(G) = k$.

Subcase 2.2. $\chi(G) = 2$.

We have, $\chi_2(G \odot H) \geq \min\{2, \Delta(G \odot H)\} + 1 = \min\{2, \Delta(G) + |V(H)|\} + 1 = 3$.

Let c' be a proper 2-coloring of G with colors $\{1, 2\}$. Extend c' to obtain $c_2 : V(G \odot H) \rightarrow \{1, 2, 3\}$ so that, for each $i \in \{1, 2, \dots, n\}$, $c_2(V(H_i)) = 3$. Let $i \in \{1, 2, \dots, n\}$. Then $|c_2(N_{G \odot H}(v_i))| = 2$ and $\min\{2, d_{G \odot H}(v_i)\} = \min\{2, d_G(v_i) + |V(H)|\} = 2$. Also, let $V(H_i) = \{u\}$. Then $|c_2(N_{G \odot H}(u))| = 1$ and $\min\{2, d_{G \odot H}(u)\} = \min\{2, 1 + d_H(u)\} = \min\{2, 1\} = 1$. It follows that, c_2 is a 2-dynamic 3-coloring of $G \odot H$, and so $\chi_2(G \odot H) \leq 3$. Thus $\chi_2(G \odot H) = 3$. This completes the proof of 2.

Proof of 3. Any r -dynamic $\chi_r(G \odot H)$ -coloring c of $G \odot H$ yields a proper $\chi_r(G \odot H)$ -coloring for G and an $(r - 1)$ -dynamic $(\chi_r(G \odot H) - 1)$ -coloring for H_i (since $c(v_i) \notin c(V(H_i))$), where $i \in \{1, 2, \dots, n\}$. Hence, $\chi(G) \leq \chi_r(G \odot H)$ and $\chi_{r-1}(H) \leq \chi_r(G \odot H) - 1$. Thus, $\chi_r(G \odot H) \geq \max\{\chi(G), 1 + \chi_{r-1}(H)\}$.

Let $k = \max\{\chi(G), 1 + \chi_{r-1}(H)\}$. Let c' be a proper $\chi(G)$ -coloring of G with colors $\{1, 2, \dots, \chi(G)\}$ and let c'' be an $(r - 1)$ -dynamic $\chi_{r-1}(H)$ -coloring of H with colors $\{1, 2, \dots, \chi_{r-1}(H)\}$. Define $c'_r : V(G \odot H) \rightarrow \{1, 2, \dots, k\}$ as follows: for $i \in \{1, 2, \dots, n\}$, $c'_r(v_i) = c'(v_i)$ and c'_r restricted to $V(H_i)$ is c'' . We now recolor $G \odot H$ as follows. For $i \in \{1, 2, \dots, n\}$, if $c'(v_i) \in \{1, 2, \dots, \chi_{r-1}(H)\}$, recolor the vertices $u \in V(H_i)$ satisfying $c''(u) = c'(v_i)$ with the color $1 + \chi_{r-1}(H)$. This yields a proper k -coloring $c_r : V(G \odot H) \rightarrow \{1, 2, \dots, k\}$ of $G \odot H$.

Let $i \in \{1, 2, \dots, n\}$. Then $|c_r(N_{G \odot H}(v_i))| \geq |c_r(V(H_i))| = \chi_{r-1}(H) \geq r = \min\{r, d_G(v_i) + |V(H)|\} = \min\{r, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $|c_r(N_{G \odot H}(u))| \geq |c_r(v_i)| + |c_r(N_{H_i}(u))| \geq 1 + \min\{r - 1, d_H(u)\} = \min\{r, 1 + d_H(u)\} = \min\{r, d_{G \odot H}(u)\}$. Hence, c_r is an r -dynamic k -coloring of $G \odot H$, and so $\chi_r(G \odot H) \leq k$. Thus $\chi_r(G \odot H) = k$.

Proof of 4. Any r -dynamic $\chi_r(G \odot H)$ -coloring c of $G \odot H$ yields a proper $\chi_r(G \odot H)$ -coloring for G and a $\Delta(H)$ -dynamic $(\chi_r(G \odot H) - 1)$ -coloring for H_i (since $c(v_i) \notin c(V(H_i))$), where $i \in \{1, 2, \dots, n\}$. This implies that, $\chi(G) \leq \chi_r(G \odot H)$ and $\chi_{\Delta(H)}(H) \leq \chi_r(G \odot H) - 1$. Thus, $\chi_r(G \odot H) \geq \max\{\chi(G), 1 + \chi_{\Delta(H)}(H)\}$.

Consider the coloring $c_{\Delta(H)+1}$ obtained in the proof of 3. Take $c_r = c_{\Delta(H)+1}$. Let $i \in \{1, 2, \dots, n\}$. Then, by hypothesis, we have $|c_r(N_{G \odot H}(v_i))| \geq |c_r(V(H_i))| \geq \chi_{\Delta(H)}(H) \geq r$. Since $r \leq \chi_{\Delta(H)}(H) \leq |V(H)|$, we have $\min\{r, d_{G \odot H}(v_i)\} = \min\{r, d_G(v_i) + |V(H)|\} = r$. Hence, $|c_r(N_{G \odot H}(v_i))| \geq \min\{r, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $|c_r(N_{G \odot H}(u))| \geq |c_r(v_i)| + |c_r(N_{H_i}(u))| = 1 + d_H(u) = \min\{r, 1 + d_H(u)\} = \min\{r, d_{G \odot H}(u)\}$, since $1 + d_H(u) \leq 1 + \Delta(H) < \Delta(H) + 2 \leq r$. Hence, c_r is an r -dynamic k -coloring of $G \odot H$, where $k = \max\{\chi(G), 1 + \chi_{\Delta(H)}(H)\}$, and so $\chi_r(G \odot H) \leq k$. Thus $\chi_r(G \odot H) = k$.

Proof of 5. $\chi_r(G \odot H) \geq \chi(G)$ and $\chi_r(G \odot H) \geq \min\{r, \Delta(G \odot H)\} + 1 = \min\{r, \Delta(G) + |V(H)|\} + 1 = r + 1$ implies that $\chi_r(G \odot H) \geq \max\{\chi(G), r + 1\}$.

Consider the coloring $c_{\Delta(H)+1}$ obtained in the proof of 3. Note that, from the proof of 4, $c_{\chi_{\Delta(H)}(H)} = c_{\Delta(H)+1}$. For each r satisfying $\chi_{\Delta(H)}(H) + 1 \leq r \leq |V(H)|$, let $k = \max\{\chi(G), r + 1\}$. We define a k -coloring c_r from c_{r-1} recursively, which satisfies that $|c_r(V(H_i))| = |c_{r-1}(V(H_i))| + 1$, for every $i \in \{1, 2, \dots, n\}$, as follows.

Observe that, for each $i \in \{1, 2, \dots, n\}$, $|c_{\chi_{\Delta(H)}(H)}(V(H_i))| = \chi_{\Delta(H)}(H)$. Hence, $|c_{r-1}(V(H_i))| = |c_{r-2}(V(H_i))| + 1 = |c_{r-3}(V(H_i))| + 2 = \dots = |c_{\chi_{\Delta(H)}(H)}(V(H_i))| + r - \chi_{\Delta(H)}(H) - 1 = r - 1 < |V(H)|$. Therefore, for each $i \in \{1, 2, \dots, n\}$, in H_i ,

there exist vertices x_i and y_i such that $c_{r-1}(x_i) = c_{r-1}(y_i)$. Modify the onto mapping $c_{r-1} : V(G \odot H) \rightarrow S$ as follows:

- If $S \setminus c_{r-1}(V(H_i)) \neq \emptyset$, then take any $\alpha \in S \setminus c_{r-1}(V(H_i))$ and assign α to y_i .
- If $S \setminus c_{r-1}(V(H_i)) = \emptyset$, then assign a new color β to y_i .

Call this modified coloring as c_r .

Let $i \in \{1, 2, \dots, n\}$. Then $|c_r(N_{G \odot H}(v_i))| \geq |c_r(V(H_i))| = |c_{r-1}(V(H_i))| + 1 = r$. Since $r \leq |V(H)|$, we have $\min\{r, d_{G \odot H}(v_i)\} = \min\{r, d_G(v_i) + |V(H)|\} = r$. Hence, $|c_r(N_{G \odot H}(v_i))| \geq \min\{r, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $|c_r(N_{G \odot H}(u))| \geq |c_r(v_i)| + |c_r(N_{H_i}(u))| = 1 + d_H(u) = \min\{r, 1 + d_H(u)\} = \min\{r, d_{G \odot H}(u)\}$, since $1 + d_H(u) \leq 1 + \Delta(H) < \Delta(H) + 2 \leq r$.

Hence, c_r is an r -dynamic k -coloring, where $k = \max\{\chi(G), r + 1\}$.

Proof of 6. Let c be an arbitrary r -dynamic $\chi_r(G \odot H)$ -coloring of $G \odot H$. Then $|c(N_G(v_i))| + |V(H)| \geq |c(N_{G \odot H}(v_i))| \geq \min\{r, d_{G \odot H}(v_i)\} = \min\{r, d_G(v_i) + |V(H)|\} = |V(H)| + \min\{r - |V(H)|, d_G(v_i)\}$ implies $|c(N_G(v_i))| \geq \min\{r - |V(H)|, d_G(v_i)\}$. So, c yields an $(r - |V(H)|)$ -dynamic $\chi_r(G \odot H)$ -coloring for G . Hence, $\chi_r(G \odot H) \geq \chi_{r-|V(H)|}(G)$. This together with $\chi_r(G \odot H) \geq r + 1$ imply that $\chi_r(G \odot H) \geq \max\{\chi_{r-|V(H)|}(G), r + 1\}$.

Let $r = |V(H)| + s$, $1 \leq s \leq \Delta(G)$. Let $c' : V(G) \rightarrow \{1, 2, \dots, \chi_s(G)\}$ be an s -dynamic $\chi_s(G)$ -coloring of G . For $i \in \{1, 2, \dots, n\}$, let $A_i = c'(N_G(v_i)) = \{\alpha_1, \alpha_2, \dots, \alpha_{d_G(v_i)}\}$ if $\min\{s, d_G(v_i)\} = d_G(v_i)$ and $A_i = \{\alpha_1, \alpha_2, \dots, \alpha_s\} \subseteq c'(N_G(v_i))$ if $\min\{s, d_G(v_i)\} = s$, and let $B_i = A_i \cup \{c'(v_i)\}$. Let $k = \max\{\chi_s(G), r + 1\}$. Define $c_r : V(G \odot H) \rightarrow \{1, 2, \dots, k\}$ as follows: For $i \in \{1, 2, \dots, n\}$, $c_r(v_i) = c'(v_i)$ and assign colors to the vertices of H_i so that c_r restricted to $V(H_i)$ is a one-to-one function from $V(H_i)$ to $\{1, 2, \dots, k\} \setminus B_i$.

Let $i \in \{1, 2, \dots, n\}$. Then $|c_r(N_{G \odot H}(v_i))| \geq \min\{s, d_G(v_i)\} + |V(H)| = \min\{s + |V(H)|, d_G(v_i) + |V(H)|\} = \min\{r, d_{G \odot H}(v_i)\}$. Also, let $u \in V(H_i)$. Then $|c_r(N_{G \odot H}(u))| = 1 + d_H(u) = \min\{r, 1 + d_H(u)\} = \min\{r, d_{G \odot H}(u)\}$. Hence, c_r is an r -dynamic k -coloring of $G \odot H$, and so $\chi_r(G \odot H) \leq k$.

This completes the proof of 6 and also the proof of the theorem. $\blacksquare$

In Section 3, we deduce the r -dynamic chromatic number of the corona products $(G_1 \vee K_1) \odot H$ and $G \odot (H_1 \vee K_1)$.

3. Corollaries

It is easy to observe that, for any graph F , if $E(F) \neq \emptyset$, then

$$\chi_s(F \vee K_1) = \begin{cases} \chi(F) + 1 & \text{for } s \in \{1, 2\}, \\ \chi_{s-1}(F) + 1 & \text{for } 3 \leq s \leq \Delta(F) + 1, \\ \chi_{\Delta(F)}(F) + 1 & \text{for } \Delta(F) + 2 \leq s \leq \chi_{\Delta(F)}(F), \\ s + 1 & \text{for } \chi_{\Delta(F)}(F) + 1 \leq s \leq |V(F)|; \end{cases}$$

if $E(F) = \emptyset$, then

$$\chi_s(F \vee K_1) = s + 1 \text{ for } 1 \leq s \leq |V(F)|.$$

Hence, from Theorem 1, we have the following corollaries 1, 2, 3 and 4.

Corollary 1. *Let $G = G_1 \vee K_1$ and $E(G_1) \neq \emptyset$.*

1. $\chi_2(G \odot H) = \chi_1(G \odot H) = 1 + \max\{\chi(G_1), \chi(H)\}$.
2. For $3 \leq r \leq \Delta(H) + 1$,
 $\chi_r(G \odot H) = 1 + \max\{\chi(G_1), \chi_{r-1}(H)\}$.
3. For $\Delta(H) + 2 \leq r \leq \chi_{\Delta(H)}(H)$,
 $\chi_r(G \odot H) = 1 + \max\{\chi(G_1), \chi_{\Delta(H)}(H)\}$.
4. For $\chi_{\Delta(H)}(H) + 1 \leq r \leq |V(H)| + 2$,
 $\chi_r(G \odot H) = 1 + \max\{\chi(G_1), r\}$.
5. For $|V(H)| + 3 \leq r \leq |V(H)| + \Delta(G_1) + 1$,
 $\chi_r(G \odot H) = 1 + \max\{\chi_{r-|V(H)|-1}(G_1), r\}$.
6. For $|V(H)| + \Delta(G_1) + 2 \leq r \leq |V(H)| + \chi_{\Delta(G_1)}(G_1)$,
 $\chi_r(G \odot H) = 1 + \max\{\chi_{\Delta(G_1)}(G_1), r\}$.
7. For $|V(H)| + \chi_{\Delta(G_1)}(G_1) + 1 \leq r \leq |V(H)| + |V(G_1)|$,
 $\chi_r(G \odot H) = \max\{r - |V(H)| + 1, r + 1\} = r + 1$.

Corollary 2. *Let $H = H_1 \vee K_1$ and $E(H_1) \neq \emptyset$.*

1. For $r \in \{1, 2, 3\}$, $\chi_r(G \odot H) = \max\{\chi(G), \chi(H_1) + 2\}$.
2. For $4 \leq r \leq \Delta(H_1) + 2$,
 $\chi_r(G \odot H) = \max\{\chi(G), \chi_{r-2}(H_1) + 2\}$.
3. For $\Delta(H_1) + 3 \leq r \leq \chi_{\Delta(H_1)}(H_1) + 1$,
 $\chi_r(G \odot H) = \max\{\chi(G), \chi_{\Delta(H_1)}(H_1) + 2\}$.
4. For $\chi_{\Delta(H_1)}(H_1) + 2 \leq r \leq |V(H_1)| + 2$,
 $\chi_r(G \odot H) = \max\{\chi(G), r + 1\}$.
5. For $|V(H_1)| + 3 \leq r \leq |V(H_1)| + \Delta(G) + 1$,
 $\chi_r(G \odot H) = \max\{\chi_{r-|V(H_1)|-1}(G), r + 1\}$.

Let $G = G_1 \vee K_1$ and $E(G_1) = \emptyset$. Then, G is the star $S_q = K_{1,q}$ with $q + 1$ vertices, where $q = |V(G_1)|$. For $q = 1$, $S_1 = K_2$. For $q \geq 2$, $\Delta(S_q) = q$ and, for $1 \leq s \leq q$, $\chi_s(S_q) = s + 1$.

Corollary 3. *Let $n \geq 1$.*

1. $\chi_1(S_n \odot H) = 1 + \chi(H)$.
2. $\chi_2(S_n \odot H) = \begin{cases} 3 & \text{if } E(H) = \emptyset, \\ \chi_1(S_n \odot H) & \text{otherwise.} \end{cases}$
3. For $3 \leq r \leq \Delta(H) + 1$, $\chi_r(S_n \odot H) = 1 + \chi_{r-1}(H)$.
4. For $\Delta(H) + 2 \leq r \leq \chi_{\Delta(H)}(H)$, $\chi_r(S_n \odot H) = 1 + \chi_{\Delta(H)}(H)$.
5. For $\chi_{\Delta(H)}(H) + 1 \leq r \leq |V(H)| + n$, $\chi_r(S_n \odot H) = r + 1$.

Corollary 4. *Let $m \geq 1$.*

1. $\chi_1(G \odot S_m) = \max\{\chi(G), 3\}$.
2. For $2 \leq r \leq m + 2$, $\chi_r(G \odot S_m) = \max\{\chi(G), r + 1\}$.
3. For $m + 3 \leq r \leq m + 1 + \Delta(G)$, $\chi_r(G \odot S_m) = \max\{\chi_{r-m-1}(G), r + 1\}$.

4. Associativity

In this section, we obtain the r -dynamic chromatic number of the corona product of three graphs.

Let G_1 , G_2 and G_3 be graphs with n_1 , n_2 and n_3 vertices, respectively. Then the graphs $G_1 \odot G_2$, $G_2 \odot G_3$, $(G_1 \odot G_2) \odot G_3$ and $G_1 \odot (G_2 \odot G_3)$, respectively, have $n_1(1 + n_2)$, $n_2(1 + n_3)$, $n_1(1 + n_2)(1 + n_3)$ and $n_1(1 + n_2(1 + n_3))$ vertices. It follows that, we consider at-least two different graphs, then $(G_1 \odot G_2) \odot G_3$ and $G_1 \odot (G_2 \odot G_3)$ are non-isomorphic and so the operation $\odot$ is not associative. Suppose that $G_1 \equiv G_2 \equiv G_3$, then the operation $\odot$ is associative.

From Theorem 1, we have the following corollaries.

Corollary 5. *Let $G' = (G_1 \odot G_2) \odot G_3$.*

1. $\chi_1(G') = \max\{\chi(G_1), \chi(G_2) + 1, \chi(G_3) + 1\}$.
2. $\chi_2(G') = \begin{cases} 3 & \text{if } \chi(G_1) \leq 2, E(G_2) = \emptyset \text{ and } E(G_3) = \emptyset, \\ \chi_1(G') & \text{otherwise.} \end{cases}$
3. For $3 \leq r \leq \Delta(G_3) + 1$,
 $\chi_r(G') = \max\{\chi(G_1), \chi(G_2) + 1, \chi_{r-1}(G_3) + 1\}$.
4. For $\Delta(G_3) + 2 \leq r \leq \chi_{\Delta(G_3)}(G_3)$,
 $\chi_r(G') = \max\{\chi(G_1), \chi(G_2) + 1, \chi_{\Delta(G_3)}(G_3) + 1\}$.
5. For $\chi_{\Delta(G_3)}(G_3) + 1 \leq r \leq |V(G_3)| + 1$,
 $\chi_r(G') = \max\{\chi(G_1), \chi(G_2) + 1, r + 1\}$.
6. $\chi_{|V(G_3)|+2}(G')$
 $= \begin{cases} |V(G_3)| + 3 & \text{if } \chi(G_1) = 2 \text{ and } E(G_2) = \emptyset, \\ \max\{\chi(G_1), \chi(G_2) + 1, |V(G_3)| + 3\} & \text{otherwise.} \end{cases}$
7. For $|V(G_3)| + 3 \leq r \leq |V(G_3)| + \Delta(G_2) + 1$,
 $\chi_r(G') = \max\{\chi(G_1), \chi_{r-|V(G_3)|-1}(G_2) + 1, r + 1\}$.
8. For $|V(G_3)| + \Delta(G_2) + 2 \leq r \leq |V(G_3)| + \chi_{\Delta(G_2)}(G_2)$,
 $\chi_r(G') = \max\{\chi(G_1), \chi_{\Delta(G_2)}(G_2) + 1, r + 1\}$.
9. For $|V(G_3)| + \chi_{\Delta(G_2)}(G_2) + 1 \leq r \leq |V(G_3)| + |V(G_2)|$,
 $\chi_r(G') = \max\{\chi(G_1), r + 1\}$.
10. For $|V(G_3)| + |V(G_2)| + 1 \leq r \leq |V(G_3)| + |V(G_2)| + \Delta(G_1)$,
 $\chi_r(G') = \max\{\chi_{r-|V(G_3)|-|V(G_2)|}(G_1), r + 1\}$.

Corollary 6. *Let $H' = G_1 \odot (G_2 \odot G_3)$.*

1. $\chi_2(H') = \chi_1(H') = \max\{\chi(G_1), \chi(G_2) + 1, \chi(G_3) + 2\}$.
2. $\chi_3(H')$
 $= \begin{cases} \max\{\chi(G_1), 4\} & \text{if } \chi(G_2) = 2 \text{ and } E(G_3) = \emptyset, \\ \max\{\chi(G_1), \chi(G_2) + 1, \chi(G_3) + 2\} & \text{otherwise.} \end{cases}$
3. For $4 \leq r \leq \Delta(G_3) + 2$,
 $\chi_r(H') = \max\{\chi(G_1), \chi(G_2) + 1, \chi_{r-2}(G_3) + 2\}$.
4. For $\Delta(G_3) + 3 \leq r \leq \chi_{\Delta(G_3)}(G_3) + 1$,
 $\chi_r(H') = \max\{\chi(G_1), \chi(G_2) + 1, \chi_{\Delta(G_3)}(G_3) + 2\}$.
5. For $\chi_{\Delta(G_3)}(G_3) + 2 \leq r \leq |V(G_3)| + 1$,
 $\chi_r(H') = \max\{\chi(G_1), \chi(G_2) + 1, r + 1\}$.
6. For $|V(G_3)| + 2 \leq r \leq |V(G_3)| + \Delta(G_2) + 1$,
 $\chi_r(H') = \max\{\chi(G_1), \chi_{r-1-|V(G_3)|}(G_2) + 1, r + 1\}$.

7. For $|V(G_3)| + \Delta(G_2) + 2 \leq r \leq \max\{\chi_{\Delta(G_2)}(G_2), |V(G_3)| + \Delta(G_2) + 1\}$,
 $\chi_r(H') = \max\{\chi(G_1), \chi_{\Delta(G_2)}(G_2) + 1, |V(G_3)| + \Delta(G_2) + 2\}$.
8. For $\max\{\chi_{\Delta(G_2)}(G_2), |V(G_3)| + \Delta(G_2) + 1\} + 1 \leq r \leq (1 + |V(G_3)|)|V(G_2)|$,
 $\chi_r(H') = \max\{\chi(G_1), r + 1\}$.
9. For $(1 + |V(G_3)|)|V(G_2)| + 1 \leq r \leq (1 + |V(G_3)|)|V(G_2)| + \Delta(G_1)$,
 $\chi_r(H') = \max\{\chi_{r-(1+|V(G_3)|)|V(G_2)|}(G_1), r + 1\}$.

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Data Availability: Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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